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PREFACE

The following pages contain a list of the additions to the Society's Library during the year 1947.

Users of this second supplement are referred to the Catalogue published in 1946 (Bulletin, vol. 52, no. 5, pt. 2) and to the preface to Supplement No. 1 (Bulletin, vol. 53, no. 1, pt. 2) for an explanation of the abbreviations and for a statement of the plan followed in reporting additions to the serial publications.

ARNOLD DRESDEN
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January 1948

SECTION I

SERIAL PUBLICATIONS

ACADEMIA nacional de ciencias exactes, físicas y naturales, Lima
Actas
1+, 1938+
R. ACCADEMIA delle scienze di Torino
Classe di scienze fisiche, matematiche e naturali
Atti
76+, 1940+
AIX-MARSEILLE, Université d'
Faculté des sciences
Annales
Ser. in 8°
13+, 1940+
Index: 1–10
BULLETIN des sciences mathématiques
s2 v65+, 1941+
CAGLIARI
Università
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Seminario scientifico
Rendiconti
10+, 1940+
ČASOPIS pro pěstování matematiky a fysiky. Prague
53–54, 1924–25
ČESKÁ společnost nauk, Prague
Třída mathematicko-přírodovedecka
Věstník
1938+
K. DANSKE videnskabernes selskab, Copenhagen
Mathematisk-fysiske meddelelser
17+, 1939+
GRENOBLE
Université
Section sciences-médecine
Annales
ns v15+, 1938+
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Universitet
Uchenye zapiski
Seriia fizicheskikh nauk
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MATHEMATICAL tables and other aids to computation. (National research council. Division of physical sciences)
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MONTEVIDEO
Universidad
Facultad de ingenieria
Boletin
2+, 1941+
MORAVSKA přírodovědecka společnost, Brünn
Prace
10+, 1936+
PERIODICO di matematiche
s4 v21+, 1941+
PHYSICA
7+, 1940+
Index: 1–10
PISA
Scuola normale superiore
Annali
Scienze fisiche e matematiche
s2 v10+, 1941+

Rozhledy matematicko přírodovědecké
  ns v11, 24+, 1931, 1944+
Sankhyā, the indian journal of statistics. (Indian statistical institute) Calcutta
  1+, 1933+
Suomalainen tiedeakatemia
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    1934+
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  Egyetem
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      Acta scientiarum mathematicarum
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SECTION II

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CORRECTIONS TO BE MADE IN CATALOGUE OF THE LIBRARY[1]

p. 50. Replace "Bovieri-Lapierre" with "Bovier-Lapierre."
p. 67. Replace entry for Durfee, W. F. with "Factor table for the sixteenth million. (Manuscript) n.p., n.d."
p. 77. Delete entry for Föppl, L.
p. 83. Replace second entry for Goldschmidt, L. with "Über den satz Eulers: $(1-x)(1-x^2)(1-x^3)\cdots=\sum_{-\infty}^{\infty}(-1)^n x^{(3n^2\pm n)/2}$. Gotha, 1892.
p. 112. Delete entry for Lekarski, I. C.
p. 140. Replace entry for Roberts, H. L. with "On the transformation group p, q, r, xp, yq, zr, x^2p, y^2q, z^2r. Diss., Virginia, 1899."
p. 165. Entry under Wassiljef, A. should be moved to p. 162 as a second item under the name Vasil'ev, A.

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[1] Bull. Amer. Math. Soc. v. 52 (1946) n5, part 2.

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

CAMBRIDGE, MASSACHUSETTS, U. S. A.,
August 30–September 6, 1950

An International Congress of Mathematicians will be held in Cambridge, Massachusetts, in 1950 under the auspices of the American Mathematical Society. The Society originally planned to act as host for a Congress in September, 1940, which was also scheduled to meet in Cambridge. At the 1936 Congress in Oslo, Norway, the invitation for the 1940 Congress was issued by the American delegation in the name of the American Mathematical Society. Plans for the 1940 Congress were practically completed when the outbreak of World War II in September, 1939, made it necessary for the Society to postpone the Congress to a more favorable date. An Emergency Committee was established to carry on in the interim and, on recommendation of this Committee, the Council of the Society voted to hold the Congress in 1950.

The 1950 Congress will be the third International Congress of Mathematicians to be held on the continent of North America. The first was held at Northwestern University in 1893 and the second at the University of Toronto in 1924. International Congresses were held at intervals of approximately four years, except when war intervened, until 1936. There has been no international gathering of mathematicians since that time and it is the sincere hope of the Organizing Committee that the gathering in 1950 will be a truly international one, that American mathematicians will attend in large numbers, and that all other countries will be well represented. The council of the American Mathematical Society has voted unanimously to hold a Congress which will be open to mathematicians of all national and geographical groups.

Time and Place. The dates for the Congress have been fixed as August 30–September 6, 1950. Harvard University will be the principal host institution. A number of other institutions in metropolitan Boston will join in the entertainment of Congress visitors by arranging special features on their campuses.

Type of Congress. In recent years mathematicians have been much impressed by the success of the conference method for presenting recent research in fields where vigorous advances have just been made or are in progress. In view of the success of mathematical conferences on special topics which have been held in Russia, France and Switzerland and, more recently, at the Princeton Bicentennial Celebration,

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the 1950 Congress will include Conferences in several fields. For the 1940 Congress, Conferences in four fields had been planned. The number of Conferences was thus restricted lest the introduction of a promising and novel feature result in failure through the dissipation of interest and energy. A subcommittee of the Organizing Committee, under the chairmanship of Professor A. A. Albert, is now studying the question of the number and the fields of the Conferences to be included in the 1950 Congress and the results of the committee's deliberations will be reported at a later date.

Following the established custom, the Organizing Committee plans to have a number of invited hour addresses by outstanding mathematicians. In addition, sectional meetings for the presentation of contributed papers not included in Conference programs will be held in the following fields: I, Algebra and Theory of Numbers; II, Analysis; III, Geometry and Topology; IV, Probability and Statistics, Actuarial Science, Economics; V, Mathematical Physics and Applied Mathematics; VI, Logic and Philosophy, History and Education.

The official languages of the 1950 Congress will be English, French, German, Italian, and Russian.

Organization. The plans for the Congress are under the supervision of an Organizing Committee which was elected by the Council of the American Mathematical Society in February, 1948. The Chairman is Professor Garrett Birkhoff of Harvard University and the Vice Chairman is Professor W. T. Martin of The Massachusetts Institute of Technology. Other members of the committee are: Professors J. L. Doob, G. C. Evans, J. R. Kline, Solomon Lefschetz, Saunders MacLane, Dean R. G. D. Richardson, Professors J. L. Synge, Oswald Veblen, J. L. Walsh, D. V. Widder, Norbert Wiener, and R. L. Wilder.

Many of the subventions promised for the 1940 Congress are still available. A Financial Committee under the chairmanship of Professor John von Neumann is endeavoring to secure additional funds. Besides support from Harvard University and The Massachusetts Institute of Technology, generous subventions have been subscribed for the Congress by the Carnegie Corporation, the Institute for Advanced Study, the National Research Council, and the Rockefeller Foundation.

An Editorial Committee under the chairmanship of Professor Salomon Bochner will assume responsibility for the publication of the Proceedings of the Congress.

Professor J. R. Kline of the University of Pennsylvania has been named Secretary of the Congress and Dr. R. P. Boas, Executive

Editor of Mathematical Reviews, has been designated Associate Secretary.

Entertainment. Harvard University has offered the use of its dormitories and dining rooms for mathematicians and their guests for the period of the Congress. The Organizing Committee hopes that it will be possible to furnish room and board without charge to all mathematicians from outside continental North America who are members of the Congress. Congress membership fees and rates for room and board will be announced well in advance of the opening of the Congress.

The Entertainment Committee, of which Professor L. H. Loomis of Harvard University is Chairman, is planning many interesting features, including a reception, garden party, symphony concert, and banquet. It is hoped that American mathematicians will be able to assist in the entertainment by putting their automobiles at the disposal of the Entertainment Committee for trips to be made out of Cambridge.

Every effort will be made to facilitate the travel at reasonable cost of foreign participants while in the United States. Previous to the Congress, opportunity will be given them to see New York City under the guidance of some mathematicians.

Information. Detailed information will be sent in due course to individual members of the American Mathematical Society and to foreign mathematical societies and academies. Others interested in receiving information may file their names in the Office of the Society, and such persons will receive from time to time information regarding the program and arrangements.

Communications should be addressed to the American Mathematical Society, 531 West 116th Street, New York City 27, U. S. A.

THE ORGANIZING COMMITTEE

MATHEMATICAL PROBLEMS IN OPERATIONS RESEARCH

PHILIP M. MORSE

During the past war a new field of application of the scientific method was developed, called operations research, or operational analysis. A number of mathematicians contributed to this development and some of their work should be of general interest to mathematicians, though reasons of military security have precluded publication of much of it. It is my intention to discuss a number of problems in this field which can now be talked about, and to indicate why further work in operations research is of some importance at present.

The term operations research has heretofore been used to connote various attempts to study the operations of war by scientific methods. From a more general viewpoint, however, operations research can be considered to be an attempt to study those operations of modern society which involve organizations of men or of men and machines. The study is behavioristic, not subjective, and the aim is to be able to predict the changes produced in the operation by changes in external conditions. The subject can be said to cover, for example, the quantitative aspects of city planning (including traffic analysis), those parts of the study of telephone traffic handling which are called systems engineering, and certain aspects of efficiency engineering. Its methods might be applied to a scientific study of peacetime operations, such as those of railroads or of other transportation systems, if this were desired. In each case one seeks to study the dependence of certain measures of effectiveness of an operation upon certain operational parameters which are subject to the control of the director of the operation. It is important to appreciate what operations research is *not*, as well as what it is: it is *not* efficiency engineering or historical analysis nor is it a branch of applied statistics. It views operations from a much more active and *experimental* point of view than do these other subjects.

It is obvious that operations research is a branch of engineering, rather than of pure science, since it is concerned with the application of scientific method to immediate and pressing practical problems. At least in the initial stages it must be developed in close connection with these practical problems and in personal contact with the administrators who have control over the operation under study. The

The twenty-first Josiah Willard Gibbs lecture delivered at Athens, Georgia, December 29, 1947, under the auspices of the American Mathematical Society; received by the editors January 9, 1948.

fundamental data concerning the operation must be obtained in part by first-hand observation and in part by judicious variation of the operational parameters involved. After the data are collected the results of their analysis are of value to the administrator in charge, in helping him plan future operations.

As with any field of engineering, a number of branches of science are called upon for help. For example, physics is required for a basic understanding of the operational possibilities of the machines involved, and biophysics and psychophysics are required in order to understand the capabilities of the human components. Mathematics is (as always) basic to all of the work, and some of the problems already encountered have required mathematical abilities of fairly high order to solve. During the war it was found that a mathematician, a physicist and a biologist together make up a research team of considerable effectiveness. The biologist is often useful chiefly for his ability to draw mechanistic inferences from a little data concerning very complex systems.

Operations research is still of considerable importance, and there are at present more positions open in the field than there are compentent workers to fill them. Because of the success of several operations research groups during the war the armed services of this country all wish to maintain groups of this sort now; and in the present state of international relations, it is probably important that such groups be maintained. In addition, a number of large industries have shown interest in applying the same techniques to improving their own operations.

The writer wishes to call the attention of mathematicians to these opportunities, and to indicate, by a group of examples, that problems of mathematical interest are encountered in operations research. The group of problems chosen for discussion here are from the military field, because up to the present, the military applications have been most intensively studied. The examples were chosen for their mathematical interest, not necessarily for their practical importance. Some of the results could have peace-time application, of course; this will be seen as the discussion progresses. Further work in peacetime operations research will certainly bring to light other problems of equal or perhaps greater mathematical interest.

To sum up this preliminary discussion in concise form:

1. Operations research (in its military application) is the quantitative study of a strategic or a tactical operation, or a part of one.

2. It gets its data from combat or trial action reports and from physical or psychological tests of the mean and machines involved.

3. It uses idealized and simplified pictures of the operation (as all branches of applied mathematics do) to serve as the basis of analytic calculation and prediction.

4. It has an amply demonstrated practical value, and it is capable of wide application (cf. mathematical economics, traffic analysis, epidemiology, systems engineering, and so on).

5. It can lead to new and challenging problems in mathematics, as some of the following discussion may indicate.

The search problem. A very large class of problems encountered in naval and air operations is related to the process of *search*; an *observer*, equipped with some means of making contact with a *target*, moves or is moved over an area or volume in some more or less regular *pattern of search*; the problem is to find the pattern which most efficiently results in contact under specified circumstances. The problem is applicable to many cases: the means of contact may be visual, by radar or sonar; the means of transport of the observer may be by aircraft or beneath the water; the "observer" may be a proximity-fused guided missile, and the "contact" desired may be the destruction of the target; peacetime applications to geological prospecting are obvious, and so on. Mathematically, the problem reduces often to one of calculus of variations, but, in many cases, with "something new added."

The problem[1] usually can be divided into a number of parts: the *contact problem*, dealing with the relationship between the physical properties of the detection equipment and the probability of contact with the target, when observer and target are in given relative positions; the *track* or *pattern problem*, dealing with the determination of the optimum pattern of search for given conditions; and the *tactical problem*, dealing with the reciprocal effects occurring when the target is also provided with detection equipment. Analyses of the tactical problem often require the techniques[2] developed by von Neumann for his theory of games. All aspects of the search problem involve fundamental concepts and techniques of the theory of probability, expressed from a point of view enough different from that of classical probability theory as to cast new light on some of its concepts and techniques.

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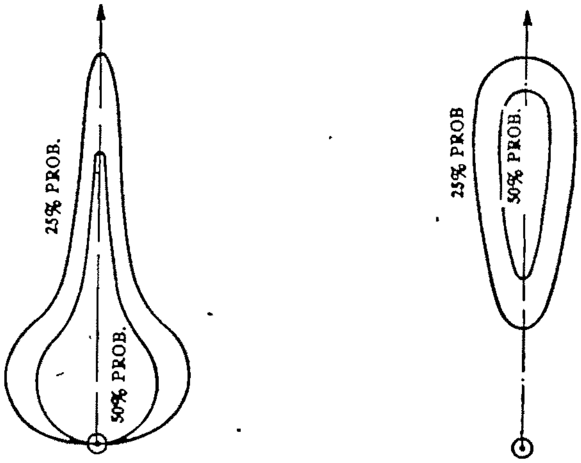
[1] Much of the discussion here follows the work of B. O. Koopman, who made major contributions to this field while working as a member of the Operations Research Group (now Operations Evaluation Group) U. S. Navy Office of the Chief of Naval Operations.

[2] *Theory of games and economic behaviour* by J. von Neumann and O. Morgenstern, Princeton University Press, 1944.

Probability of contact. The mathematical aspects of the contact problem start with the definition of the *instantaneous probability of contact*, g. Nearly all detection equipment searches by means of "glimpses": a radar set sends out a sequence of pulses, which may or may not return a measurable echo from the target; a sonar set behaves similarly, though with longer intervals between pulses; even the human eye searches the horizon in a series of momentary "fixations." The probability contours for single-glimpse detection of an object on the surface of the ocean by eye or by radar are plotted in polar diagrams in Fig. 1, as functions of range and horizontal angle. By way of contrast, the contours for probability of "contact" (that is, of destruction) for a single bomb would be concentric circles, the probability g here being simply a function of r, the distance between the target and the point of impact of the bomb.



VISUAL GLIMPSE

RADAR PULSE

FIG. 1. Polar contours of instantaneous probability of detecting target on ocean surface for single eye fixation and for single radar pulse. Observer is at O.

These instantaneous probabilities are next combined to obtain the probability of detection (or destruction) by any of a number of glimpses (or bombs). It turns out that the individual probabilities are independent of each other in most of the cases studied, so the usual rules for combination of probabilities can be applied. The probability that a contact is made in at least one of n glimpses is

$$
\begin{aligned}
P &= 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_n) \\
&= 1 - \exp\left[- \sum_{m=1}^{n} \Phi_m \right]; \qquad \Phi_m = - \ln (1 - g_m).
\end{aligned} \tag{1}
$$

As long as the glimpse probabilities are independent the quantities Φ_m are additive; for this reason they are called *contact potentials*.

This equation suffices for the basis of a large number of problems connected with gunnery and area bombardment. Combination of probabilities of destruction with probabilities representing gunnery or bombing errors will enable over-all probabilities to be computed and optimum patterns for different targets to be determined. In some cases of practical importance (as in rapid-fire gunnery) successive probabilities are *not* independent, and the above equation is not valid.

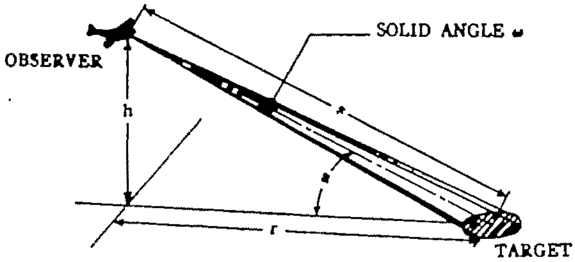


FIG. 2. Visual detection of wake on ocean surface by observer in aircraft.

In the case of actual search for a target (on the surface of the ocean, for instance) we must take into account the fact that the search goes on continuously as the observer moves about. During a time dt which is short compared to the total time of search but which is long compared to the average time between glimpses, the eye or radar set scans the horizon about equally well in all directions. Therefore the probability of making contact within the time interval dt can be written as $\gamma(r)dt$, where the quantity γ is usually independent of the angle between the line of sight and the direction of motion of observer or target. The quantity γ depends on r, the horizontal distance between observer and target, and on the visibility conditions (atmospheric haziness, state of repair of radar set, and so on) at the time of search.

For instance, within rather narrow limits, the probability rate γ for visual detection of a distant object is proportional to the solid angle subtended by the object at the eye. If the object searched for is a wake of constant area on the surface of the ocean and if the observer is in an aircraft at height h and horizontal range r from the target, study of Fig. 2 will show that if $r \gg h$,

$$\gamma = \kappa\omega = (B/s^2)\sin\alpha = (Bh/s^3) \simeq (Bh/r^3), \tag{2}$$

where the quantity B depends on the area and contrast of the wake, the amount of haze in the air, and so on. This is known as the *inverse cube law* for contact rate. This equation is only valid for search for small objects in conditions of good visibility, but it serves as a good example, which can be handled analytically.

Lateral range and search width. Next we must consider the effect of the motion of the observer with respect to the target. As shown in Fig. 3, we picture this most easily by considering the observer to be at rest and we assume that the relative velocity is constant both in magnitude, v, and in direction, as long as the two are within detection range. For a certain initial configuration the distance of closest approach will be x (which is called the *lateral range*). The position of the target along its path of relative motion is given by y.

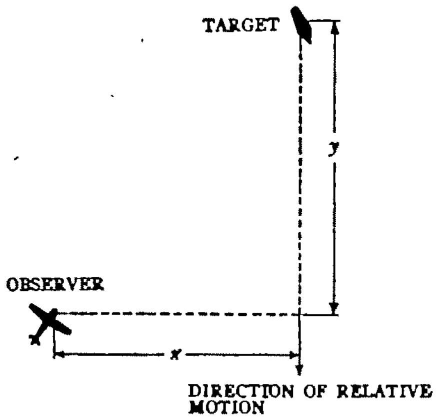


FIG. 3. Relative motion of target past observer, relative velocity is v and lateral range is x. Probability of contact during transit.

It should be obvious that the probability of contact during the time while the observer is passing by the target is given by the formula

$$p(x) = 1 - \exp\left[-\frac{1}{v}\int_{-\infty}^{\infty}\gamma((x^2+y^2)^{1/2})dy\right]. \tag{3}$$

For instance, for the inverse cube law given in Equation 2, the contact probability as function of lateral range turns out to be

$$p(x) = 1 - e^{-2\beta/x^2}; \qquad \beta = (Bh/v). \tag{4}$$

Possible curves for contact probability $\gamma(r)$ and corresponding lateral range probability are shown in Fig. 4. In the first case, the probability of contact increases so rapidly at range R that there is almost certainty of detection whenever the lateral range is less than R. This is called the *definite range case*. The other case shown is that for the inverse cube law.

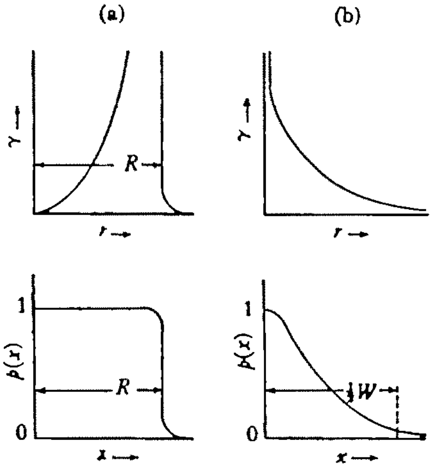


FIG. 4. Curves for instantaneous contact rate $\gamma(r)$ together with the corresponding lateral range probability, $p(x)$.

Case (a) is the definite range case.

Case (b) is for the inverse cube law.

Now suppose that the target is at rest somewhere along a line of length D (long compared to the range of detection) at right angles to the motion of the observer. If the observer's track crosses this line at some randomly chosen point, independent of the position of the target, then the probability that contact will be achieved turns out to be

$$P = \frac{W}{D}; \qquad W = \int_{-\infty}^{\infty} p(x)dx \tag{5}$$

where W is called the ***effective search width*** for the particular observer-target pair under study. For the definite range case, $W=2R$, where R is the definite range. For the inverse cube law,

$$W = 2(2\pi\beta)^{1/2} = 2(2\pi(Bh/v))^{1/2}.$$

As might be expected, the search width is a useful measure of the effectiveness of a piece of detection equipment. The quantity vW, called the ***search rate***, gives the effective number of square miles which can be searched over in an hour (if W is in miles and v in miles per hr.). For the inverse cube law the search rate is proportional to the square root of the velocity and of the altitude of the plane (assuming a cloudless day). In actual practice the dependence of search rate on velocity is not as simple as this example would indicate.

Search pattern. A problem of some interest, and one which is basic to the question of search pattern mentioned earlier, is: a single target is placed at random somewhere inside an area A, whose dimensions are large compared to the search width W; what is the probability that, after T hours of search by the observer, the target is found? The answer will, of course, depend on how the search is carried out, that is, on the pattern of search. We shall obtain the answer for two different sorts of search pattern.

As one extreme, we assume that the search is completely random; the track of the observer (or observers, for several identical observers may take part) consists of pieces of straight line, of various lengths, placed at random within the area A, and covering it in a more or less uniform manner. For this search pattern the probability that the target be passed at a lateral range x is independent of x, and the chance of finding the target in time dT is just the search rate, vW, times dT, divided by A. Therefore the probability of making contact at some time during the search is

$$P = 1 - e^{-WL/A} \tag{6}$$

where $L=vT$ is the total length of track passed over by the observer (or observers) during the search.

Another search pattern, somewhat more efficient, is to cover area A with straight, parallel tracks in as uniform a manner as possible. If the total track length is to be L, the spacing between parallel tracks is $S=(A/L)$. If the target is a lateral distance x from one of the tracks it can be shown that the probability of contact is

$$P(x, S) = 1 - e^{-\Phi(x,S)},$$

$$(7) \qquad \Phi(x, S) = -\sum_{n=-\infty}^{\infty} \ln\left[1 - p(x - nS)\right]$$
$$= \prod_{n=-\infty}^{\infty} \frac{1}{v} \int_{-\infty}^{\infty} \gamma((y^2 + (x - nS)^2)^{1/2})\,dy$$

where $p(x)$ is the lateral range probability defined in Equation 3.

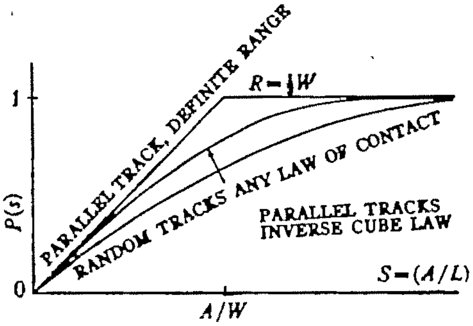


FIG. 5. Probability of contact during search of track length L, for single target at random in area A.

Since the target is placed at random in the area the probability of contact is the average value of P:

$$P(S) = \frac{1}{S} \int_0^S P(x, S)\,dx.$$

For the definite range case, where $p(x)$ is zero for $x > R$ (where $2R = W$) and unity for $x < R$, $P(S)$ is equal to $(2R/S) = (WL/A)$ whenever S is larger than $2R$ (in this case the searched-over paths do not completely cover the area); and is equal to unity when S is smaller than $2R$ (in this case the area searched covers A completely). For the inverse cube law, discussed earlier, the result is

$$(8) \qquad P(S) = \operatorname{erf}(\pi(2\beta)^{1/2}/S) = \operatorname{erf}(\pi^{1/2}W/2S).$$

Inspection of Fig. 5, which shows the contact probability as function of total track length L for the three cases discussed, displays the amount of advantage to be gained by care in arranging the pattern of search. Purely random search introduces inevitably a certain amount of overlapping of search effort, which is eliminated by the use of parallel, equidistant tracks. For the definite range case, no overlap occurs until the track spacing S becomes smaller than $2R$. The inverse cube law case represents an intermediate case, in that a certain amount of overlap of search effort occurs even with parallel tracks. In actual practice it is quite difficult to follow exactly parallel, exactly equidistant search tracks, and careful navigation is required to obtain search efficiencies appreciably larger than the efficiency corresponding to the curve for random search.

Of course it is easy to design a search pattern which would be considerably less effective than the curve for purely random search. Concentration of search effort in one part of the area, for instance, will reduce the probability (as long as the target is equally likely to be anywhere). On the whole, however, any search effort which attempts to cover the area in a reasonably uniform manner will produce results approximately equal to the random search case, and considerable care in navigation will enable markedly better results to be attained. In the succeeding parts of this paper the formula for probability of contact for random search, given in Equation 6, will be used as a minimum attainable measure of effectiveness; any reasonably well designed and executed search pattern should do better than this, none should do worse unless they are badly designed.

We next consider a somewhat more complicated case, where the target may be in one or the other of two areas, A_1 and A_2. The probability that it is in A_1 is p_1, with its whereabouts within the area unknown; the probability that it is in A_2 is $p_2 = 1 - p_1$. The problem is as follows: suppose we have a certain amount of searching effort, represented by the track length L; how do we distribute this effort between A_1 and A_2 to have the best chance of finding the target?

If we try assigning a length of track x to area A_1 and distribute it uniformly over the area (so we can use Equation 6 for the chance of contact), then a track-length $(L-x)$ will be left to be used in area A_2 and the contact probability will be

$$P(x) = 1 - p_1 e^{-(W/A_1)x} - p_2 e^{-(W/A_2)(L-x)}. \tag{9}$$

We have therefore to make $P(x)$ a maximum, or $p_1 e^{-(W/A_1)x} + p_2 e^{-(W/A_2)(L-x)}$ a minimum, in the range $0 \leqq x \leqq L$.

To make the discussion more concrete, let us assume that the probability density (p_1/A_1) for the target in the first area is greater than that in the second area. We start with a small allowed track length L and investigate how the best distribution of search effort changes as we have more and more track length available. For a small enough value of L, we have $(p_2/A_2) \leqq (p_1/A_1)e^{-(WL/A_1)}$. When this is true the

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quantity $p_1e^{-(W/A_1)x}+p_2e^{-(W/A_2)(1-x)}$ ***has no minimum*** for x in the range $0<x<L$, its least value is for $x=L$ corresponding to all search in A_1. In other words, if the available search effort is small enough we should ***concentrate this effort in the most promising area and disregard the other area entirely***. This is a result which is not often encountered in classical calculus of variations problems, but which often turns up in operations research problems.

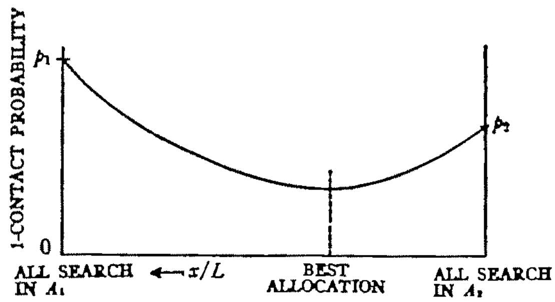

FIG. 6. Calculation of optimum distribution of search effort between two areas A_1 and A_2 with p_1 the probability the target is in A_1 and p_2 the probability it is in A_2. Total track length L, track length in A_1 is x. Probability of contact for random search.

If now the available path length L is sufficiently increased there will be a maximum for $P(x)$ in the range $0<x<L$ as shown in Fig. 6, and the usual process of maximization will obtain the proper answer for x. It is of interest to note that, if L is large enough, the solution indicates that more time should be spent in A_2 than in A_1, although A_2 is to be ignored when L is small. If there is plenty of effort available it is worthwhile to spend a lot of it on the less rewarding area.

These very simple examples will perhaps show how the search-pattern problem can be attacked, and may perhaps indicate the difficulties which are encountered as soon as less simple cases are tackled. The difficulties, it should be brought out, are mathematical ones and not ones of boring detail. The solution of a search-pattern problem which includes, for instance, a certain distribution in velocity on the part of the target requires mathematical skill of no mean order. For that matter, the comparatively simple problem of determining the optimum pattern of search track when the probability density of the target varies from point to point is beyond the present capacities of the calculus of variations. The integral giving the probability of detection can be written down for each chosen pattern of track; but if one attempts to compute the optimum pattern, an integral equation is obtained which so far defies even formal solution.

And yet these more complex problems should be solved. We have spoken here only of two-dimensional search problems, typical of a surface fleet. But if "push-button warfare" should become a reality, three-dimensional search will be a matter of life or death, and the time allowed to find and destroy a guided missile headed for our shores is likely to be limited indeed.

A point of interest in theory of probability can be mentioned here. In practice one needs the value of the effective search width W before one can lay out search patterns. This value is obtained from data collected from past operations. It is, of course, an average value, averaged over the fluctuations of W which occur in normal operation. These fluctuations are of two types: the short-term fluctuations which occur during an individual search, due to rolling and pitching of the target, momentary fluctuations in attention of the observer, and so on; and long-term fluctuations, which occur from trip to trip, due to changes in atmospheric properties or instrument maintenance, etc. The value of W obtained from past operational data is usually an average over both long and short-term fluctuations. But *is this the best value of W to use in laying out the next individual search*? If the atmosphere or equipment is in top form one day is there some way of finding this out and/or taking advantage of it? Here is a point which strikes close to the fundamentals of probability theory.

Measure and countermeasure. In operations research one often encounters the following sort of problem:[3] if A uses one tactic, B should use a certain defense, but if A uses another tactic, B should do something else; what are the best tactics for both A and B? Problems of this sort are discussed by von Neumann and Morgenstern[4] in their *Theory of games*. In some of the problems they discuss the answer is definite; there is one particular action which A must take, and likewise for B, in order that neither A nor B shall lose too much. In other words, if A took any other action than the recommended one, B could take advantage of him, and vice versa.

In other problems the answer is indefinite; any one action on the part of A will eventually lead to trouble if the "game" is played several times; what is needed is a variation of tactics, in a random manner, so that B is never sure which action A will take next time. This sort of situation often turns up in operations research. The enemy may have a countermeasure to some weapon of ours which renders the weapon a liability in certain cases. Should we use the weapon or not?

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[3] The following examples are taken from some of the work done by the author and G. E. Kimball while they were with the Operations Research Group, U.S.N.

An example of this type of analysis can again be taken from the theory of search. A submarine wishes to pass through a channel without being detected by a patrol aircraft, which flies back and forth across the channel each day. The submarine can escape detection by submerging, but we assume that it can only travel a total distance a submerged and that a is shorter than the channel length. It can surface and dive again several times, but the total length of path submerged is no larger than a.

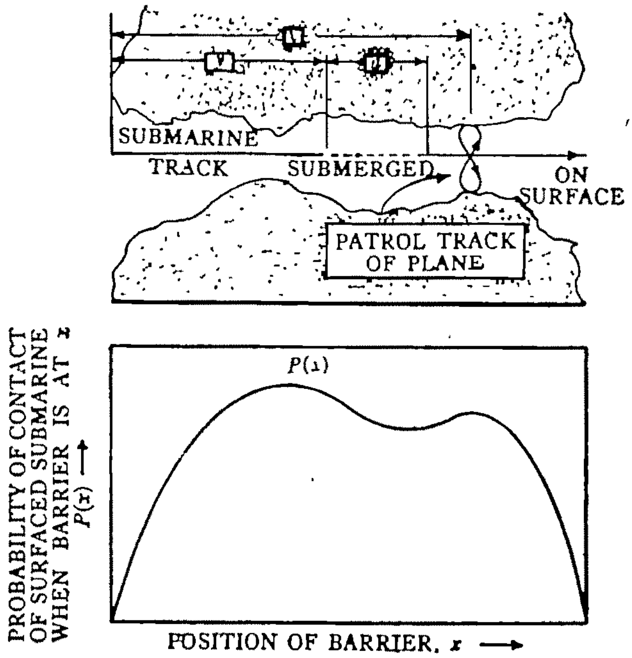

FIG. 7. Barrier patrol versus diving submarine. $P(x)$ = Probability of contact if barrier is at x and submarine is surfaced.

The channel is supposed to be wide enough so that only a few trips are possible by the plane each day. Consequently each day the plane must choose which point along the channel it is to patrol across. If it chooses a wide part the number of traverses will be fewer and the chance of detection, even if the submarine is surfaced at that point, is less than if the plane were patrolling across a narrower part. In fact, one can plot a curve giving the probability $P(x)$ that the plane will discover the submarine if it patrols across the portion a distance x from one end of the channel and if the submarine is not submerged when passing this part. Such a curve is given in Fig. 7.

It might seem, at first sight, that the aircraft should always patrol where P is largest. But if this action were taken the submarine would soon learn to dive when it goes past that point and the plane would *never* see the submarine. Obviously the plane must spread out its effort, one day patrol at one value of x, next day at another and so on, with the frequency of patrol at x governed by some predetermined function $\phi(x)$, which is called the *density of the barrier* at x. This patrol effort should not be distributed from point to point in a regular manner, or the opponent could take advantage of this; the assignment of patrols should be in a random manner, subject to the density function $\phi(x)$.

Similarly, if the submarine is always submerged in one region the plane could patrol outside that region and the submarine would lose the advantage of its submergence. It must vary its regions of submergence in a random manner, with a relative frequency according to a density function $\psi(x)$. To recapitulate, equation-wise:

$$\begin{aligned} &\text{Density of patrol} = \phi(x); && \int \phi(x)dx = 1, \\ (10)\qquad &\text{Prob. of submergence} = \psi(x); && \int \psi(x)dx = a, \\ &\text{Prob. of detection } G = \int P(x)\phi(x)[1 - \psi(x)]dx. && \end{aligned}$$

First let us consider the way in which the submarine commander must analyze the situation to decide on the submergence density $\psi(x)$. He must choose ψ so that if the opposing side learned what ψ he was using they could not gain an advantage, but if he learned the patrol plans of the plane (the values of ϕ) he might be able to gain an advantage.

If the airplane patrol-squadron commander found out that the submergence density function for the submarine was to be $\psi(x)$, he could make the probability of contact G a maximum by flying his patrol plane at the point x where $P(x)[1-\psi(x)]$ has its maximum value. The submarine commander would, of course, like to make the quantity $P(x)[1-\psi(x)]$ equal to zero, everywhere, but this is not possible, since the channel is longer than the maximum range of submergence, a. The next best thing for the submarine commander to do is to arrange so that $P(x)[1-\psi(x)]$ has no single maximum, that is, is flat along the top. This is done as follows:

Safe tactics for submarine. Over as great a range of x as possible, choose $\psi(x)$ so that

$$P(x)[1-\psi(x)] = H, \quad \text{a constant, that is}$$

$$(11) \quad \psi(x) = \begin{cases} 1 - \dfrac{H}{P(x)}, & \text{when } P(x) > H \text{ (over range } B \text{ of } x), \\ 0, & \text{when } P(x) < H \text{ (outside range } B \text{ of } x). \end{cases}$$

The value of H is determined by the submergence limitations of the submarine,

$$\int \psi dx = \int_B \left(1 - \frac{H}{P(x)}\right) dx = a, \quad \text{where the integration is only over range } B \text{ of } x.$$

If the submarine follows these submergence tactics then no matter what tactics the patrol plane follows, the probability of contact will never be greater than H, and it may well be less than H if the patrol plan is badly placed. The integral determining H ensures that H is as small as possible, consistent with the submergence capabilities of the submarine.

These are the safe tactics for the submarine. If the submarine commander learns the plans of the aircraft patrol, he can alter ψ so that G, the contact probability, is less than H; but if he is not sure about the air patrol plans then a choice of the tactics of Equation 11 will ensure that the probability of contact is never larger than H even if the opposing side has a perfect spy system.

The details of the solution are shown in the left-hand side of Fig. 8. We plot the reciprocal of $P(x)$ and draw a horizontal line of height $(1/H)$ so adjusted that the area between this line and the curve $(1/P)$ is equal to (a/H). The difference between this line and the $(1/P)$ curve, multiplied by H, will then be ψ which is plotted in the lower left-hand figure. If the air patrol is placed anywhere between x_0 and x_1 the probability of contact G is equal to H: if it is placed outside this range G will be less than H.

We note that if a is small enough, or if the value of $P(x)$ fluctuates more widely, the line at height $(1/H)$ may cut the line $(1/P)$ at more than two points. When this is the case the region for submergence is divided into two or more parts; those regions for which $(1/P)$ is larger than $(1/H)$ are excluded from the diving schedule ($\psi=0$ there) and for those regions where $(1/P)$ is smaller than $(1/H)$, ψ will equal $1-(H/P)$ as before. This illustrates again a property of many solutions of operational problems: if the resources available are limited they should be used only in the more promising regions, the less promising regions *should be disregarded entirely.* The same result turned up in the example, quoted earlier, of the search of two areas. It turns up in many problems discussed by von Neumann and Morgenstern.[3]

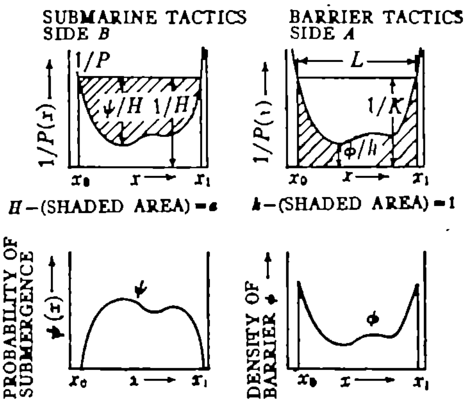

FIG. 8. Solutions of problem shown in figure 7 for safe tactics for both sides.

This property of solutions of operational problems should not be surprising to us (though it is not a usual result in the classical examples of the calculus of variations) for many of these problems are minimax problems with definite boundaries imposed on the parameters by reason of force limitations. Hence it often happens that the minimax is on one of the boundaries (where one of the parameters is zero, for instance) instead of inside the boundary.

Now let us see what the patrol squadron commander must do to ensure that the contact probability G be as *large* as possible. We assume that he knows that the submarine can only travel a distance a under water. If he knew, in addition, that the submarine was submerging according to the schedule ψ he could fly his patrol anywhere between x_0 and x_1 and get the same value of G. But flying the patrol always at one point (or even at several points) is a dangerous thing to do, for if the submarine commander were to learn this he could alter ψ so as to reduce G. The aircraft commander must fly his patrol at random along x, with a frequency predetermined by a probability density factor $\phi(x)$ (see Equation 10), such that if the submarine commander found out the shape of ϕ he would not gain any advantage (he could not lower G) but if the aircraft commander found out the submarine's schedules, a change of patrol might enable G to be increased. This requirement, we see, is symmetric to that for the submarine. As with the submarine, the barrier density ϕ must be adjusted so that the quantity $P(x)\phi(x)$ which comes into the integral for G has no maxima, but is flat on top. Therefore no barrier will be flown for those ranges of x where P is less than some limiting value K; and where P is larger than K the barrier density ϕ will be inversely proportional to P, so that $P\phi$ be a constant in those regions. Therefore we have:

Safe tactics for patrol aircraft. Choose $\phi(x)$ so that $P(\phi)$ equals some constant h over a range of x,

$$(12)\quad \phi(x) = \begin{cases} \dfrac{h}{P(x)}, & \text{when } P(x) > K \text{ (over range } C \text{ of } x\text{)}, \\ 0, & \text{when } P(x) < K \text{ (outside range } C \text{ of } x\text{)}, \end{cases}$$

where h is chosen so that $\int \phi dx = 1$, that is, $(1/h) = \int_C (1/P)dx$, with the integral taken only over range C of x. The length L of the region patrolled is the length of range, $\int_C dx = L$.

Assuming that the submarine submerges within range C of x, the probability of contact G would be $h(L-a)$; in any case G would never be less than $h(L-a)$ for the patrol schedule ϕ given in Equation 12. We have not yet chosen the value of K, and therefore the length L of range C of x. It certainly would not be advisable to make L smaller than a, for then the submarine could dive under the whole barrier. On the other hand it would not do to make L too large, for then h would become too small. In fact, we must adjust L and h so that $h(L-a)$ is a maximum.

A certain amount of algebraic manipulation is required to show that the requirement for $h(L-a)$ to be maximum is equivalent to the requirement that $K = H$, or that range C for the aircraft is identical with range B for the submarine submergence. This is of course reasonable, for any lack of coincidence on the part of either side would represent wasted effort.

The graphical method of determining the patrol schedule $\phi(x)$ is shown in the right-hand side of Fig. 8. One plots again the reciprocal of $P(x)$, draws the line of height $(1/K) = (1/H)$ and determines the range $C(=B)$ of x. We choose the normalizing factor h, such that h times the shaded area equals unity. Then the best the submarine can do if the aircraft patrol is alert, and the best the aircraft can do if the submarine commander is clever, is the set of tactics outlined in Equations 11 and 12 and in Fig. 8, resulting in the probability of contact $G=H=h(L-a)$. Any departure from these tactics by one side will enable the other side to gain an advantage if the departure is ascertained.

The problem considered here is an extremely simplified example of a large class of problems which turn up continuously in tactical studies and in many peacetime applications of operations research. The difficulties of solving such problems are not ones of tedious detail, but are often due to lack of fundamental techniques. Much more basic research must be carried out before many problems of practical importance can be solved.

Lanchester's laws. One final example, from the field of strategy this time, deals with the relation[4] between size of opposing forces and the casualty rates in combat of various types. Suppose that at time t there are m units of one side engaging n units of the other side. The time rate of decrease of m and n as the battle continues will depend on the nature of the engaging units and the types of weapons used. For some types of engagement the relations are:

$$\frac{dm}{dt} = -A, \quad \frac{dn}{dt} = -EA, \quad \frac{dn}{dm} = E, \qquad (13)$$
$$n_0 - n = E(m_0 - m)$$

where the constant E, called the *exchange rate*, is a measure of the relative efficiency of the weapons used by the two sides. This relationship is called *Lanchester's linear law* of combat.

Another type of engagement corresponds to the equations

$$\frac{dm}{dt} = -\alpha n, \quad \frac{dn}{dt} = -\alpha E m, \quad \frac{dn}{dm} = (mE/n) \qquad (14)$$
$$n_0^2 - n^2 = E(m_0^2 - m^2).$$

This relationship is called *Lanchester's square law* of combat. It is more often applicable to engagements of modern warfare (where gun fire rather than hand-to-hand combat predominates) than is the linear law. Both sets of equations are related mathematically to the equations governing biological equilibria, epidemiology and theory of evolution.[3]

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[4] See *Aircraft in warfare* by F. W. Lanchester, London, Constable, 1916, for a discussion of some of the relations referred to here.

The solutions of these and related differential equations have been obtained and the consequences discussed in some detail.[4] For instance, from the solution of the square law the strategic advantages of the concentration of force are at once apparent, for a doubling of the number of units in any engagement is worth as much as the *quadrupling* of the weapon effectiveness.

Solutions of the differential equations do not tell the whole story, however, for they give only the results of a sort of "average engagement," and do not indicate the sort of fluctuations which might be expected in this non-regular world. What is needed is a detailed statistical analysis of the problem.

For the linear law this is not difficult. We define a "unit combat" as that portion of the engagement during which, on the average, $1/(1+E)$ red units (m's) are lost and $E/(1+E)$ blue units (n's) are lost. After T combats the probability that α red units and β blue units are actually lost is, of course,

$$P(\alpha, \beta) = \frac{T!}{\alpha!\beta!} = \frac{E^{\alpha}}{(1+E)^{T}}; \qquad T = \alpha + \beta; \quad \alpha < m_0; \quad \beta < n_0. \tag{15}$$

Special consideration is required to obtain expressions for $P(\alpha, n_0)$, the probability that the engagement is ended with all of the blue units lost and $(m_0 - \alpha)$ red units left, and for the symmetrically related $P(m_0, \beta)$ corresponding to annihilation of the red forces. The results are

$$P(\alpha, n_0) = \frac{(\alpha + n_0 - 1)!}{\alpha!(n_0 - 1)!} \frac{E^{\alpha}}{(1+E)^{\alpha+n_0}}, \quad P(m_0, n_0) = 0. \tag{16}$$

From these probabilities one can calculate the chances of any deviation from the solution of the differential equation 13.

A similar analysis of the square law is more difficult, for here we cannot eliminate the time from explicitly entering the formulas. The probability function here is $P(m, n, t)$, the probability that at time t there are m red units and n blue units still unharmed. We change the time scale so that during dt, the chance that a red unit is hit is

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[3] See, for instance, V. Volterra, *The theory of the struggle for life*, Gouthiers-Villars, Paris, 1931, and A. V. Lotka, *Elements of physical biology*, Williams and Wilkins, Baltimore, 1925.

$(ndt/E^{1/2})$ and the chance that a blue unit is hit is $(mdtE^{1/2})$. Then a consideration of the interrelationship of the functions P shows that

$$\begin{aligned}
&\frac{d}{dt}P(m, n, t) = (mE^{1/2})[P(m, n+1, t) - P(m, n, t)] \\
&\qquad\qquad\qquad + (n/E^{1/2})[P(m+1, n, t) - P(m, n, t)], \\
(17)\quad &\frac{d}{dt}P(m, 0, t) = (mE^{1/2})P(m, 1, t), \\
&\frac{d}{dt}P(0, n, t) = (n/E^{1/2})P(1, n, t).
\end{aligned}$$

This set of equations should be solved subject to the initial condition that $P(m_0, n_0, t)$ is unity and all other P's are zero at $t=0$.

Investigation of these and similar equations for other types of combat have not received the detailed study they merit. Various approximate and asymptotic solutions of Equations 15 and 17 are needed in order to show general trends more clearly than can the very complicated exact solutions. Other equations, corresponding more closely to specific strategic problems, should be investigated.

Conclusion. Perhaps the foregoing examples will serve to indicate that the mathematical problems encountered in operations research are not trivial, and that while many *important problems can be formulated in precise mathematical form, a precise solution of all but the very simplest of these is beyond the capabilities of mathematics at present.* It is to be hoped that more of the work completed during the war can be made available to the mathematical public, and that students can be trained in this and related subjects, so that they may contribute to the peace-time applications of this new field of applied mathematics and so that they may be available to contribute in important ways to our defense in case of war.

BROOKHAVEN NATIONAL LABORATORY

THE APRIL MEETING IN NEW YORK

The four hundred thirty-fourth meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, April 16–17. The attendence was over three hundred, including the following three hundred six members of the Society:

C. R. Adams, E. B. Allen, C. B. Allendoerfer, R. L. Anderson, R. G. Archibald, L. A. Aroian, Emil Artin, Natascha Artin, R. N. Ascher, Max Astrachan, M. C. Ayer, W. L. Ayres, Joshua Barlaz, P. T. Bateman, G. E. Bates, M. F. Becker, Richard Bellman, Stefan Bergman, P. G. Bergmann, Lipman Bers, A. A. Blank, W. E. Bleick, M. L. Boas, R. P. Boas, Salomon Bochner, H. W. Bode, H. A. Bohr, G. L. Bolton, Samuel Borofsky, Paul Boschan, C. B. Boyer, A. D. Bradley, H. R. Brahana, Paul Brock, A. B. Brown, E. F. Buck, R. C. Buck, Hobart Bushey, J. H. Bushey, S. S. Cairns, W. R. Callahan, J. D. Campbell, P. G. Carlson, Alan Cathcart, K. Chandrasekharan, K. K. Chen, K. T. Chen, Sarvadaman Chowla, Alonzo Church, Randolph Church, Edmund Churchill, J. A. Clarkson, L. W. Cohen, R. M. Cohn, T. F. Cope, Richard Courant, M. D. Darkow, D. A. Darling, F. H. Davidson, M. D. Davis, J. B. Díaz, M. P. Dolciani, J. L. Doob, C. H. Dowker, Y. N. Dowker, Arnold Dresden, Roy Dubisch, James Dugundji, William H. Durfee, Jacques Dutka, Aryeh Dvoretzky, J. E. Eaton, A. W. Ebin, Samuel Eilenberg, Churchill Eisenhart, Paul Erdös, G. C. Evans, R. M. Exner, J. M. Feld, William Feller, Emanuel Fischer, Irwin Fischer, D. A. Flanders, Erling Folner, R. M. Foster, M. F. Freeman, Bernard Friedman, K. O. Friedrichs, Orrin Frink, P. R. Garabedian, L. L. Gavurin, H. M. Gehman, Abe Gelbart, P. W. Gilbert, B. P. Gill, Leonard Gillman, Wallace Givens, O. E. Glenn, Sidney Glusman, V. D. Gokhale, H. H. Goode, R. E. Goodman, M. J. Gottlieb, H. J. Greenberg, Harriet Griffin, M. M. Gutterman, P. R. Halmos, Carl Hammer, R. W. Hamming, G. H. Handelman, O. G. Harrold, K. E. Hazard, A. E. Heins, M. H. Heins, Alex Heller, Eric Hemmingsen, L. A. Henkin, L. H. Herbach, A. D. Hestenes, Einar Hille, W. M. Hirsch, I. I. Hirschman, Banesh Hoffmann, R. B. Hofstra, E. M. Hull, Witold Hurewicz, L. C. Hutchinson, M. A. Hyman, R. E. Ingram, R. L. Jeffery, Fritz John, R. E. Johnson, A. W. Jones, Mark Kac, Aida Kalish, Samuel Kaplan, M. E. Kellar, J. B. Kelly, P. W. Ketchum, H. S. Kieval, R. S. Kingsbury, F. L. Kiokemeister, S. A. Kiss, J. W. Kitchens, J. R. Kline, Morris Kline, E. R. Kolchin, B. O. Koopman, H. L. Krall, M. S. Kramer, H. G. Landau, R. E. Langer, Mario Laserna, P. D. Lax, Solomon Lefschetz, Howard Levene, Howard Levi, Samuel Linial, M. A. Lipschutz, Marie Litzinger, Charles Loewner, E. R. Lorch, Lee Lorch, A. N. Lowan, Eugene Lukacs, Brockway McMillan L. A. MacColl, G. W. Mackey, G. R. Magee, Dis Maly, Murray Mannos, A. J. Maria, M. H. Maria, M. H. Martin, W. T. Martin, Imanuel Marx, J. L. Massera, D. G. Mead, A. E. Meder, Paul Meier, A. N. Milgram, K. S. Miller, W. H. Mills, S. H. Min, Don Mittleman, Morris Monsky, J. T. Moore, T. W. Moore, K. A. Morgan, C. R. Morris, Marston Morse, I. R. Moses, H. H. Mostafa, G. D. Mostow, T. S. Motzkin, G. W. Mullins, F. J. Murray, C. A. Nelson, O. E. Neugebauer, John von Neumann, P. B. Norman, Morris Ostrofsky, E. R. Ott, O. G. Owens, J. C. Oxtoby, Edward Paulson, A. M. Peiser, Anna Pell-Wheeler, Everett Pitcher, Harry Pollard, E. L. Post, M. H. Protter, Hans Rademacher, H. W. Reddick, Irving Reiner, Eric Reissner, Daniel Resch, R. G. D. Richardson, John Riordan, J. F. Ritt, I. F. Ritter, H. E. Robbins, G. de B. Robinson, S. L. Robinson, I. H. Rose, M. E. Rose, J. B.

Rosenbach, Edward Rosenthall, L. R. Rubashkin, Herman Rubin, Herbert Ruderfer, C. W. Saalfrank, Charles Saltzer, H. E. Salzer, Arthur Sard, A. T. Schafer, R. D. Schafer, Robert Schatten, Albert Schild, I. J. Schoenberg, Lowell Schoenfeld, Abraham Schwartz, G. E. Schweigert, C. H. W. Sedgewick, H. N. Shapiro, Max Shiffman, S. S. Shu, D. N. Silver, James Singer, L. L. Smail, P. A. Smith, J. J. Sopka, E. H. Spanier, W. H. Spragens, George Springer, J. J. Stachel, N. E. Steenrod, S. K. Stein, Fritz Steinhardt, Wolfgang Sternberg, J. J. Stoker, R. W. Stokes, M. H Stone, Walter Strodt, M. M. Sullivan, Fred Supnick, J. H. Taylor, J. M. Thomas, D. L. Thomsen, C. J. Titus, M. M. Torrey, C. C. Torrance, H. I. Treiber, A. W. Tucker, J. W. Tukey, J. R. Van Andel, H. E. Vansant, S. I. Vrooman, B. L. van der Waerden, J. L. Walsh, R. M. Walter, J. B. Walton, W. G. Warnock, Alan Wayne, M. A. Weber, J. V. Wehausen, Alexander Weinstein, Louis Weisner, F. P. Welch, M. E. White, A. L. Whiteman, A. M. Whitney, Hassler Whitney, John Williamson, Jacob Wolfowitz, H. A. Wood, M. A. Wurster, Bertram Yood, J. W. Young, Arthur Zeichner, J. J. Zeig, Daniel Zelinsky, J. A. Zilber, H. J. Zimmerberg, Leo Zippin, O. J. Zobel.

On Friday evening Professor O. E. Neugebauer of Brown University gave an address on *Mathematical methods in ancient astronomy.* Professor P. A. Smith presided.

On Saturday afternoon Professor Charles Loewner of Syracuse University gave an address on *Some classes of functions defined by difference or differential inequalities.* Professor G. C. Evans presided.

On Friday afternoon, at the kind invitation of the International Business Machines Corporation, members of the Society inspected the I.B.M. Selective Sequence Electronic Calculator.

On Saturday morning there were two sections, one for papers in Analysis in which Professor Salomon Bochner, President Hille, and Professor G. C. Evans presided, and one for papers in Algebra in which Professor Hans Rademacher presided. On Saturday afternoon there were two sections, one for papers in Geometry, Topology, and Applied Mathematics, in which Professor Samuel Eilenberg presided, and one for papers in Statistics and Probability, in which Professor J. L. Doob presided.

The Council met at 4:00 P.M. on Friday in the Faculty Club and at 12:00 M. on Saturday in the School of Business.

The Secretary announced the election of the following forty-six persons to ordinary membership in the Society:

Professor Andrew Wilson Ashburn, Texas State College for Women, Denton, Tex.;
Mr. Howard William Baeumler, University of Buffalo;
Mr. Albert Walter Bailey, Casper Junior College, Casper, Wyo.;
Miss Aurora Belsky, Polytechnic Institute of Brooklyn;
Mr. William Sheffield Bryan, University of Washington;
Mr. José Angel Colon, University of Puerto Rico;
Mr. Ralph Pat Crouch, Texas Technological College;
Mr. Roy Edward Dawson, Jacksonville Junior College;

Mr. Burton Victor Dean, Columbia University;
Mr. J. T. Duprat, University of Ottawa;
Dr. Fred G. Elston, Seton Hall College, South Orange, N. J.;
Mr. Marvin Phelps Epstein, University of California;
Mr. Avner Herman Ferester, Seton Hall College, South Orange, N. J.;
Mr. Robert Fitzpatrick, University of Pittsburgh;
Mr. Herbert Parrish Galliher, Yale University;
Mr. Sidney Glusman, Seton Hall College, South Orange, N. J.;
Professor Victor Goedicke, Ohio University, Athens, Ohio;
Professor David Cann Haley, Acadia University, Wolfville, N. S.;
Dr. Theodore Edward Harris, Douglas Aircraft Company, Santa Monica, Calif.;
Mr. Henry John Harrje, Harrje Associates, Architects and Engineers, Jacksonville Beach, Fla.;
Mr. William Walden Kester, Yakima Valley Junior College, Yakima, Wash.;
Professor Vivian Wright Kline, Northeast Missouri State Teachers College, Kirksville, Mo.;
Professor Carl Robert Kossack, State Teachers College, New Haven, Conn.;
Mr. Saul Kravetz, Brooklyn, N. Y.;
Miss Leonore Morey Laderman, New York, N. Y.;
Mr. Gordon Eric Latta, University of British Columbia;
Mr. John Burr Lennes, University of Oklahoma;
Mr. Arthur Davenport Lyles, Presbyterian Junior College, Maxton, N. C.;
Mr. Ralph A. Mauller, Central College, Fayette, Mo.;
Mr. Edward Francis O'Shea, University of Scranton;
Mr. Windsor Lewis Sherman, Providence, R. I.;
Mr. Jordan H. Siedband, Illinois Institute of Technology;
Mr. Thomas Francis Singleton, Regis College, Denver, Colo.;
Mr. Thomas Harrison Slook, Temple University;
Mr. Otis Sheldon Spears, University of Oklahoma;
Miss Vivian Spurgeon, Southwest Baptist College, Bolivar, Mo.;
Mr. William Duane Stahlman, Massachusetts Institute of Technology;
Mr. Walter Boyd Stovall, Jr., University of Florida;
Mr. C. Robert Swenson, Georgia School of Technology;
Mr. Andrew J. Terzuoli, Polytechnic Institute of Brooklyn;
Mr. Francis Bernard Taylor, Manhattan College;
Mr. William Bell Thompson, University of Toronto;
Miss Bess B. Vogel, New York University;
Miss Mamie Smith Ware, Clark College, Atlanta, Ga.;
Ina Warren Welmers (Mrs. E. T.), University of Buffalo;
Mr. Arthur Zeichner, Harvard University.

It was reported that the following had been elected to membership on nomination of institutional members as indicated:

Haverford College: Mr. Murray Fox Freeman.
Institute for Advanced Study: Drs. Sarvadaman Chowla, Robert Gustave Debever, Paul Turán.

The Secretary reported that a reciprocity agreement had been established between the Society and the Wiskundig Genootschap te Amsterdam.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: London Mathematical Society: Dr. Wolfgang Heinrich Fuchs, University of Liverpool; Professor Philip Gerard Gormley, University College, Dublin; Matematisk Forening i København: Dr. Mogens Pihl, Institute for Theoretical Physics, Copenhagen; Wiskundig Genootschap te Amsterdam: Dr. Christoffel Jacob Bouwkamp, N. V. Philips' Laboratory, Eindhoven, Netherlands; Professor Adriaan Cornelis Zaanen, University of Indonesia, Bandoeng, Java.

The following appointments by President Einar Hille were reported: Professor H. J. Ettlinger as representative of the Society at the inauguration of William Richardson White as President of Baylor University on April 13, 1948; Professor E. J. Purcell as representative of the Society at the inauguration of James Byron McCormick as President of the University of Arizona on May 5, 1948; Professor Eric Reissner as representative of the Society on the Cooperating Committee for the International Congress of Theoretical and Applied Mechanics, to be held in London in September, 1948; Professor H. A. Meyer as representative of the Society at the inauguration of Richard Vernon Moore as President of Bethune-Cookman College on March 19, 1948; Professor John von Neumann as Chairman of the Financial Committee for the International Congress of Mathematicians, to replace Professor M. H. Stone (Professor Stone to continue as member of committee until May 1, 1948); Professor Jerzy Neyman as representative of the Society at the Sierpinski Jubilee of the Polish Mathematical Society on September 20–23, 1948; Dr. R. P. Boas (Chairman), Professors Samuel Eilenberg, D. H. Lehmer, William Prager, and G. Y. Rainich as a committee on the project of translation of Russian mathematical articles; Professors Samuel Eilenberg (Chairman), Lipman Bers, P. R. Halmos, G. Y. Rainich, and A. E. Ross as a committee to prepare a pamphlet for aid to mathematicians in reading Russian mathematical articles.

It was reported that Professor B. P. Gill had been added as an ex officio member of the Financial Committee for the International Congress.

The Secretary reported that the Mathematics Branch of the Office of Naval Research had approved a request from the Society for funds to inaugurate a project whereby translations of articles in Russian and other unfamiliar languages will be made available and a pamphlet to aid mathematicians in reading Russian mathematical articles prepared. The request has been referred to the Contracts Division of the Office of Naval Research for negotiation.

The Council elected Professor J. W. T. Youngs as Associate Secretary for the midwest, to fill the unexpired term of Professor R. H. Bruck. This term is to begin after the 1948 Summer Meeting and to extend to December 31, 1949. (This election was subsequently approved by the Board of Trustees.)

The Council voted to invite Professor Casimir Kuratowski of the University of Warsaw to act as Visiting Lecturer of the Society during the academic year 1948–1949.

A report from the Committee on Reorganization was presented by Dean W. L. Ayres, Chairman. This committee was authorized by the Council at the 1947 Summer Meeting to study the present organization of the Society. President Hille subsequently appointed the following committee: Dean W. L. Ayres (Chairman), Professors S. S. Cairns, B. P. Gill, J. R. Kline, W. T. Martin, P. A. Smith, and M. H. Stone. Changes in the organization of the Society, as recommended by this committee and adopted by the Council, appear in the following excerpts from the report of the Committee on Reorganization:

I. *President Elect.* A new office of President Elect shall be created and this office shall be filled during the even numbered years only. The person designated as President Elect shall automatically become President of the Society at the end of his one year term as President Elect. He shall be a member of the Council and of its Executive Committee during this one year term and thus should be well acquainted with the affairs of the Society at the time he assumes office as its President.

II. *Executive Committee of the Council.* An Executive Committee of the Council shall be formed and this Committee shall be empowered to act for the Council on (1) matters which have been delegated to the Executive Committee by the Council (such as election of members, approval of dates of meetings, etc.) and (2) such other matters of policy or administration of Society business that require immediate decision, it being understood that if three members of the Executive Committee request postponement of any matter until the next meeting of the Council the matter shall be so postponed. It may consider the agenda for the meetings of the Council and make recommendations on important matters to the Council. It shall be responsible to the Council and shall report its actions to the Council. The Executive Committee shall consist of seven members, the President, Secretary, President Elect (even numbered years) and Ex-President (odd numbered years), and four members of the Council elected in January for two year terms with staggered periods of office. These elected members of the Executive Committee shall be elected directly by the Council, and any member of the Council shall be eligible for election to this Committee. In case an elected member of the Executive Committee shall be elected to serve for a term beyond his term of membership on the Council he shall automatically continue as a member of the Council during his period on the Executive Committee.

The Executive Committee shall meet at the call of the President or of any two of its members. The members of the Executive Committee shall have expenses paid by the Society in attending these meetings and, in general, it would be expected that members of the Executive Committee would attend all meetings unless prevented by illness or duties of extreme importance.

Comment: The proposed Executive Committee should form a small responsible group which can meet several times a year to guide the policies of the Society. There are many times in the affairs of the Society when a decision must be made before the next possible meeting of the entire Council. Such a small sub-committee could be called together quickly or could be consulted by mail or telegram. The payment of travel expenses is believed necessary to insure full attendance at these meetings, particularly of those members of the committee at some distance from the office of the Society.

III. *The Council.* The membership of the Council shall be changed as follows:

1. The number of Associate Secretaries shall be changed from four to three, eliminating the Associate Secretary who has been in charge of financial matters.
2. The President of the Society shall serve as a member of the Council and its Executive Committee for one year following his period of office.
3. Each Secretary of the Society shall serve as a member of the Council for two years following his term of office.
4. At the time of this change the Ex-Presidents and Ex-Secretaries who are members of the Council shall continue to the end of their usual term.

All members of the Council shall be voting members of the Council under the following provisions: in case of a roll call vote, the group of Associate Secretaries, the editorial committees of the Society publications which now have representation on the Council, and the representatives on the Board of Editors of the American Journal of Mathematics shall be entitled to one vote each. This vote shall be divided among the members of the committee who are present at the Council meeting. No member of the Council shall have more than one vote, and if a member of an editorial committee is also entitled to a vote on the Council due to another office, the remaining members of that editorial committee present shall divide the vote equally between them.

Comment: Your Committee has heard considerable criticism of the existence of membership of the Council without vote and the above proposal is intended to improve this situation. The objection raised is that non-voting members of the Council resent this status and do not take as active a part in the deliberations of the Council. Many of these individuals are persons of considerable experience in the affairs of the Society and the Council needs the benefit of their wisdom.

Several years ago the Society established the principle that the members of the Council elected by the membership of the Society should have a voting majority in the Council. The Vice-Presidents of the Society and the members-at-large of the Council are so elected. The reduction of the number of Ex-Presidents on the Council and the specification of the term of the Ex-Secretaries, together with the proposed method of voting of the editorial members of the Council are designed to maintain this principle. The above proposal would give the elected members eighteen votes as compared with fifteen ex-officio votes.

IV. *The Secretariat.* The present office of Secretary shall be divided into two offices, an Executive Director and a Secretary.

The Executive Director shall be a full-time paid employee of the Society. He shall be in charge of the central office of the Society and shall be charged with the general administration of the affairs of the Society once policies have been set by the Council and its Executive Committee. He will work under the immediate direction of the Council and its Executive Committee. He will attend meetings of the Council and its Executive Committee, but not be a member of these bodies. The person holding this office shall be nominated by the Council and approved by the Trustees. Due to the nature of his work it seems desirable that he be a mathematician.

The Executive Director shall take over the functions of the present Associate Secretary who deals with financial affairs, and this office shall be abolished. Three individuals have held this Associate Secretaryship and all three believe that the office should be abolished and the duties performed by an officer located in the main office of the Society.

All abstracts shall be mailed directly to the office of the Society and the programs will be arranged by the Executive Director subject to review by the Associate Secretary in charge of the meeting.

The Secretary of the Society will continue to be the principal officer of the Society concerned with policy making. He will work with the President, the Council, and its committees in setting the policies of the Society. While the Secretary and Executive Director must cooperate closely in their work, the division of their functions can be described briefly in the two phrases "policy making" and "administration." Once the Council has set the policy of the Society it will be the duty of the Executive Director to carry out the administration.

Comment: The growth in membership and activities of the Society has produced a pressure on the Secretary, which must be relieved. The present Secretary estimates that he devotes well over fifty per cent of his time to the affairs of the Society, and there is every indication that this pressure will continue to grow in the immediate years ahead. Your committee feels that it is imperative that a new full-time office be planned at this time to relieve the Secretary of many of the present duties of his office, permitting him to concentrate on policy matters. There would remain the type of position that a professor as Secretary could reasonably be expected to handle along with his university work.

On recommendation of the Committee on Reorganization, the Council voted to refer the question of the possible establishment of a federation of various mathematical organizations to the Policy Committee for Mathematics.

The President was authorized to appoint: (1) a committee to formulate the changes in By-Laws made necessary by the adoption of the above changes in organization; (2) a committee to nominate to the Council an Executive Director of the Society.

The Council voted to schedule annual meetings in the southeastern section of the country, the meetings to be held at a time most likely to encourage southeasterners to attend both southeastern and national (Summer, Annual) meetings.

The Birkhoff Editorial Committee reported it had completed the compilation and arrangement of the material to be published as the *Collected Mathematical Papers* of G. D. Birkhoff.

On recommendation of the Committee on Reorganization of the Society's Programs, the Council voted to advance by one week the deadlines for the submission of contributed papers, to consult the Committee on Applied Mathematics regarding the titles of sections on abstract blanks, to recommend to the Committees to Select Hour

Speakers that they take advantage of opportunities to invite distinguished foreign visitors who happen to be in the United States, to allow publishers to display books at or above the graduate level at meetings of the Society.

It was reported that Professor R. D. James would deliver an hour address at the June, 1948, meeting at the University of British Columbia.

The Secretary reported plans for the publication of the Canadian Journal of Mathematics by the Canadian Mathematical Congress, the first number of which will appear in January, 1949. Members of the Society will receive a discount of fifty per cent on subscriptions to this Journal.

The Council adopted the following resolution regarding the Committee on Un-American Activities of the House of Representatives which was subsequently released to the newspapers and sent to appropriate members of Congress:

> The Council of the American Mathematical Society, a national scientific organization comprising 3700 members, at the 434th meeting of the Society in New York adopted the following resolution:
>
> Whereas, the Committee on Un-American Activities of the House of Representatives has recently made public a statement derogatory to a distinguished American scientist, Dr. Edward U. Condon, the present Director of the Bureau of Standards in the Department of Commerce; and
>
> Whereas, a careful examination of these statements in the light of public information discloses that the Committee has made insinuations unsubstantiated by the evidence submitted, and resorted (in the furtherance of its own purposes) to misrepresentation of even the most innocent acts, and
>
> Whereas, it is inevitable that the procedure adopted by the said Committee in attacking Dr. Condon will, if tolerated, prove an effective obstacle to the entry of scientists into public service; and
>
> Whereas, furthermore these procedures will, if continued, enable congressional committees to usurp and abuse functions properly reserved under our Constitution to the executive and judicial branches of our government; now therefore be it resolved
>
> That, this Council publicly express its grave concern over the implications of the actions followed by the Committee on Un-American Activities of the House of Representatives in attacking Dr. Edward U. Condon; and
>
> That the Council petition the House of Representatives and the Senate of the United States forthwith to adopt rules of procedure for their committees which will safeguard the constitutional separation of powers, secure our citizens in their individual right to just treatment by their government, and reflect Congressional understanding of the necessity of self-restraint in the exercise of the powers of legislative inquiry.

The Council approved the recommendation of the Organizing Committee for the International Congress that Russian be added to the list of official languages as stated in the Eisenhart report (the fundamental document on which the organization of the Congress is based, adopted by the Council in 1937). It was also agreed that the Organizing Committee need not adhere to the time limits set in the Eisenhart report, due to the difference in the interval between the inauguration of the plans and the opening date of the Congress.

Abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 241 was read by Dr. Chandrasekharan, paper number 249 by Professor Frink, and paper number 252 by Professor Schatten.

ALGEBRA AND THEORY OF NUMBERS

226*t*. Eckford Cohen: *Sums of an odd number of squares in* $GF[p^n, x]$.

Let F be a polynomial of $GF[p^n, x]$, $p>2$, of degree $f<2k$, and let $\alpha_1, \cdots, \alpha_{2s+1}$ be nonzero elements of $GF(p^n)$. The object of the author is to find the number of solutions N^k_{2s+1} of $F=\sum_{i=1}^{2s+1} \alpha_i X_i^2$ in polynomials X_i $(i=1, 2, \cdots, 2s+1)$ of degree less than k. The solution of the present problem is obtained in terms of Artin's σ-function, $\sigma_i(F)=\sum(F/H)$, where (F/H) indicates the quadratic character of F with respect to H and the summation ranges over primary H of degree i. Although the formula obtained is rather complicated, it simplifies greatly for certain special cases. For example, when $k>f$ and F is quadratfrei, $N^k_{2s+1}=\sum_{i=f}^{k} p^{-sni}(\sigma_i(\theta F)-p^n\sigma_{i-1}(\theta F)) +(p^n-1)\sum_{i=f}^{k-1} p^{-sni}\sigma_i(\theta F)$ where $\theta=(-1)^s\alpha_1\alpha_2\cdots\alpha_{2s+1}$. (Received March 11, 1948.)

227. R. M. Cohn: *Manifolds of difference polynomials*.

The author studies the manifold of a single difference polynomial and the theory of elimination among systems of difference polynomials. Let A be an algebraically irreducible difference polynomial in an unknown y, with coefficients in an abstract difference field. Let A be of order n and degree r in y_n and let y_0 appear effectively in A. It is shown that A has at least one, and at most r, essential irreducible manifolds not held by polynomials of order less than n. A may also possess essential irreducible manifolds held by polynomials of order less than n, and termed, by analogy with differential polynomials, essential singular manifolds. Thus, if $A=yy_2-y_1$, then $yA_1+A=y_1(yy_2-1)$, so that $y=0$ constitutes an essential irreducible manifold. In the theory of elimination and the structure of difference fields theorems are developed bearing a close analogy to those known in algebra, and the theory of differential equations, and culminating in the proof that, given a prime reflexive difference ideal Σ, one may find a resolvent unknown w, in terms of which the unknowns of Σ may be expressed uniquely. It is a peculiarity of difference equations that the system of equations which furnishes this unique expression need not be linear. (Received February 16, 1948.)

228. R. E. Johnson: *Equivalence rings*.

If in a ring R two elements are related by $a=pbq$ for some p, q in R, then $a\leq b$. R is called an equivalence ring if for every pair of elements a, b of R either $a\leq b$ or $b\leq a$. In case $a\leq b$ and $b\leq a$ for every pair of nonzero elements of R, R is called a 1-equivalence ring. The relation $<$ gives rise to a linearly-ordered lattice, L, of subsets of R.

The ideals of R are elements of L if L is well-ordered. A commutative equivalence ring has a unit element and a nonzero radical if it is not a field. A simple nonradical ring is either a regular equivalence ring or a subring of a regular 1-equivalence ring. (Received March 12, 1948.)

229t. E. R. Kolchin: *Existence theorems connected with the Picard-Vessiot theory of homogeneous linear ordinary differential equations.*

It is shown how a theorem on algebraic differential equations due to J. F. Ritt (Trans. Amer. Math. Soc. vol. 48 (1940) pp. 543–544) yields two results concerning the differential equation $L(y)=y^{(n)}+p_1y^{(n-1)}+\cdots+p_ny=0$ ($p_1,\cdots,p_n$ in an ordinary differential field $\mathfrak{F}$ of characteristic 0 with algebraically closed field of constants $\mathfrak{C}$). 1. *$L(y)=0$ always has a fundamental system of solutions $\eta_1,\cdots,\eta_n$ such that the field of constants of $\mathfrak{F}\langle\eta_1,\cdots,\eta_n\rangle$ is $\mathfrak{C}$.* This solves a problem proposed by R. Baer (pp. 332–334 of F. Klein's *Vorlesungen über hypergeometrische Funktionen*, Berlin, 1933). 2. *If $L(y)$ is linearly irreducible* [that is, not properly of the form $L_1(L_2(y))$] *over $\mathfrak{F}$ and if $L(y)=0$ has one solution contained in an extension of $\mathfrak{F}$ by integrals, exponentials of integrals, and algebraic functions, then $L(y)=0$ has a fundamental system of solutions $\eta_1,\cdots,\eta_n$ such that $\mathfrak{F}\langle\eta_1,\cdots,\eta_n\rangle$ is a liouvillian extension of $\mathfrak{F}$* (that is, an extension by integrals, exponentials of integrals, and algebraic functions, of which the field of constants is $\mathfrak{C}$). (Received March 11, 1948.)

230t. E. R. Kolchin: *On certain concepts in the theory of algebraic matric groups.*

For an algebraic matric group $\mathfrak{G}$ over an algebraically closed field of characteristic p the concepts "component of the identity," "solvable," "anticompact," and "quasicompact" (previously discussed by the author, Ann. of Math. (2) vol. 49(1948) pp. 1–42) are further studied. Among the new results are the following: 1. $\mathfrak{G}$ is anticompact if and only if $\mathfrak{G}$ is reducible to special triangular form. 2. A connected $\mathfrak{G}$ is quasicompact if and only if $\mathfrak{G}$ is reducible to diagonal form. 3. $\mathfrak{G}$ is solvable as an algebraic group if and only if $\mathfrak{G}$ is solvable as an abstract group. 4. The component of the identity $\mathfrak{G}^0$ of $\mathfrak{G}$ is the smallest subgroup of $\mathfrak{G}$ of finite index (provided $p=0$; for $p\neq 0$ purely group-theoretic characterization of $\mathfrak{G}^0$ is impossible). 5. "Quasicompact and connected" is characterized in terms of dim $\mathfrak{G}$ and purely group-theoretic concepts (no purely group-theoretic characterization exists). (Received March 11, 1948.)

231t. Joseph Lehner: *Divisibility properties of the Fourier coefficients of the modular invariant $j(\tau)$.*

The following theorem is proved: let c_n ($n=0, 1, 2, \cdots$) be the Fourier coefficients of the absolute modular invariant $j(\tau)=x^{-1}+744+196{,}884x+\cdots=x^{-1}+c_0+\sum_{n=1}^{\infty}c_nx^n$, $x=\exp 2\pi i\tau$ ($j((a\tau+b)/(c\tau+d)))=j(\tau)$ for every integral a, b, c, d, with $ad-bc=1$). Then for $\alpha=1, 2, 3, \cdots$, and $n=1, 2, 3, \cdots$, $5^\alpha | n$ implies $5^{\alpha+1} | c_n$; $7^\alpha | n$ implies $7^\alpha | c_n$. Also for $\beta=1, 2$, and $n=1, 2, 3, \cdots$, $11^\beta | n$ implies $11^\beta | c_n$. The methods used are those of the theory of modular functions and are described in the author's earlier paper (Amer. J. Math. (1943)). The difficulty in extending the congruences to powers of 11 beyond the second comes from the fact that the subgroup $\Gamma_0(n)$ of the full modular group (defined by $c\equiv 0 \pmod{n}$), which occurs in an essential way in the proof, is of genus one when $n=11$, whereas it is of genus zero when $n=5, 7$. (Received February 9, 1948.)

232t. N. H. McCoy: *Prime ideals in general rings.*

Some of the known results about prime ideals in commutative rings are extended to arbitrary rings. In place of the multiplicative systems used in the commutative case an important role is played by *m-systems*. An m-system M of elements of the ring R is a set with the property that $a\in M$, $b\in M$ imply the existence of an x in R such that $axb\in M$. It is shown that a two-sided ideal $\mathfrak{p}$ in R is a prime ideal if and only if its complement in R is an m-system. The radical of an ideal $\mathfrak{a}$ is defined, and shown to be the intersection of all prime ideals which contain $\mathfrak{a}$. The methods are based on those of Krull (Math. Ann. vol. 101 (1929) pp. 729–744). The radical N of the zero ideal has all the familiar "radical-like" properties, and is actually a generalization of the classical radical of a ring. The Jacobson radical (Amer. J. Math. vol. 67 (1945) pp. 300–320) is the intersection of a certain class of prime ideals, as is also the radical defined by Brown and McCoy (Amer. J. Math. vol. 69 (1947) pp. 46–58). (Received March 11, 1948.)

233. I. F. Ritter: *Solution of algebraic equations.*

This paper describes an iterative process yielding approximate values for the roots, complex or real, of an algebraic equation of any degree. Practically useful approximations with "slide rule accuracy" are obtained by it, using as tool nothing more elaborate than the slide rule. Such tools as logarithms or an adding machine make the process yield as many correct significant digits of the roots as wanted. The effort required is less than that needed in Graeffe's or other methods. It is not at all increased by the presence of complex roots and the iteration involved in the process makes most steps in it self-checking. These two facts are the main reasons for the practical advantage of this new method. (November 6, 1947.)

234. G. de B. Robinson: *On the representations of the symmetric group.*

If $[\alpha]$ be any irreducible representation of the symmetric group $\mathfrak{S}_n$, then the Kronecker product $[\alpha]\times[\alpha]\times\cdots$ (m factors) on m different sets of variables yields a representation of the sub-group $H=\mathfrak{S}_n\times\mathfrak{S}_n\times\cdots$ (m factors) which induces a representation of $\mathfrak{S}_{mn}$; the problem under consideration is the reduction of this representation arising from the interchangeability of the m factors. The case is parallel to Schur's derivation of the irreducible representations of the full linear group of order m. The components, here reducible, have been designated $[\alpha]\otimes[\beta]$, where $[\beta]$ is an irreducible representation of S_m. For $[\alpha]=[n]$ (cf. D. E. Littlewood, Proc. London Math. Soc. vol. 49 (1947) pp. 282–306), the irreducible representation $[\lambda]$ of S_{mn} appears in $[n]\otimes[\beta]$ with a multiplicity $(1/m!)\sum_S\chi_\beta(S^*)(n!)^{-m}\phi_\lambda^{HS}$ where ϕ_λ^{HS} denotes the sum of the characters of the elements of the coset HS in $[\lambda]$, and S is a substitution of $\mathfrak{S}_{mn}$ which permutes the m sets of variables (or symbols used to define H); $\{S\}$ is a subgroup of $\mathfrak{S}_{mn}$ simply isomorphic to $\mathfrak{S}_m^*$. This formula generalizes to the case of an arbitrary $[\alpha]$. (Received March 12, 1948.)

235. R. D. Schafer: *The Wedderburn principal theorem for alternative algebras.*

The following generalization of the so-called Wedderburn principal theorem for associative algebras is proved. Let $\mathfrak{A}$ be an alternative algebra over an arbitrary field $\mathfrak{F}$ which is neither semisimple nor nilpotent, and let $\mathfrak{N}$ be the radical of $\mathfrak{A}$. Then, if

$\mathfrak{A}-\mathfrak{R}$ is separable (that is, semisimple for every scalar extension), $\mathfrak{A}=\mathfrak{S}+\mathfrak{R}$ where $\mathfrak{S}$ is equivalent to $\mathfrak{A}-\mathfrak{R}$. (Received January 26, 1948.)

236. Jean B. Walton: *Theta series in the Gaussian field.*

The functions $\vartheta_r(\tau, \rho, Q)=\sum_{\mu\equiv\rho(Q)}^{\infty}\mu^r e^{\pi i\tau\mu\mu'/Q}$ for μ and ρ integers in the Gaussian field, Q a positive rational integer, τ a complex number with positive imaginary part, are defined and transformation formulae derived showing their behavior when $Q=1$ under arbitrary modular substitutions. Positive values of r are considered. The cases with $r=0, 1$ have been discussed by Hecke (Math. Ann. vol. 97 (1927) pp. 210–242). The functions are all invariant in the principal congruence subgroup modulo 4. If the concept of invariance is generalized to include as multiplicative factor a root of unity, the functions $A(\vartheta_r(\tau, 0, 1)+\vartheta_r(\tau, 1+i, 1))$ if $r\equiv0$ mod 4 and $\vartheta_r(\tau, 1, 1)$ if $r\equiv2$ mod 4 are found to be the only linear combinations of the theta series being invariant in a larger subgroup than before. By means of a general method suggested by Hecke (*Lectures on Dirichlet series, modular functions and quadratic forms*, Edwards Brothers, 1938) formulae are derived expressing these functions in terms of Jacobi's theta functions. (Received March 12, 1948.)

237*t*. Daniel Zelinsky: *On ordered loops.*

In a previous paper (Bull. Amer. Math. Soc. vol. 54 (1948) pp. 175–183) was stated the following necessary and sufficient condition for an ordered loop L to be the value loop of some (not necessarily associative) algebra with a unity quantity and of finite order over a field: L contains in its center a subloop G of finite index. In the present paper the author determines all such loops L as special loop extensions of finite centrally nilpotent loops (L/G) by order abelian groups (G). The central theorem proves the central nilpotence of the quotient loop L/G and of L itself by exhibiting a uniquely defined central series for L. The proof leans heavily on the fact that L is a topological loop in the interval topology. (Received March 2, 1948.)

238. Daniel Zelinsky: *Topological characterization of fields with valuations.*

Necessary and sufficient conditions for a topological field F to have a nonarchimedean valuation (values in any ordered group) preserving the topology of the field are (i) some neighborhood of zero in F generates a bounded additive group; and (ii) if $A\subset F$ and A is bounded away from zero, then A^{-1} is bounded. The fields satisfying (i) alone are the quotient fields of integral domains, topologized by designating the nonzero ideals of the integral domain as neighborhoods of zero. (Received March 2, 1948.)

ANALYSIS

239. Lipman Bers: *Isolated singularities of minimal surfaces.*

Isolated singularities of functions satisfying the minimal surface equation (1) $(1+\phi_y^2)\phi_{xx}-2\phi_x\phi_y\phi_{xy}+(1+\phi_x^2)\phi_{yy}=0$ are classified. Only functions possessing single-valued gradients are considered. It turns out that, unlike harmonic functions, solutions of (1) possess only relatively simple singularities. Single-valued solutions possess at finite points only removable isolated singularities. A multiple-valued solution behaves in the neighborhood of a finite isolated singularity like the velocity potential of a flow due to a vortex. A solution ϕ regular in the neighborhood of infinity behaves there either like a harmonic function regular at infinity or like the velocity potential of a flow due to a source-vortex-doublet; in either case ϕ_x and ϕ_y possess finite limits at infinity. Parametric representations for these singularities are derived and a new proof is given for S. Bernstein's theorem which asserts that a solution of (1) regular at all finite points is a linear function. The author uses Chaplygin's gas-dynamical interpretation of the minimal surface equation, and some properties of one-to-one mappings by means of a pair of nonconjugate harmonic functions. It seems probable that similar results hold for a wide class of nonlinear partial differential equations. (Received March 11, 1948.)

240*t*. R. P. Boas, R. C. Buck, and Paul Erdös: *The set on which an entire function is small.*

Let $f(z)$ be an entire function, and let E_λ be the set of points for which $\log |f(z)| \leqq (1-\lambda) \log M(r)$, where $\lambda > 0$. Denote by $D^*(S)$ the upper density of the measurable set S, and by $D_*(S)$ the lower density. Then, there is a number K, dependent only upon λ, such that $D^*(E_\lambda) \leqq K$; moreover, $c^2/(1+c)^2 \leqq K \leqq \lambda^{-1}$, where c is the positive root of $c(2+c)^{\lambda-1}=1$. It is also shown that $D_*(E_\lambda)=o(\lambda^{-1})$ as $\lambda \to \infty$. (Received March 3, 1948.)

241. Salomon Bochner and K. Chandrasekharan: *Gauss summability of trigonometric integrals.*

If $f(x_1, \cdots, x_k)$ is of period 2π in each variable and belongs to the Lebesgue class L_1, and if $f_p(t)$ and $S^\delta(R)$ denote respectively the pth spherical mean of f and the δth Riesz mean of its Fourier series at a given point $(x_1, \cdots, x_k)$, where $p \geqq 0$, $\delta \geqq 0$, then it is known that for $\delta > p+(k-1)/2$, $S^\delta(R) = R\int_0^\infty f_p(t)H(tR)dt$ where $H(x) = x^{\nu-1-2\delta}J_\nu(x)$, $\nu = p+\delta+k/2$ and J_ν is the Bessel function of order ν; it is also known that for large p and δ, $S^\delta(R)$ and $f_p(t)$ could be interchanged in the formula (Proc. Indian Acad. Sci. vol. 24 (1946) pp. 229–232). It is now proved that $S^\delta(R) = \lim_{\epsilon\to 0} R\int_0^\infty \exp(-\epsilon^2 t^2)f_p(t)H(tR)dt$ for *every* $\delta \geqq 0$, and that the inverse formula could also be similarly summed for every $p \geqq 0$. (Received March 4, 1948.)

242*t*. Orrin Frink: *A ratio test.*

The series $\sum a_n$ is absolutely convergent if $\lim\sup |a_n/a_{n-k}|^n < e^{-k}$. If $|a_n/a_{n-k}|^n \geqq e^{-k}$ for n sufficiently large, $k \neq 0$, then it is *not* absolutely convergent. If $\lim |a_n/a_{n-1}|^n = r$, then the abscissa of absolute convergence of the Dirichlet series $\sum a_n n^{-s}$ is $1+\log r$. (Received March 11, 1948.)

243*t*. P. R. Garabedian: *Schwarz's lemma and the Szegö kernel function.* Preliminary report.

Let D be a finite domain of connectivity n bounded by analytic curves C. The regular function $F_0(z)$ maximizing $|F'(z_1)|$ under the conditions $F(z_1)=0$, $|F(z)| \leqq 1$ in D is found directly by a method of contour integration analogous to that introduced by Grunsky. The zeros $z_2, \cdots, z_n$ of $F_0(z)$ are located by means of a Jacobi inversion problem. The function $f_0(z) = (z-z_1)^{-2} + a_{-1}(z-z_1)^{-1} + a_0 + a_1(z-z_1) + \cdots$ minimizing $l(f) = \int_C |f(z)|\, ds$ for functions with a pole of the type shown is found by the same method, using the fundamental relation $iF_0 f_0 z' < 0$ on C, where z' is the derivative of the parametric representation $z(s)$ of C. The Szegö kernel function of D (Math. Zeit. vol. 9 (1921) pp. 218–270) is found to be given by $4\pi^2 K(z, z_1)^2 = F_0(z)^2 f_0(z)$, and this leads to a formula for $K(z, z_1)$ in terms of Green's function and harmonic measures. The identities $2\pi |F_0'(z_1)| = l(f_0) = 4\pi^2 K(z_1, z_1)$ are proved. Thus there is a close relationship between the Szegö kernel function and Schwarz's lemma in multiply-connected domains. An application of these results is made to the problem of Painlevé concerning the existence of bounded analytic functions in the exterior of a compact set. (Received March 5, 1948.)

244t. P. R. Garabedian and M. M. Schiffer: *Identities in the theory of conformal mapping.*

The Green's function, Neumann's function, harmonic measures and canonical mapping functions associated with a multiply-connected domain of the complex plane and their derivatives satisfy various simple boundary conditions. These conditions are used in a method of contour integration similar to that exploited by Grunsky in his thesis and analogous to that employed in the classical theory of Abelian integrals on a Riemann surface to prove a large number of identities relating the domain functions named. A few of these identities have already appeared in the papers of Bergman and Schiffer concerning the theory of kernel functions. Here a systematic treatment of the identities is given, while at the same time geometric and extremal properties of the functions involved are discussed. Applications of the results of the paper are made to the theory of the variation of domain functions with their domain of definition. In particular, it is shown how variations of a multiply-connected domain can be made which preserve conformal type or which preserve boundary configurations. (Received March 4, 1948.)

245t. A. P. Hillman: *On the reality of zeros of Bessel functions.*

A necessary and sufficient condition for the existence of real zeros on any branch of a Bessel function of real order is obtained. This condition is applied for special cases such as real cylinder functions and Bessel functions of the first kind, of the second kind, of integral order, and of rational order. (Received March 8, 1948.)

246. Witold Hurewicz: *On stability of nonlinear operators.*

Let T be a continuous linear transformation of Hilbert space E_∞, defined over a neighborhood of the origin 0, and let $T(0)=0$. Suppose T is "differentiable" at the point 0, that is, there exists a continuous linear transformation L such that $\|L(x)-T(x)\|/\|x\|$ tends uniformly to zero when $x\rightarrow 0$. Let S be the spectrum of L (S is a point set in the complex plane). Assume that $\lambda\in S$ implies $\lambda\neq 0$ and $|\lambda|\neq 1$. Under these conditions the following theorem holds: If $|\lambda|<1$ for every $\lambda\in S$, the transformation T is *stable* at the origin, that is, $T^n(x)\rightarrow 0$ with $n\rightarrow\infty$, for all points x sufficiently near to the origin. If $|\lambda|>1$ for at least one point $\lambda\in S$, T is *unstable*, that is, given any pair of positive numbers (ϵ, δ) there exists a point $x\in E_\infty$ and a positive integer n, such that $0<\|x\|<\delta$, $\|T^n(x)\|>\epsilon$ (or $T^n(x)$ undefined). This result can be regarded as an extension of classical results of Poincaré-Liapounoff to the case of infinite number of dimensions. The proof is based on recent results of E. R. Lorch concerning linear operators. (Received March 22, 1948.)

247t. Meyer Karlin: *Note on the expansion of confluent hypergeometric functions in terms of Bessel functions of integral order.* Preliminary report.

An expansion of the confluent hypergeometric series $F(\alpha, \gamma; x)$ was obtained in terms of modified Bessel functions of the first kind $I_k(x)$ of integral orders by means of a recursion formula, that is well adaptable for computational purposes. This expansion has decided advantages over the power series, especially for large values of α/γ. An illustration is cited where eight terms in this expansion yield five significant figures; whereas hundreds of terms in the power series are required for a similar accuracy. (Received March 19, 1948.)

248t. V. K. Klee: *On the support property of a convex set in a linear normed space.*

Suppose that C is a closed convex set in the linear normed space S. The following theorems are proved: (1) if C is a cone $\neq S$, then C has at least one plane of support. (2) If C has an interior point, then C is supported at each of its boundary points. (3) If any one of the following conditions is satisfied, C is supported at a set of points dense in its boundary: (i) C is weakly compact; (ii) E is weakly compact, $S=E^*$, and C is weakly compact as a set of functionals; (iii) $S=E^*$ and C is transfinitely closed in S. (1) is used to prove (2) and (3). (2) is already known, but (1) and (3) have previously been proved only under considerably more restrictive assumptions. (Received March 12, 1948.)

249. H. L. Krall and Orrin Frink: *A new class of orthogonal polynomials: the Bessel polynomials.*

The *Bessel polynomials* are solutions of the differential equation $x^2y''+(2x+2)y'=n(n+1)y$, and are closely related to the Bessel functions of half-integral order. They are orthogonal with weight function $e^{-2/x}$ and path of integration the unit circle in the complex plane. This paper contains a detailed treatment of their theory, including the derivation of their recurrence relations, generating function, Rodrigues' formula, and normalizing factors. Applications to the theory of traveling waves are also given. In the second part of the paper a similar theory is developed for the *generalized Bessel polynomials*, which satisfy the differential equation $x^2y''+(ax+b)y'=n(n+a-1)y$. (Received March 9, 1948.)

250. J. L. Massera: *On the existence of periodic solutions of differential equations.*

Systems of the form $\dot{x}=X(x, t)$, where x is a vector of m-dimensions and X depends periodically on t with period 1, are considered. It is proved that: (a) If $m=1$, the existence of a solution which is bounded "in the future" (that is, for $0\leq t<\infty$) implies the existence of a periodic solution of period 1; and any almost periodic solution is actually periodic of period 1; (b) If $m=2$, the existence of a bounded solution is not sufficient for the existence of a periodic solution of period 1, but it is so if, moreover, all the solutions exist in the future; (c) If $m>2$, the existence of a bounded solution and the existence of all the solutions in the future is not sufficient for the existence of a periodic solution. Furthermore, it is shown that the only necessary conditions on the number of the different harmonics and subharmonics are the compatibility with Theorems (a) and (b) and the obvious restriction that the number of solutions of period p should be a multiple of p. (Received March 5, 1948.)

251t. C. N. Moore. *On the relationship between the integral formulas of Cauchy and Poisson.*

A method is given of deriving Poisson's formula directly from that of Cauchy. By taking real parts on both sides of Cauchy's formula a preliminary expression for $u(x, y)$ is obtained in terms of an integral involving both u and v. The v is then eliminated by means of known relationships between conjugate functions. (Received March 12, 1948.)

252. Robert Schatten and John von Neumann. *The cross-space of linear transformations.* III.

The present notation is that of Bull. Amer. Math. Soc. Abstracts 51-9-162 and 52-1-22. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ denote two Banach spaces and $\mathfrak{B}$ a Banach space of some linear transformations A from $\mathfrak{B}_1$ into $\mathfrak{B}_2^*$. An element A in $\mathfrak{B}$ possesses a norm $\|A\|$ which in general is different from its bound $|||A|||$. $\mathfrak{B}$ is termed an "ideal" if: (i) $A\in\mathfrak{B}$ implies $YAX\in\mathfrak{B}$ for any pair of linear transformations X and Y on $\mathfrak{B}_1$ and $\mathfrak{B}_2^*$ respectively, and (ii) $\|YAX\|\leqq|||Y|||\ \|A\|\ |||X|||$. The trace-class, the Schmidt-class, the space of all linear transformations on a Hilbert space $\mathfrak{H}$ furnish such ideals. The class of crossnorms α for which $(\mathfrak{B}\otimes_\alpha\mathfrak{B}_2)^*$ is an ideal is characterized. Furthermore, $(\mathfrak{H}\otimes_\alpha\mathfrak{H})^*$ is an ideal if and only if α is unitarily invariant. If in addition $\alpha\leqq\sigma$ (where σ represents the Schmidt-crossnorm on $\mathfrak{H}\odot\mathfrak{H}$) we have $(\mathfrak{H}\otimes_\alpha\mathfrak{H})^*=\mathfrak{H}\otimes_{\alpha'}\mathfrak{H}$ whenever for a complete orthonormal set (ϕ_i) and every sequence of real numbers $\{a_i\}$ with $\lim a_i=0$, the condition $\lim\alpha'(\sum_{i=1}^{n}a_i\phi_i\otimes\phi_i)<+\infty$ as $n\to\infty$ implies $\lim\alpha'(\sum_{i=m}^{n}a_i\phi_i\otimes\phi_i)=0$ as m and $n\to\infty$. In particular, we have $(\mathfrak{H}\otimes_\lambda\mathfrak{H})^*=\mathfrak{H}\otimes_\gamma\mathfrak{H}$. (Received February 13, 1948.)

253*t*. I. M. Sheffer. *On the theory of sum-equations.*

A system of sum-equations is one for the form $\sum_{j=0}^{\infty}a_{n,j}x_{n+j}=c_n$ $(n=0, 1, \cdots)$. The k-periodic case, that is, where there is a positive integer k such that $a_{p+sk,j}=a_{p,j}$ for $p=0, 1, \cdots, k-1; j, s=0, 1, \cdots$ was treated earlier, as also the k-periodic case with perturbations. It is now shown: (a) that solutions of the k-periodic case can be expressed in closed form by means of contour integrals; and (b) that under certain conditions the nonperiodic case has a solution obtained from (a) by a limiting process. (Received March 12, 1948.)

254. W. H. Spragens. *On the absolute Cesàro summability of double Fourier series.*

Let $\sum a_{mn}$ be a double series and denote by $s_{mn}^{\alpha\beta}$ the m, nth Cesàro partial sum of order α, β. The series is said to be summable $|C, \alpha, \beta|$ if $\sum|s_{mn}^{\alpha\beta}-s_{m-1,n}^{\alpha\beta}-s_{m,n-1}^{\alpha\beta}+s_{m-1,n-1}^{\alpha\beta}|$ converges. Sufficient conditions and a necessary condition on $f(x, y)$ for the absolute summability of its double Fourier series are obtained. These results are analogues of those for single series proved by Bosanquet (Proc. London Math. Soc. (2) vol. 41 (1936) pp. 519–527) and Hyslop (Proc. London Math. Soc. (2) vol. 43 (1937) pp. 475–483). (Received March 11, 1948.)

255*t*. J. L. Walsh. *On the critical points of real rational functions.*

Let the circular region C: $|z-bi|\leqq r$ and $\bar{C}$: $|z+bi|\leqq r$ contain all zeros of a real rational function $R(z)$ whose poles are all real. The critical points of $R(z)$ lie on the set consisting of the axis of reals plus (i) the closed interiors of C and $\bar{C}$ if we have $b\geqq 2r$; (ii) the closed interiors of the circles of the coaxal family determined by C and $\bar{C}$ passing through the points $\pm i[(b^2-2r^2)/2]^{1/2}$ if we have $2r>b>2^{1/2}r$. (Received April 8, 1948.)

256. Alexander Weinstein: *Separation theorems for the eigenvalues of partial differential equations.*

It is shown by a variational method introduced in previous papers (A. Weinstein, C. R. Acad. Sci. Paris (1935) and Mémorial des Sciences Mathématiques, No. 88, 1937, Bull. Amer. Math. Soc. vol. 49 (1943) p. 368) that some results connected with the oscillation theorems of the theory of ordinary differential equations can be extended to the field of partial differential equations. Let S be a domain in m dimensions and let f be a given function on its boundary C with a vanishing mean value $M(f)$. It is shown that the eigenvalues $\lambda=\lambda_k$ of the equation $\Delta u+\lambda u=0$ with the boundary conditions: (1) $M(fu)=0$; (2) $\partial u/\partial n$ congruent zero modulo f, separate the eigenvalues $\lambda=\omega_k$ of the same equation with the boundary condition $\partial u/\partial n=0$, namely that $\omega_k\leqq\lambda_k\leqq\omega_{k+1}\leqq\lambda_{k+1}$. For $m=1$ this theorem reduces to the corresponding result about the eigenvalues of the periodic eigenfunctions of the equation $u''+\lambda\mu=0$. (Received March 9, 1948.)

Applied Mathematics

257*t*. Garrett Birkhoff: *Dimensional analysis and its generalizations.*

Buckingham's pi theorem is proved without any restrictions on the functions involved. The assumption of differentiability in Bridgman's characterization of homogeneous variables by the property of "absolute invariance of relative magnitude" is replaced by a weak continuity assumption. Dimensional analysis can be applied to problems whose formulation is invariant under a group of transformations of the form $q_i\to\lambda_i q_i$ $(i=1, \cdots, n)$. It ordinarily permits a reduction by n in the number of variables or parameters which need be considered. It is shown that this is true of any group, and that the same principle can be applied to systems of partial differential equations. This general principle, whose explicit formulation seems to be new, is then applied to obtain as special cases well known solutions of the heat equation and nonlinear partial differential equations of fluid dynamics. (Received February 4, 1948.)

258*t*. Garrett Birkhoff and Milton Plesset: *Wall corrections for cavity flow.*

Using formulas previously derived by Rethy, Cisotti, and Mises, wall corrections are obtained for two-dimensional flow with cavity ("wake") past a vertical flat plate in the middle of a water tunnel with fixed walls, in a free jet, and in a jet issuing from a channel. It is shown that the method of Mises can be extended to any flow in which the hodograph is a circular sector, and the diagram in the complex potential plane is a plane, half-plane, or infinite strip, with or without cuts. The indefinite integrals involved are the same, and can be evaluated explicitly in terms of complex logarithms if the central angle of the circular sector is commensurable with 2π. The problems of numerical computation are discussed, and a "principle of stability of the pressure coefficient" is obtained in the fixed wall case. (Received March 19, 1948.)

259. A. E. Heins: *The effect of a submerged barrier on water waves in a channel of finite depth.*

The author is concerned with the solution of (*) $\nabla^2\phi=0$ in the strip $-\infty<x<\infty$, $0\leqq y\leqq a$ subject to the conditions (i) $\partial\phi/\partial y=0$ for all x, and $y=0$, (ii) $\partial\phi/\partial y=0$, $y=b$ $(b<a)$, $x<0$, (iii) $\partial\phi/\partial y=\beta\phi$ $(\beta>0)$ $y=a$, for all x. A travelling wave solution in the regions $b\leqq y\leqq a$, $x<0$, and $0\leqq y\leqq a$, $x>0$, is found. The plane $y=b$, $x<0$ is the submerged barrier. The problem is formulated as an integral equation of the Wiener-Hopf type in terms of a Green's function kernel which may be exhibited explicitly. An explicit solution as an infinite series or a contour integral is derived and the asymptotic properties of the solution are discussed. (Received March 1, 1948.)

260. Brockway McMillan: *Realizability of passive networks*. Preliminary report.

A $2n$-pole is a network of lumped elements having n terminal pairs T_r, T_r' $(1 \leqq r \leqq n)$ across which external generators may be connected. If the currents j_re^{pt} in these external meshes are unrestricted, then the $2n$-pole has an impedance operator $Z(p)$ such that $f = Z(p)j$, where f_re^{pt} is the voltage between T_r and T_r', and the f_r and j_r are components of vectors f and j in a unitary space. R. M. Foster (1924) gave conditions necessary and sufficient that a scalar $Z(p)$ be the impedance of a 2-pole of reactances only. O. Brune (1932) extended these to a general passive 2-pole. Using ideal transformers: (i) W. Cauer (1932) proved that the operator $Z(p)$ is the impedance of a reactance $2n$-pole if and only if $\phi_j(p) = [Z(p)j, j]$ satisfies Foster's scalar criterion for every vector j; (ii) in the present work the author proves that $Z(p)$ is the impedance of a general passive $2n$-pole if and only if $\phi_j(p)$ satisfies Brune's criterion for every j. Some simple side conditions common to (i) and (ii) are unstated. (Received March 12, 1948.)

261*t*. H. E. Salzer: *Coefficients for facilitating trigonometric interpolation.*

The Gauss formula (1) $f(x) = \sum_{i=0}^{2n} [\prod_{j=0}^{2n}{}' \sin 2^{-1}(x-x_j) / \prod_{j=0}^{2n}{}' \sin 2^{-1}(x_i-x_j)]f_i$ for trigonometric interpolation, the symbol $\prod'$ denoting the absence of a factor having $j=i$, involves a large amount of computational labor, even when the points x_i are equally spaced. By generalization of a method originally given by W. J. Taylor for Lagrangian interpolation only, the work in employing (1) is reduced considerably with the aid of the auxiliary coefficients $A_i^{(2n+1)} = 1/\prod_{j=0}^{2n}{}' \sin 2^{-1}(x_i-x_j)$. Unlike the corresponding quantities for Lagrangian interpolation, separate sets of $A_i^{(2n+1)}$ are required for each interval in x. The coefficients $A_i^{(2n+1)} = A_{2n-i}^{2n+1}$ are tabulated to $8S$ for the 3-, 5-, 7-, 9-, and 11-point cases, all at intervals in x equal to 1.0, 0.5, 0.2, 0.1, 0.05, 0.02 and 0.01; also $A_i^{(2n+1)}$ are given to $8S$ for functions tabulated at $2n+1$ equally spaced points over a range of π and $\pi/2$, for $2n+1 = 3(2)11$. The extension of this method to interpolate for other special types of functions besides polynomials and trigonometric sums is briefly indicated. (Received February 2, 1948.)

262*t*. H. E. Salzer: *Formulas for complex Cartesian interpolation of higher degree.*

In interpolating by Lagrange's formula for an analytic function tabulated over a square grid, $f(z) = f(z_0 + Ph)$ where h is the length of a square, if in $P = p + iq$ either p or q is finer than 0.1, or if the interpolation requires eight or nine points, no Lagrangian coefficients (polynomials in P) are available, and their calculation involves a prohibitive amount of labor. But generalization of a scheme originated by W. J. Taylor, to hold for complex interpolation based upon any set of fixed points, leads to a shorter method of interpolation which employs auxiliary coefficients A_k. These quantities A_k are tabulated for various configurations of the fixed points z_k, ranging from the three- to nine-point cases. In the five- to nine-point cases, the quantities A_k are given for at least two configurations which are applicable everywhere in a table, and for one to three additional configurations (for each degree) which will locate the argument z more centrally with respect to the z_k's. The quantities A_k are also useful for (a) checking purposes, since they are the coefficients of f_k in the complex divided differences, and for (b) performing complex extrapolation. (Received February 13, 1948.)

263*t*. Alexander Weinstein: *On surface waves.*

The mixed boundary value problems considered by H. Lewy (Bull. Amer. Math. Soc. vol. 52 (1946) p. 737), J. J. Stoker (Quarterly of Applied Mathematics vol. 5 (1947) p. 1) and others in connection with the theory of surface waves are reduced by a conformal transformation to the following problem: to find all harmonic functions vanishing on the boundaries of a parallel strip (see A. Weinstein, ***Zum Phragmen-Lindeloefschen Ideenkreis***, Abh. Math. Sem. Hamburgischen Univ., 1928). This procedure allows one to find a more general type of solutions than those considered previously. (Received March 18, 1948.)

GEOMETRY

264*t*. Edward Kasner and John DeCicco: *Generalization of Appell's transformation.*

The authors extend Appell's theorem concerning the transformation theory of the dynamical trajectories of a field of force from positional fields to generalized fields where the force vector depends not only on the position of the point but also on the direction through the point. The required transformations for generalized fields are those of the mixed projective group of eight-parameters consisting of collineations and correlations together with a time factor. This time factor depends in the case of a collineation only on the particular collineation up to an arbitrary function of two variables; and in the case of a correlation, it depends not only on the particular correlation but also on the particular generalized field under consideration up to an arbitrary function of two variables. This is a new phenomenon. If both fields are positional, the authors' theorems give Appell's theorem as a special case. Then the only appropriate transformations are the collineations and the time factor depends only on the collineation used up to an arbitrary constant factor. (Received December 30, 1947.)

265*t*. Edward Kasner and John DeCicco: *Physical curves in generalized fields of force.*

The authors study the geometry of systems S_k of ∞^3 curves in a generalized field of force where the force vector depends not only on the position of the point but also on the direction through the point. A system S_k consists of curves along which a constrained motion is possible so that the pressure P is proportional to the normal component N of the force vector. The important systems of physical interest are (a) the system S_0 of dynamical trajectories, (b) the system S_1 of generalized catenaries, (c) the system S_{-3} of generalized brachistrochrones, (d) the systems S_∞ of generalized velocity systems. Every system S_k is given by a differential equation of the type (G) and thus possesses the Geometric Property I. Velocity systems are curvature trajectories. The ratio of the curvature of the curves of S_k at the points where the speed of the particle is zero to the curvature of the tangent line of force is studied. Finally the transformation theory of systems S_k is developed. (Received February 4, 1948.)

266*t*. Edward Kasner and John DeCicco: *Supplementary theorems in dynamics.*

The authors discuss geometric properties I, II, III of triply infinite families of trajectories in an arbitrary positional field of force. New proofs in parametric form of the known properties are given and additional properties are discussed. The distinction between actual and virtual trajectories is carefully considered. (Received February 4, 1948.)

267*t*. Edward Kasner and John DeCicco: *The osculating conics of physical families of curves.*

Consider the ∞^3 curves of a system S_k connected with any arbitrary positional field of force in the plane. A curve of a system S_k is a locus along which a constrained motion is possible so that the pressure P is proportional to the normal component N of the force vector. Thus $P=kN$. The four important physical subcases are (1) the systems S_0 of trajectories, (2) the systems S_1 of catenaries, (3) the systems S_{-1} of brachistochrones, (4) the systems S_∞ of velocity curves. Construct the osculating conics of fourth order contact to the ∞^3 curves of a system S_k. Through any general lineal-element E, there pass ∞^1 conics. Their centers describe a conic and also the envelope is a conic. By varying k, it is found that the central conics and also the enveloping conics form quadratic families. The system S_0 of trajectories are of special interest. The osculating conics of S_0 passing through a given lineal-element E always touch two fixed lines which intersect on the line of the force vector. The centers describe a straight line. (Received March 17, 1948.)

268. Don Mittleman: *The unions of trajectorial series of lineal elements generated by the plane motion of a rigid body.*

The motion of a plane, rigid body moving freely in a plane under the influence of any force which depends solely on the position of the body may be described by the motion of a lineal element composed of the center of gravity and a principal axis of inertia. If the maximum diameter of the body approaches zero, limiting forms of the equations of motion of the rigid body, called the equations of motion of an *element-particle*, are obtained. The concept of *microscopic body* is introduced also, and the equations of motion for such a body are determined. A trajectory in each of the three cases is a series of lineal elements. For each case, the totality of trajectories in a given field is ∞^5 series; among these series there may be trajectorial unions. The unions associated with the motion of an element-particle are studied in detail: for no field of force can one find ∞^3 unions among the ∞^5 series; in some fields of force one can find ∞^2 unions. Necessary and sufficient conditions that ∞^1 curves be trajectorial unions are given geometrically. (Received March 10, 1948.)

LOGIC AND FOUNDATIONS

269. E. L. Post: *Degrees of recursive unsolvability.* Preliminary report.

The author's canonical sets (Amer. J. Math. vol. 65) are generalized to S-canonical sets by hypothetically adding primitive assertions representing the membership or non-membership of 1, 2, 3, $\cdots$ in set of positive integers S. Set S_1 of positive integers is proved (Turing) reducible to S (Post, Bull. Amer. Math. Soc. vol. 50) when and only when S_1 and its complement are S-canonical sets. A "complete" S-canonical set S' is set up, and it is proved that each S-canonical set is reducible to S', while S' is not reducible to S. With S, say, the null set K_0, a scale of increasing degrees of unsolvability is thus furnished by $K_1, K_2, K_3, \cdots, K_{n+1}=K_n'$. The main completed result is a strengthened Kleene Theorem II, Trans. Amer. Math. Soc. vol. 53: each class of sets in Kleene's $(n+1)$st column includes a set of higher degree of unsolvability than any set common to both classes. Work is in progress on further equivalence proofs, further applications of the K_n-scale, incomparable degrees of unsolvability related to the K_n-scale, extension of the main theorem to Mostowski, Fund. Math. vol. 34, and extension of the K_n-scale into the constructive transfinite. (Received March 17, 1948.)

270*t*. A. R. Schweitzer: *On the principles of correspondence and continuation in heuristic mathematics.*

In his monograph, *Theorie und Anwendung der Laplace-Transformation* (Berlin, 1937; New York, 1943) Gustav Doetsch mentions two principles which guide the development of his discussion, namely, (1) "Abbildungsprinzip" (p. 146; see also pp. 174, 222, 223), (2) "Fortsetzungsprinzip" (pp. 285–286). In this paper principle (1) is found to be similar to the principle of correspondence stated by the author in his article, *Les idées directrices de la logique génétique des mathématiques* (Rev. de Mét. et de Mor., Paris, 1914, pp. 174–196) and which is attributed by the author to W. K. Clifford (loc. cit. p. 189). Principle (2) is interpreted as a special case of the author's principle of continuation (loc. cit. p. 188) to which the preceding principle of correspondence is subordinated. The effect of these relationships is to extend the application of the author's principle of continuation to mathematical physics as outlined by Doetsch in his treatise. Reference is made to the author's interpretation of the principle of Dirichlet (loc. cit. p. 192; Doetsch, loc. cit. p. 378) in the theory of the potential. Also the author's principle of continuation is again seen to act as a bond connecting analysis with other branches of mathematics. (Received March 9, 1948.)

STATISTICS AND PROBABILITY

271*t*. T. W. Anderson: *The theory of testing serial correlation.*

Tests of significance of $\lambda=0$ are considered when the observations have a normal distribution with exponent $-2^{-1}\alpha\{(X-M)'\Psi(X-M)+\lambda(X-M)'\Theta(X-M)\}$, where X relates to the vector of observations, M is its expected value (a regression on vectors of "fixed variates"), and Ψ and Θ are specified matrices such that for values of λ under consideration $\Psi+\lambda\Theta$ is positive definite. The class of all similar regions is deduced. The uniformly most powerful tests for one-sided alternatives and the B_- test for two-sided alternatives are derived under the condition that the fixed variate vectors are characteristic vectors of Θ normalized with respect to Ψ. This theory is applied to cases where Ψ and Θ are defined so that the density is one of a serially correlated population. If the fixed variate vectors consist of certain trigonometric terms, the best tests for a circular population are given by inequalities on the circular serial correlation coefficient of residuals. In each of two other cases a serial correlation coefficient treated in the literature is obtained for defining the best tests. Significance levels are given for two cases of special interest. (Received March 10, 1948.)

272*t*. Garrett Birkhoff: *Note on Poincaré's recurrence theorem.*

Kac has recently extended Poincaré's recurrence theorem to stochastic processes involving discrete states (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1002–1010). The recurrence theorem is now extended to arbitrary stochastic processes, in the following form. Let T be any transition operator, with $eT=T$ and $0\leqq f\leqq e$. Then

$\lim_{m\to\infty} f \cap m \limsup_{n\to\infty} \{fT^n\} = f$. This is obviously equivalent to the classical recurrence theorem in the deterministic case. The simpler statement $f \cap \limsup_{n\to\infty} \{fT^n\} = f$ is unfortunately not generally true, except in the deterministic case. (Received March 19, 1948.)

273t. K. L: Chung: *Asymptotic distribution of the maximum cumulative sum of independent random variables.*

Certain distributions considered by Erdös-Kac and Wald are derived from a classical formula due to DeMoivre. An estimate of the remainder is obtained in each case. For Bernoullian variables this is of the order of magnitude $n^{-1/2}$. For more general cases a recent theorem due to H. Bergström is used and the order of magnitude is a certain negative power of n. This seems amply sufficient for any "strong" theorem we may wish to prove. (Received March 1, 1948.)

274. Paul Erdös: *On a theorem of Robbins and Shu.*

Let $f_1(x)$, $f_2(x)$, $\cdots$ be an infinite sequence of independent functions in (0, 1) all having the same distribution function $G(x)$. Assume that $\int_{-\infty}^{\infty} x\,dG(x) = 0$, $\int_{-\infty}^{\infty} x^2 dG(x) < \infty$. A theorem of Robbins and Shu state that if M_n denotes the measure of the set in x for which $\left| \sum_{k=1}^{n} f_k(x) \right| > n$ then $\sum_{n=1}^{\infty} M_n$ converges. They also conjecture the converse, namely that if $\sum_{n=1}^{\infty} M_n$ converges then $\int_{-\infty}^{\infty} x\,dG(x) = 0$ and $\int_{-\infty}^{\infty} x^2 dG(x) < \infty$. The author succeeded in proving this conjecture and also obtained a simpler proof of the theorem of Robbins and Shu. (Received March 9, 1948.)

275. Bernard Friedman: *A simple type of Markoff chain.* Preliminary report.

Let W_n, X_n ($n = 1, 2, \cdots$) be random variables such that $W_{n+1} = W_n + X_n$ and such that the probability distribution of X_n depends linearly on W_n. If $f_n(t)$ is the generating function of W_n, we find that $f_{n+1}(t) = m_0(t) f_n(t) + m_1(t) f_n'(t)$ where $m_0(t)$ and $m_1(t)$ are known functions. A method for solving this difference-differential equation is given. The results can be specialized to obtain the probabilities in an urn problem considered by Pólya and Eggenberger (Zeitschrift für Angewandte Mathematik und Mechanik vol. 3 (1923) p. 279) and also the probabilities in a random walk with elastic restoring force (M. Kac. Amer. Math. Monthly vol. 54 (1947) p. 369). (Received March 12, 1948.

276t. Wassily Hoeffding and Herbert Robbins: *A central limit theorem for dependent random variables.*

A sequence of random variables or vectors is called m-dependent if any two sets of terms are independent whenever they are separated by more than m intervening terms, $m \geq 0$. Under certain mild conditions (*not* involving conditional expectations) the central limit theorem for sums holds for m-dependent sequences. (Received March 19, 1948.)

277t. W. G. Madow: *On the limiting distributions of statistics based on samples of finite populations.*

Let us suppose that a random sample is selected without replacement from a finite population. Then under very general conditions concerning the "growth" of the population, it is shown that the limiting distribution of the sample mean is normal. This result is then applied to obtain the limiting distribution of more general statistics. The method of obtaining the basic distribution is through the calculation of the moments. Once the basic distribution is obtained conclusions are drawn concerning the convergence of the characteristic functions of other statistics. (Received March 11, 1948.)

278t. F. J. Murray: *On locally dependent chance variables.* Preliminary report.

Consider a set R of chance variables ρ_r such that each ρ_r is independent of all but a certain subset $U(\rho_r)$ of R. The number of elements in $U(\rho_r)$ is supposed to be bounded by a number k which is small relative to N, the number of elements in R. Let Σ denote the sum of the ρ_r. The objective of the present paper is to extract the Nth root of the characteristic function of Σ and for this purpose it is shown that the semi invariants of Σ are essentially proportional to N under these circumstances. Convergence questions are also considered. This result is applicable to the order-disorder problem in a crystal lattice and this connection is also given. (Received April 12, 1948.)

279. Edward Paulson. *Some limiting distributions associated with the two box problem.*

Let balls be thrown into either box $B^{(1)}$ or $B^{(2)}$ with probability p of falling in $B^{(1)}$. Let n denote the random number of throws required to reach a prescribed pattern of balls. McCarthy (Ann. Math. Statist. vol. 18 (1947) pp. 349–384) obtained the moments of n for several prescribed patterns. The limiting distribution of n required to get at least k_1 in $B^{(1)}$ and at least k_2 in $B^{(2)}$ is now shown to be normal. In the same problem, let n_1 equal number of throws needed until one box contains required number, and n_2 the additional number needed to complete experiment; the limiting distribution of n_1 and n_2 is shown to be bivariate normal. Consider probability $C_{n-1,k-1}p^k(1-p)^{n-k}$ that n throws are required to get k balls in $B^{(1)}$. Put $p=1-1/s$, let k be a random variable depending on parameter s in such a manner that $E(k)=As$, $\sigma^2(k)=Bs$ (A, B constants), and the limiting distribution of k as $s\to\infty$ is normal. It is shown that $\lim_{s\to\infty}$ [Probability $\{n-k=i\}]=e^{-A}A^i/i!$. In same situation, keep p fixed, then the limiting distribution of n is normal. This last result has been generalized. (Received February 28, 1948.)

280. Herman Rubin: *Some extensions of the central limit theorem to dependent variables.*

The method of Lindeberg and Lévy is applied to the distribution of real-valued functions of the values of a discrete stochastic process with values in an arbitrary space. The special case in which the space is the unit interval with Borel field the ordinary Borel sets is first considered, and the results are obtained in that case by a modification of the proofs given by Lévy and Loève. The results in the general case follow immediately from those of the special case. (Received March 10, 1948.)

281. J. W. Tukey: *Asymptotic moments and expectations.*

Let $Z_n\|I$ denote the chance quantity Z_n *restricted* to I, that is to say, the chance quantity whose values all lie in I with probabilities proportional to those of Z_n. If $\Pr\{Z_n \text{ not in } I\}$ vanishes more rapidly than the variance of $Z_n\|I$, the variance of the sequence $\{Z_n\|I\}$ is the *asymptotic variance* of $\{Z_n\}$. The choice of another such *adequate* interval does not change the order of the asymptotic variance. If every open interval containing z_0 is adequate for $\{Z_n\}$, then, under mild conditions: (I) the asymptotic variance of $\{g(Z_n)\}$ is the product of the asymptotic variance of $\{Z_n\}$ and $(g'(z_0))^2$; (II) the *asymptotic mean* (in a similar sense) of $\{g(Z_n)\}$ depends on $g(z_0)$, $g'(z_0)$, and $g''(z_0)$ in a well defined way; (III) the higher *asymptotic moments* vanish (to the order to which they are unique); (IV) if the *asymptotically standardized* distributions corresponding to $\{Z_n\}$ converge to a distribution with unit variance, then the asymptotically standardized distributions corresponding to $g\{Z_n\}$ converge to the same limiting distribution. These results are extended to the case of a sequence of functions $g_n(Z)$, and applied to the special case of estimation of scale by equating the average pth power of its expected value. (Received March 15, 1948.)

TOPOLOGY

282*t*. Jean M. Boyer and D. W. Hall: *On a certain Peano space.*

The authors show that any Peano space M can be regarded as the continuous image of a rather simple Peano space L containing the unit interval I in such a way that I maps under the transformation onto a preassigned Peano space P contained in M, while the image of each of the at most denumerable collection of components of $L-I$ has at most one point in common with P. Applications to structure theory of Peano spaces are drawn including a very simple proof of the cyclic connectivity theorem along lines suggested by W. L. Ayres. (Received March 25, 1948.)

283. C. H. Dowker: *Retracts of metric spaces.*

A metric space is called an absolute neighborhood retract (ANR) if it is a neighborhood retract of every metric space containing it as a closed subset. Let R be a metric (but not necessarily separable) space. The following conditions hold if and only if R is an ANR: (1) Every mapping into R of a closed subset A of a metric space B can be extended over a neighborhood of A in B. (2) There is a homeomorphism f of R into a generalized Hilbert space H such that $f(R)\times 0$ is a neighborhood retract of the subspace $f(R)\times 0+H\times(0, 1]$ of $H\times[0, 1]$. (3) For every $\epsilon>0$ there is a natural polytope P and mappings $f: R\rightarrow P$, $g: P\rightarrow R$ such that $gf: R\rightarrow R$ is ϵ-homotopic to the identity. (4) Every open covering σ of R has a refinement τ such that a partial realization of a singular complex in τ (that is, with each partial cell contained in a set of τ) can be extended to a full realization in σ. (Received March 17, 1948.)

284*t*. Beno Eckmann: *Coverings and Betti numbers.*

Let the finite polyhedron P be a regular covering of the polyhedron $\overline{P}$, and let G be the corresponding group of covering transformations of P. The nth homology group H_n of P with real coefficients is a real vector group of finite rank, and G acts on H_n as a group of linear transformations. It is proved that the homology group $\overline{H}_n$ of $\overline{P}$ with real coefficients is determined by H_n and by the operation of G on H_n. Namely, when $s_n(x)$, $x\in G$, denotes the character of the linear representation of G in H_n, and g the order of G, the nth Betti number $\bar{p}_n$ of $\overline{P}$ is equal to $1/g\sum_{x\in G}s_n(x)$. As an application, relations are established between the Betti numbers of a closed nonorientable manifold $\overline{M}$ and those of a two-sheeted orientable covering of $\overline{M}$. (Received February 24, 1948.)

285*t*. C. J. Everett and S. M. Ulam: *On the problem of determination of mathematical structures by their endomorphisms.*

As examples of the general problem the following are noted: (1) Given the transformations $n \rightarrow kn$ for all integers n, the endomorphisms of the additive group of integers, what are all abelian groups definable on the set of integers, admitting the given class of transformations as endomorphisms? A general class of such groups is obtained. The same problem can be studied for the case of the rational, real, and complex fields. (2) Given the class of all homeomorphisms of a topological space, what other topologies exist on the same set which have these transformations as the class of all their homeomorphisms? It is easy to find examples (including the line) where the topology is uniquely determined by the class of its homeomorphisms. The problem in an algebraic formulation originated in conversations with E. Teller and J. v. Neumann. (Received March 12, 1948.)

286*t*. M. K. Fort: *A unified theory of semi-continuity*. Preliminary report.

Let M be a metric space with metric ρ, and let P be the set of all positive real numbers. A *continuity structure* for M is an ordered triple $(\bigcirc, F, \Box)$, such that: $\bigcirc$ is one of the relations $>$ or $<$, F is a function whose domain is $P \times M$, $\Box$ is a relation which partially orders the range of F, and $F(h, x) \Box F(k, y)$ for all $h, k \in P$ and all $x, y \in M$ such that $h \bigcirc k$ and $\rho(x, y) < |h-k|$. A function f on a topological space into M is $(\bigcirc, F, \Box)$-continuous at a point p if corresponding to each pair $h, k \in P$ such that $h \bigcirc k$ there is a neighborhood V of p such that $F(h, f(p)) \Box F(k, f(q))$ for all $q \in V$. Some theorems about $(\bigcirc, F, \Box)$-continuous functions are proved. If the proper continuity structures are chosen, this theory specializes to yield some of the classical theorems about semi-continuous real-valued functions (including a theorem concerning the category of the set of continuity points of a semi-continuous function). It is possible to choose other continuity structures and, by making use of hyperspaces, obtain theorems about semi-continuous set-valued functions (including semi-continuous decompositions). (Received February 27, 1948.)

287. A. N. Milgram: *Isomorphisms of the semigroup of continuous functions on a bicompact Hausdorff space.* II.

Let X be a bicompact Hausdorff space and $C(X)$ the semigroup of continuous functions with multiplication defined pointwise; $f \cdot g(x) = f(x) \cdot g(x)$. If $p(x)$ is a positive function in $C(X)$ denote by E_p the automorphism E_p: $f \rightarrow f$ where $f(x) = \operatorname{sgn} f(x) \cdot |f(x)|^{p(x)}$. Call such an automorphism an exponential automorphism of the semigroup $C(X)$. Let $\tau_1, \cdots, \tau_n$ be n automorphisms of the semigroup of the real numbers, and let $x_1, \cdots, x_n$ be n isolated points of X. Define the automorphism T of $C(X)$ by $T: f \rightarrow \bar{f}$ where $\bar{f}(x_i) = \tau_i(f(x_i))$ and $\bar{f}(x) = f(x)$ for all $x \neq x_1, \cdots, x_n$. Call T a finite automorphism. If H is a homeomorphism of X on X', define the isomorphism H^* of $C(X)$ on $C(X')$ by $H^*: f \rightarrow f'$, $f(x) = f'(H(x))$. The theorem can now be stated: If X and X' are spaces satisfying the first denumerability axiom of Hausdorff and σ is an isomorphism of $C(X)$ on $C(X')$ there exists a homeomorphism H of X on X', a finite automorphism T, and an exponential automorphism E_p such that $\sigma = T \cdot E_p \cdot H^*$. (Received March 11, 1948.)

288. C. W. Saalfrank: *Retraction properties for normal Hausdorff spaces.*

The author develops a retraction theory for normal Hausdorff spaces, NHS. AR and ANR are defined for NHS and are characterized as homeomorphs of retracts and closed neighborhood retracts, respectively, of Tychnoff cubes. It is shown that the topological product of any set of AR or any finite set of ANR is an AR or an ANR, respectively. It is proved that AR have the fixed point property and that for ANR any null-homotopic map has a fixed point. Borsuk's theorem is extended as follows. Let C be a closed subset of a compact Hausdorff space X and let B be a retract of an AR set A. Then for any map f such that f: $(X\times B)\cup(C\times A)\rightarrow N$, where N is an ANR, there exists an extension F of f over $X\times A$ such that F: $X\times A\rightarrow N$. A similar result is proved in which N is any topological space, but C is a closed neighborhood retract of an ANR set X and $X\times A$ is perfectly normal. The author exhibits a finite-dimensional, compact, locally contractile, Hausdorff space which is not an ANR and thus shows that a rather natural conjecture for the characterization of finite-dimensional ANR is impossible. (Received February 24, 1948.)

J. R. KLINE,
Secretary

THE APRIL MEETING IN ANN ARBOR

The four hundred thirty-fifth meeting of the American Mathematical Society was held at the University of Michigan, Ann Arbor, Michigan, on April 16–17, 1948. The attendance was approximately 200, including the following 175 members of the Society:

L. U. Albers, A. A. Albert, W. R. Allen, D. N. Arden, R. C. F. Bartels, J. H. Bell, Felix Bernstein, R. H. Bing, H. L. Black, C. J. Blackall, W. M. Boothby, D. G. Bourgin, J. W. Bradshaw, Richard Brauer, F. L. Brown, L. M. Browne, R. H. Bruck, W. K. Burroughs, R. E. Carr, E. D. Cashwell, Abraham Charnes, R. V. Churchill, Nathaniel Coburn, C. J. Coe, H. J. Cohen, A. H. Copeland, Max Coral, H. S. M. Coxeter, C. C. Craig, Jane Cronin, H. B. Curry, M. M. Day, D. J. Dickinson, Flora Dinkines, C. L. Dolph, Ben Dushnik, P. S. Dwyer, John Dyer-Bennett, W. F. Eberlein, B. J. Eisenstadt, Benjamin Epstein, Ky Fan, C. H. Fischer, K. W. Folley, J. S. Frame, R. E. Fullerton, Casper Goffman, Michael Golomb, S. H. Gould, L. M. Graves, L. J. Green, E. B. Grennan, P. E. Guenther, William Gustin, S. W. Hahn, Marshall Hall, G. E. Hay, Ruth Heinsheimer, R. G. Helsel, I. N. Herstein, Fritz Herzog, Edwin Hewitt, J. F. Heyda, T. H. Hildebrandt, J. D. Hill, D. L. Holl, L. A. Hopkins, C. C. Hsiung, H. K. Hughes, H. D. Huskey, R. E. Huston, A. W. Jacobson, Meyer Jerison, P. S. Jones, G. K. Kalisch, L. H. Kanter, Irving Kaplansky, Wilfred Kaplan, Leo Katz, D. E. Kibbey, W. M. Kincaid, C. F. Kossack, D. M. Krabill, Benjamin Lapidus, C. G. Latimer, H. D. Larsen, K. B. Leisenring, J. H. Levin, Madeline Levin, D. J. Lewis, B. J. Lockhart, E. D. McCarthy, S. W. McCuskey, H. B. Mann, Morris Marden, J. W. Marshall, L. E. Mehlenbacher, E. J. Mickle, D. D. Miller, J. M. Mitchell, E. E. Moise, D. C. Morrow, S. B. Myers, C. J. Nesbitt, K. L. Nielsen, E. A. Nordhaus, M. J. Norris, D. A. Norton, J. A. Nyswander, F. C. Ogg, J. M. H. Olmsted, E. J. Olson, E. H. Ostrow, G. K. Overholtzer, F. W. Owens, H. B. Owens, R. S. Pate, M. H. Payne, P. M. Pepper, Sam Perlis, C. L. Perry, George Piranian, E. C. Pixley, V. C. Poor, G. B. Price, Gustave Rabson, Tibor Rado, G. Y. Rainich, E. D. Rainville, M. O. Reade, O. W. Rechard, Arthur Rosenthal, A. E. Ross, E. H. Rothe, H. J. Ryser, Hans Samelson, W. C. Sangren, A. C. Schaeffer, H. M. Schaerf, K. C. Schraut, W. R. Scott, I. E. Segal, M. E. Shanks, A. S. Shapiro, L. W. Sheridan, Seymour Sherman, M. F. Smiley, F. C. Smith, G. W. Smith, W. F. Smith, W. N. Smith, T. H. Southard, R. D. Specht, R. H. Stark, H. E. Stelson, Rothwell Stephens, B. M. Stewart, Helen Sullivan, R. L. Swain, R. M. Thrall, C. W. Topp, Leonard Tornheim, G. L. Walker, M. S. Webster, M. T. Wechsler, D. V. Wend, N. A. Wiegmann, R. L. Wilder, J. E. Wilkins, C. S. Williams, M. A. Woodbury, J. B. Wright, G. S. Young, P. M. Young, J. W. T. Youngs.

At 2:00 P.M. Friday Professor H. S. M. Coxeter delivered an address entitled *Self-dual configurations and regular graphs*, with Professor T. H. Hildebrandt presiding. At 9:00 A.M. Saturday Professor Irving Kaplansky gave an address entitled *Topological rings*, with Professor Marshall Hall presiding.

Two sessions for short research papers were held at 10:00 A.M. with presiding officers Professors A. C. Schaeffer and Richard Brauer. There was also a session at 3:15 P.M. Friday and one at 10:15 A.M.

Saturday with presiding officers Professors Morris Marden and S. B. Myers, respectively.

The University of Michigan was host to those attending the meeting at an afternoon tea which took place at 4:30 P.M. Friday in the Rackham Building. At 6:30 P.M. Friday a dinner was held at the Smith Catering Service with an attendance of approximately 150. Professor T. H. Hildebrandt acted as toastmaster. The guests were welcomed in a warm and amusing manner by Dean Hayward Kenniston, of the University of Michigan. Professor Tibor Rado gave a serious and interesting talk on the language problem in Puerto Rico, recounting his impressions and conclusions during his stay on the island.

Local arrangements for the meeting were in charge of Professor S. B. Myers.

There follow the complete abstracts of the papers presented at this meeting. Papers whose abstract number is followed by the letter "*t*" were presented by title. Paper no 291 was presented by Dr. Ryser, no. 298 by Professor Perlis, no. 308 by Professor Mickel.

Algebra and Theory of Numbers

289. A. A. Albert: *On right alternative algebras.*

An algebra A over a field F is called right alternative if $y(xx)=(yx)x$ for every x and y of A. We show that if F has characteristic not 2 then every such algebra of degree two over F is alternative. If e is an idempotent of A the right multiplications R_e satisfy the usual relation $R_e^2=R_e$. However the left multiplications only need satisfy $(L_e^2-L_e)^2=0$. Examples are given of right alternative algebras which are not alternative. Let F be a nonmodular field and $t(x, y)$ be the trace of L_{xy}. Then we show that the maximal nilideal of A is the set of all x such that $t(x, y)=0$ for every y and coincides with the radical of the attached algebra $A^{(+)}$. We call A semisimple if this radical is zero and prove that every semisimple right alternative algebra over a nonmodular field is actually alternative. (Received February 16, 1948.)

290*t*. Reinhold Baer: *Direct decompositions into infinitely many summands.*

In this note certain results concerning themselves with direct decompositions of operator loops into a finite number of direct summands are extended to direct decompositions into infinitely many direct summands. (Received February 16, 1948.)

291. R. H. Bruck and H. J. Ryser: *On the number of points in a finite projective plane.*

The following result is obtained: Let the positive integer n be such that (i) $(-1, n; p)^k=-1$ for at least one odd prime p, where $(-1, n; p)=(-1, n)_p$ is the Hilbert symbol and $2k=n(n+1)$. Then there exists no projective plane geometry with $n+1$ points on each line. (Condition (i) is equivalent to the following pair of conditions: (ii) n is of the form $4m+1$ or $4m+2$; (iii) at least one prime factor of the form $4m+3$ occurs in n to an odd power.) The proof uses the Minkowski-Hasse theory of equivalence of rational quadratic forms. The result seems to be new except for the case $n=6$ (for critical comments see F. W. Levi, *Finite geometrical systems*, University of Calcutta, Calcutta, 1942). (Received February 21, 1948.)

292. J. S. Frame: *Congruence relations between the traces of matrix powers.*

A congruence $S(p^\beta)\equiv S(p^{\beta-1})$ mod p^β involving the sums of powers of roots of unity is shown to hold under the following conditions. We let $S(m)$ denote the trace of the mth power of a matrix A of finite degree $S(0)$ over a field of characteristic zero. The congruence is implied by the two conditions: (1) $A^n=1$ and (2) $S(k)=S(1)$ for every k such that $(k, n)=1$. These conditions imply that $S(1)$ is a rational integer, but not every matrix with rational integral trace satisfies (2). The congruence is a powerful aid in constructing the table of characters of a finite group, and generalizes the well known special case $\beta=1$. (Received March 18, 1948.)

293*t*. Franklin Haimo: *An inverse limit group.*

Let G_∞ be the inverse limit group of the groups G/nG, $n=1, 2, \cdots$, for an additive abelian group G; and let G' be the subgroup of completely divisible elements of G. (Cf. S. MacLane, Bull. Amer. Math. Soc. Abstract 54-1-6.) G_∞ has no nonzero elements which are completely divisible. Moreover, G/G' is isomorphic with a subgroup G'' of G_∞ in an obvious fashion. However, if G can admit a topology which makes it a compact topological group, then $G''=G_\infty$. If G is the group of integers, it is easy to show that $G''\neq G_\infty$. From this follows the well known fact that the group of integers is never a compact topological group. (Received March 12, 1948.)

294. Marshall Hall: *Studies in free groups.*

A standard representation for a subgroup U of a free group F was given in a recent paper written jointly by T. Rado and the author. This representation is not in general unique. In this paper a canonical representation is found. This will be unique and will have certain properties not possessed by other representations. A number of applications of this representation are made. By a generalization of a process due to J. Nielson it may be shown that if the decision problem is solvable for a subgroup U, then it will be solvable within a restricted set of operations. Special properties of normal subgroups U are investigated. (Received March 12, 1948.)

295. C. G. Latimer: *On zero ternary quadratic forms.*

Let $f=\sum a_{ij}x_ix_j$ (a_{ij} integers, $a_{ji}=a_{ij}$) be a properly primitive indefinite ternary form of determinant $D>0$ and with a properly primitive adjoint F. If (u_1, u_2, u_3) is a solution of $f=0$ and $U_i=\sum_j a_{ij}u_j$ $(i=1, 2, 3)$ then (U_1, U_2, U_3) is a solution of $F=0$. If the g.c.d. of the u's is unity and the same is true of the U's, (u_1, u_2, u_3) is called a doubly primitive solution of $f=0$. It is shown that such a solution exists if and only if every generic character of f is equal to unity. If (u_1, u_2, u_3) is a doubly primitive solution there are certain integers t_i, such that $\sum t_iU_i=1$. We determine two transformations, each of determinant unity, which transform f and F into $\phi=y_1^2-y_2^2-Dy_3^2$ and $\Phi=DY_1^2-DY_2^2-Y_3^2$ respectively. The coefficients of these transformations are integers which are determined explicity in terms of the u's and t's. (Received March 11, 1948.)

296*t*. Saunders MacLane: *Operator homomorphisms of kernels.*

Let (K, θ) be a Q-kernel with center H and obstruction $k^3 \in H^3(Q, H)$, in the sense of Eilenberg and the author (Ann. of Math. vol. 48 (1947) pp. 326–341). Its graph Γ consists of all pairs (x, α), with $x \in Q$ and the automorphism α in the automorphism class $\theta(x)$; Γ has a natural homomorphism $\phi(x, \alpha) = x$ onto Q. If G is an abelian group with operators in Q, consider the group A of all operator homomorphisms of the kernel (K, θ) into G, reduced modulo those operator homomorphisms which are obtained by "cutting down" crossed characters of Γ into G. THEOREM: If Γ is a free group, then A depends only on H, G, Q, and the obstruction k^3 is in fact isomorphic to the group $H^2(Q, H, k^3, G)$ introduced by Eilenberg and the author in the study of the second cohomology group of a space (Proc. Nat. Acad. Sci. U. S.A. vol. 32 (1946) pp. 277–280). (Received February 23, 1948.)

297*t*. Saunders MacLane: *The second homotopy kernel of a polyhedron.*

Let P be a polyhedron, S the one-dimensional skeleton of a sufficiently fine triangulation of P, π_1 and π_2 the first and second homotopy groups of P. The second relative homotopy group $\pi_2(P, S)$ has π_2 as center and has the free group $\pi_1(S)$ as a group of operators. Using the fact that $\pi_1(S)$ has a canonical homomorphism onto π_1, it follows that $\pi_2(P, S)$ is a π_1-kernel; its graph is the free group $\pi_1(S)$, and its obstruction is the 3-dimensional cocycle $k^3 \in Z^3(\pi_1, \pi_2)$ associated with P. The expression of the cohomology groups of P in terms of k^3, found by Eilenberg and the author, can then be formulated as follows: The second cohomology group of P, with coefficients in G, is the group of all operator homomorphisms of the kernel $\pi_2(P, S)$ into G, modulo those homomorphisms which can be obtained by cutting down crossed homomorphisms of $\pi_1(S)$. (Received February 23, 1948.)

298. Sam Perlis and G. L. Walker: *Finite Abelian group algebras.* I.

Let G be a finite group and F a field. The determination of all groups H such that the group algebras G_F and H_F are isomorphic is solved for the case in which F is non-modular and G is abelian of prime power order. The result is a corollary of the solution of another problem in which G and H are given and one seeks all fields F making G_F isomorphic to H_F. For each pair G and H (abelian of prime power order p^e) there is determined a unique field L, contained in the field of p^eth roots of unity, such that $G_F \cong H_F$ if and only if $F \geqq L$. The field L is defined in terms of the invariants of G and H. For cyclic G of arbitrary order n and for any F of characteristic not dividing n, the structure of G_F is expressed very simply in terms of the factorization of $x^n - 1$ over F. (Received March 11, 1948.)

299. A. E. Ross and Thomas Matthews: *On a constructive development of Galois theory.*

Pursuing further the ideas of S. Shatunovsky (*Algebra as the study of congruences with respect to functional moduli*, Odessa, 1913) the authors consider a separable polynomial $f(x)$ of degree n, without multiple factors in a field K, and they represent the splitting field K_n of $f(x)$ as the residue class ring $K[x_1, \cdots, x_n]/I$ where I is the ideal in $K[x_1, \cdots, x_n]$ whose basis consists of polynomials which are factors of $f(x)$ usually used to carry out successive extensions leading to the splitting field K_n. They denote these factors by $\mu_1(x_1), \cdots, \mu_n(x_1, \cdots, x_n)$ and show that a permutation π of the roots $\bar{x}_1, \cdots, \bar{x}_n$ ($\bar{x}_i$ is the residue class of x_i modulo I) of $f(x)$ in K_n determines an automorphism σ of K_n over K if and only if $\mu_i(\xi_1, \cdots, \xi_i)=0$ implies $\mu_i(\sigma(\xi_1), \cdots, \sigma(\xi_n))=0$ for $i=1, \cdots, n$. This observation leads to a direct determination of the number of admissible permutations of the roots, that is, of the order of the group G of automorphisms σ of K_n over K. It follows at once that K is the fixed field of G, and the fundamental theorem of Galois theory appears as a direct consequence of a simple lemma on substitutions. (Received March 18, 1948.)

300. R. M. Thrall: *Some generalisations of quasi-Frobenius algebras.*

Consider algebras $\mathfrak{A}$ over a field k and with 1 element. $\mathfrak{A}$ is called a quasi-Frobenius (QF) algebra if every primitive left ideal is dual to a primitive right ideal. Some of the most important properties of QF algebras do not characterize these algebras, but occur in more extensive classes, thus leading to the following definitions. $\mathfrak{A}$ is said to be a QF-1 algebra if every faithful representation is its own second commutator. $\mathfrak{A}$ is said to be a QF-2 algebra if every primitive left (right) ideal of A has a unique minimal subideal. $\mathfrak{A}$ is said to be a QF-3 algebra if it has a unique minimal faithful representation (a faithful representation is called minimal if deletion of any direct constituent leaves it unfaithful). Some of the elementary properties of these classes of algebras are investigated with special attention to relations between them; for example, the class of QF-3 algebras properly contains the class of QF-2 algebras. (Received February 13, 1948).

ANALYSIS

301. W. F. Eberlein: *Abstract ergodic theorems.*

Mean ergodic theorems for a semi-group G of linear transformations are studied. It is shown that relaxation of the uniform boundedness and countability restrictions customarily imposed on G leads to a more general theory, which not only embraces a significantly wider domain of phenomena but subsums previous results in a sharper and more transparent form. In addition to the standard mean ergodic theorems, the theory contains as special cases the most general mean ergodic theorem for Abelian semi-groups, the existence and uniqueness of the mean for almost periodic functions, Fejer's $(C, 1)$ summability theorem, and applications to the abstract theory of Markoff processes. (Received March 11, 1948.)

302. Caspar Goffman: *On Lebesgue's density theorem.*

The density theorem of Lebesgue may be stated in the following form: If S is a measurable linear point set, the metric density of S exists and is equal to 0 or 1 almost everywhere. We prove the converse that for every set Z of measure 0 there is a measurable set S whose metric density does not exist at any point of Z. We note, however, that in order for Z to be the set of points for which the metric density of some set S exists but is different from 0 or 1, Z must be both of measure 0 and of first category. As a converse, we show that for every F_σ type set of measure 0 (thus, of first category) there is a measurable set S whose metric density exists but is different from 0 or 1 at every point of Z. Several related results are given for functions. (Received March 9, 1948.)

303. L. M. Graves: *Minima of functionals in abstract spaces.*

Let $J(x)$ and $\phi(x)$ be defined near $x_0 \in X$, values of ϕ in Y, X and Y Hilbert spaces. If $J(x_0)$ is a minimum subject to $\phi(x)=0$, then there exist $\lambda_0 \geq 0$ and a linear functional L on Y such that the first variation of $F(x)=\lambda_0 J(x)+L(\phi(x))$ vanishes at

x_0. A generalization of the Weierstrass condition is proposed, and it is shown that for each "strong variation," and for each "weak variation" satisfying $\delta\phi(x_0;\,\xi)=0$, there exist multipliers λ_0, L, for which the Weierstrass condition holds, and the second variation $\delta^2F(x_0;\,\xi)\geqq 0$. Normality is not required (cf. Goldstine, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 142–149). A condition used both here and by Goldstine requires that the range C of the transformation $\delta\phi(x_0;\,\xi)$ is closed in the space Y. An example shows how easily this property may be destroyed by a transformation of the side conditions $\phi(x)=0$. However, at least in the case when the functions ϕ are linear, the norm in the subspace C of Y may always be so chosen that the required conditions are satisfied. From this it follows that when the equations $\phi=0$ are linear partial differential equations, and the functional J is a multiple integral with suitable properties, then multipliers λ_0, L exist for which the first variation δF vanishes at the point x_0. (Received March 17, 1948.)

304t. Deborah T. Haimo: *Singularities of mappings of E^n into E^n.*

Let f be a mapping of E^n into E^n. A point p is singular if the determinant of order n, $J=\left|\partial f_i/\partial x_j\right|$, vanishes at p. The rank of a singular point is that of the matrix defined by J. Let F_s denote the class of all mappings f for which $\partial J/\partial x_1, \cdots, \partial J/\partial x_n$ do not vanish simultaneously at any singular point of rank s. One shows that if f is a mapping of class C^3 of a region R of E^n into E^n, then there exists a function f' in F_{n-1} which approximates f arbitrarily closely through the second derivative. Let p be a singularity of rank $n-1$ of a mapping f in F_{n-1}. Then p is a fold, if, after a choice of a coordinate system making $\partial f/\partial x_1\big|_p=0$ (always possible), $\partial J/\partial x_1\big|_p\neq 0$. If $\partial J/\partial x_1\big|_p=0$, then p is a cusp. For $n=2$ any mapping may be approximated by one every singular point of which is either a fold or a cusp. If the origin is a fold of a mapping f in F_{n-1} of class C^4 then there exist curvilinear coordinate systems about 0 and $f(0)$ in terms of which f has the form $y_1=x_1^2$, $y_i=x_i$, $i=2, 3, \cdots, n$. The form for a cusp is less simple. For the image space E^m, $m\geqq 2n-1$, see H. Whitney (Duke Math. J. vol. 10 (1943) pp. 161–172; Ann. of Math. vol. 32 (1937) p. 818; ibid. vol. 37 (1936) p. 654). (Received March 12, 1948.)

305. L. H. Kanter: *On the zeros of the Jacobi polynomials.*

Applying the notation and method of Abstract 53-5-211 to the Jacobi polynomials, it is shown that for a weight function $(1-x/a)^{\alpha}(1+x/a)^{\beta}$, with $a\geqq 2$, $-1<\alpha(\tau)\leqq 0$, $-1<\beta(\tau)\leqq 0$, $\alpha'(\tau)\geqq 0$, $\beta'(\tau)\geqq 0$, $\alpha'(\tau)+\beta'(\tau)>0$, $\lambda_\nu(\tau)$ is a monotonically decreasing function of τ. Moreover, if the condition is changed to $a\geqq 2$, $\alpha(\tau)>\beta(\tau)>-1$, $\alpha'(\tau)\geqq 0$, $\beta'(\tau)\geqq 0$, $\alpha'(\tau)+\beta'(\tau)>0$, $x_\nu(\tau)>0$, $\lambda_\nu(\tau)$ will be a decreasing function of τ provided that $x_\nu(\tau)$ is an increasing function of τ. For the ultraspherical polynomials the conclusion is that $\lambda_\nu(\tau)$ is a monotonically decreasing function of τ if $-1<\tau\leqq 0$. By making use of equation (15.3.2), p. 343, Szego, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, it is shown that for the ultraspherical polynomials of degrees 2 and 3, respectively, $\lambda_\nu(\tau)$ is a decreasing function of τ for $-1<\tau$. (Received March 6, 1948.)

306. Morris Marden: *On the zeros of rational functions with prescribed poles.*

Biernacki and Dieudonné have proved that, if A is an arbitrary complex number and the m_j are positive integers and if all the points z_j lie in a circle C of radius r, then $F(z)=A+\sum_{j=1}^{n} m_j/(z-z_j)$ has at least $n-1$ zeros in the concentric circle of radius $2^{1/2}r$. This theorem was recently shown by Nagy to be valid also when the m_j are arbitrary positive numbers. In the present paper the theorem is generalized to functions $F(z)=\sum_{j=0}^{p} A_j z^j+\sum_{j=1}^{n} m_j/(z-z_j)$ in which the A_j are arbitrary complex numbers and the m_j are also complex. The main result is the following. Let $F(z)$ be a rational function which has at $z=\infty$ a pole of order p and has at the n finite points z_j $(j=1, 2, \cdots, n)$ simple poles with residues m_j such that $a\leqq\arg m_j\leqq a+b<a+\pi$. Let K be the smallest convex region containing all the z_j. Then $F(z)$ has at least $n-1$ zeros in the star-shaped region comprised of all points from which K subtends an angle of at least $(\pi-b)/(p+2)$. The proof is based upon the lemma (apparently new) that, if $Z_1, Z_2, \cdots, Z_{p+2}$ are any zeros of $F(z)$, then $\sum_{j=1}^{n}[m_j/(Z_1-z_j)(Z_2-z_j)\cdots(Z_{p+2}-z_j)]=0$. In the special case that the m_j are all positive integers, the theorem reduces to a result on the zeros of the function $P(z)f(z)+f'(z)$ where $f(z)$ is a given polynomial of degree n and $P(z)$ is an arbitrary polynomial of degree not exceeding p. (Received March 13, 1948.)

307*t*. H. L. Meyer: *Multiple integral problems in the calculus of variations*. I.

This paper considers the following problem $\mathcal{P}_{m,n}$ $(m>1, n>1)$ in the calculus of variations: to find conditions on a surface z_0 *sufficient* for z_0 to provide $I(z)=\int_G f(x, z, p)dx$ with relative minimum on a class of surfaces z having common boundary values with z_0. Here $x\in E_m$, $z\in E_n$ is a function of x, and $p=(\partial z^i/\partial x^j)$. General sufficiency theorems for strong and semi-strong minima are established for different classes of surfaces. These theorems use strengthened versions of the Weierstrass condition $E(p, p+\pi)>0$ and the Legendre condition $Q(\pi)>0$ which require these inequalities to hold for π's of arbitrary rank; the strengthened condition involving the second variation is taken in the form $I_2(\eta)>0$ or $I_2(\eta)\geqq k\int_G|\eta|^2dx$ for some $k>0$ and suitable variations η. The proofs are indirect, after a fashion introduced by McShane (Trans. Amer. Math. Soc. vol. 52 (1942) pp. 344–379) and developed by Hestenes (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 93–118) for simple integral problems. Unpublished results of Hestenes and Karush are exploited. (Received March 11, 1948.)

308. E. J. Mickle and Tibor Rado: *On relationships between cyclic additivity theorems*.

Rado has given a generalization of a previous result of his on cyclic additivity (*A general cyclic additivity theorem*, abstract 54-7-313). Call the new result cyclic additivity in the strong sense and the previous result cyclic additivity in the weak sense. In this paper it is shown that in certain cases cyclic additivity in the weak sense implies cyclic additivity in the strong sense. In particular, this is so if the "initial space" is a 2-cell or a 2-sphere, or more generally any unicoherent Peano space. As a consequence, the result applies to the Lebesgue area of surfaces both in the 2-cell case and the 2-sphere case. (Received March 4, 1948.)

309. Josephine M. Mitchell: *Orthonormal systems defined on nonschlicht domains*.

The aim of this paper is to prove the existence of closed orthonormal systems on nonschlicht domains in the case of one and two complex variables. In the case of one complex variable the domain, which lies on a Riemann surface, may be locally uniformized at every point, and the closed orthonormal system constructed by a generalization of the method used by Bergman for schlicht domains (*Partial differential equations, Advanced Topics*, Brown University, Summer Session, 1941, Chapter VI). As an example, a closed orthonormal system is constructed for the doubly-covered domain defined by $w^2+z^2=a^2$, $|w|<1$, $0<a<1$. Similar results may be obtained for four-dimensional, nonschlicht domains embedded in the space of two complex variables, if it is assumed that the domains may be locally uniformized at every point (cf. Bergman, Mémorial des Science Mathématique, vol. 106). Closed orthonormal sets are constructed for several examples of such four-dimensional, nonschlicht domains. (Received March 10, 1948.)

310. V. C. Poor: *On residues of polygenic functions.*

This is a sequel to a paper published in vol. 32 of the Trans. Amer. Math. Soc., in which paper residues at a point of such functions were considered. Here our purpose is to extend these results to areas, generalizing the notion of classical residues of holomorphic functions. The general theory is developed and applied to several special cases. The Cauchy contour integral loses its meaning for polygenic functions. Two difficulties arise, the discontinuity and a continuous distribution over the area. In the "limit definition" used in the previous paper these difficulties were overcome by contracting to a point the contour integral considered as a circle. In the present paper these difficulties are obviated by subtracting an integral over the area, which integral contains the areal (or mean) derivative in its integrand. These new definitions conform better to the classical definition for residues. (Received January 14, 1948.)

311*t*. V. C. Poor: *On the two-dimensional derivative of a complex function.*

The two-dimensional derivative of a polygenic function treated by Cioranescu is monogenic, in that it is independent of direction. Two papers by Haskell and by Reade in recent numbers of the Bulletin consider this same subject. These papers are analyzed and synthesized in this paper using a new and different point of view. In extending these results, this class of monogenic functions is more explicitly characterized. Further, a second class of monogenic functions is defined and a dual relation between these two classes of functions is established. (Received January 14, 1948.)

312. G. B. Price: *Researches in the theory of functions of several real variables.* III. *Elements of a theory of functions.*

This paper contains applications of the theory of differentiation which was developed in Part II to the study of functions (mappings from an n-dimensional space onto an m-dimensional space). Among other topics it treats (a) extreme points and critical points, (b) the relation of the topological index to the derivative, (c) the law of finite increments, (d) a general theory for the evaluation of Riemann integrals, (e) the law of the mean and Rolle's theorem, and (f) a new Riemann-Stieltjes integral. The theory developed for the evaluation of multiple Riemann integrals includes, as special cases, Green's theorem and their evaluation by iterated integrals. Rolle's theorem, trivial in the one-dimensional case, has several forms in higher dimensions and is not simple. The treatment throughout the paper is based on topological methods and results. (Received March 11, 1948.)

313. Tibor Rado: *A general cyclic additivity theorem.*

The theorem proved in this paper is a generalization of a previous theorem of the writer (see Trans. Amer. Math. Soc. vol. 58, p. 452, §4.12). The type of generalization may be illustrated by stating an application. Notations: Q is the unit square $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$. M is an arbitrary Peano space. E_3 is Euclidean three-space. C is a generic notation for a proper cyclic element of M, and r_C is the monotone retraction from M onto C. f is a continuous mapping from Q into M, and g is a continuous mapping from M into E_3. S and S_C are the surfaces defined by the representations gf and gr_Cf respectively (products of mappings are read from right to left). A denotes Lebesgue area. Then we have the formula $A(S) = \sum A(S_C)$, $C \subset M$, if M is not a dendrite, and $A(S) = 0$ if M is a dendrite. The statement remains valid if Q is a 2-sphere. Previously known results correspond to the special case when g is light and f is monotone and onto. (Received February 16, 1948.)

314. E. H. Rothe: *Weak topology and nonlinear integral equations.*

Existence theorems for pairs of nonlinear integral equations of the form $y^*(s) + \int_D K(t, s) f(t, y)(t))dt = 0$, $y(s) + \int_D K(s, t) f(t, y^*(t))dt = 0$ and for systems of such pairs are derived by considering their left members as differentials of certain "scalars" in a suitable Hilbert space and applying to these scalars results about the existence of extrema which had been obtained in a previous paper (*Gradient mappings and extrema in Banach spaces*, Theorems 4.1 and 4.2, to appear soon in the Duke Math. J.) by the use of a "weak" topology. Specialized to the case of a symmetric and positive definite kernel $K(s, t)$ the results of the present paper contain as special cases theorems by Hammerstein (Acta Math. vol. 54 (1930), Theorems 1 and 2) and a theorem by M. Golomb (Publications Mathématiques de l'Université de Belgrade, vol. 5, 1936, Theorem 1). (Received February 16, 1948.)

315. H. M. Schaerf: *General problem of an invariant measure.* Preliminary report.

The problem is: Given an equivalence relation $\sim$ defined in a σ-field S of subsets of an arbitrary set, and given a family N of sets of S, to construct a nontrivial measure, defined on S, invariant under $\sim$ and vanishing exactly on N. A necessary and sufficient solubility condition is established if (i) $A \sim B$ implies that for any disjoint countable partition of A into sets $A_i \in S$, there is an analogous partition of B into sets $B_i \sim A_i$; (ii) for any sets A, B in $S - N$ ("positive" sets), there is a set $A' \sim A$ meeting B in a positive set. Then the solution is unique (to within a multiplicative constant) and any two sets of equal measure are almost congruent by countable partition. Any equivalence either under a group satisfying Weil's condition M or under σ-isomorphisms considered by D. Maharam fulfills (i) and (ii). These results are obtained by constructing an isomorphism of an algebraic system with countable summation onto a set of non-negative numbers. This partially solves a problem, the solubility of which had been doubted by Tarski. (Received March 31, 1948.)

316t. H. M. Schaerf: *On the functional equation* $f(\sum x_i) = \sum f(x_i)$. Preliminary report.

Assume that to every sequence $\{t_i\}$ of elements of an arbitrary set T there is ascribed an element $\sum t_i$ in T so that the following conditions are satisfied: (A) $\sum$ is associative and commutative; (B) T contains a zero-element θ; (C) $x \geqq y$ and $y \geqq x$ imply $x = y$ ($x \geqq y$ means $x = y + t$ for some t in T); (D) $\sum_1^n x_i \leqq y$ for $n = 1, 2, \cdots$ implies $\sum_1^\infty x_i \leqq y$; (E) for any two elements x, y different from θ, there exists an element $s \neq \theta$ such that $s \leqq x$ and $s \leqq y$. A necessary and sufficient condition for the existence of a nontrivial, non-negative solution of the functional equation $f(\sum t_i) = \sum f(t_i)$ is established. This solution is unique to within a multiplicative constant and its values are either multiples of a positive number or fill an interval. These results have applications in measure theory (see the preceding abstract). (Received March 31, 1948.)

317. W. R. Scott: *On the essential multiplicity function.*

Let $T: z = t(w)$ be a continuous plane transformation from the unit square in the w-plane into the z-plane. The essential multiplicity $\kappa(z, T)$ has been defined by Rado and Reichelderfer (Trans. Amer. Math. Soc. vol. 49 (1941) pp. 258–307). The corresponding definition of $\kappa(y, f)$ for continuous real-valued functions $y = f(x)$, $0 \leqq x \leqq 1$, is similar. The following inverse problem is considered. What are the necessary and sufficient conditions that a function $g(y)$ be equal to the essential multiplicity function $\kappa(y, f)$ for some continuous function $y = f(x)$, $0 \leqq x \leqq 1$? This question is answered. Next, a generalization of $\kappa(y, f)$ is given for general real-valued functions $y = f(x)$, $0 \leqq x \leqq 1$. The corresponding inverse problem is again solved. In the two-dimensional case, however, only a partial answer is obtained to the inverse problem. (Received February 26, 1948.)

APPLIED MATHEMATICS

318. Nathaniel Coburn: *Two-dimensional non-steady irrotational flows of a compressible gas.*

In a previous paper (to be published in the Proceedings of the First Symposium of Applied Mathematics), Professor C. L. Dolph and the author developed some tensor methods for obtaining characteristic systems for the steady irrotational supersonic flow of a gas in three dimensions. The present paper indicates the manner in which the previous theory must be modified in order to treat the two-dimensional, non-steady, irrotational flow of a compressible gas. First, the potential equation of the flow is obtained. The characteristic surfaces associated with this potential equation lie in space-time. If one family of such surfaces consists of cylinders with generators parallel to the time axis, then the flow is called "proper supersonic." Such flows are characterized by the fact that the Mach number is independent of time. Further, it is shown that: (1) the flow is supersonic; (2) the characteristic system is an immediate generalization of that system for the steady case; (3) simple waves do not exist. Finally, by writing the potential equation in invariant form with respect to space-time coordinate transformations, one can obtain a characteristic system for general flows. (Received March 11, 1948.)

319. H. D. Huskey: *An example concerning rounding errors.*

The equation $y'' = y$ was integrated on the ENIAC (Electronic Numerical Integrator and Computer, now at Aberdeen Proving Ground) over an interval of approximately one radian. Increments ranged from 0.001 to 0.000005 radians. An instance was found where the digit dropped in the process of rounding-off changed slowly from zero to four and back to zero again over about 250 steps of the integration. A second order formula was used in the integration process. (Received March 15, 1948.)

320. A. W. Jacobson: *A generalized convolution for the finite Fourier transformation.*

Consider a function $F(x, y)$ integrable over the square $0 \leqq x \leqq \pi$, $0 \leqq y \leqq \pi$. A generalized Fourier convolution $F^*(x)$ of $F(x, y)$ corresponding to the iterated finite sine transformation $S\{S\{F(x, y)\}\} = \int_0^\pi \int_0^\pi F(x, y) \sin nx \sin n'y \, dx \, dy = \bar{f}_s(n, n')$ is defined to be $-\int_{-\pi}^{\pi} F_1(x-y, y)dy$, where F_1 is an odd periodic extension of F with respect to x and an odd extension with respect to y. When $n' = n$, it is shown that $S\{S\{F(x, y)\}\} = C\{F^*(x)/2\}$. In case $F(x, y) = F_1(x)G_1(y)$, the function $F^*(x)$ is the ordinary convolution of the two functions F_1 and G_1. Similar results are obtained for the iterated cosine, as well as for the sine-cosine, transformation. With the aid of this generalized convolution very general steady state boundary value problems are resolved into problems with simpler boundary conditions and source functions. Moreover, this method enables us to extend the Duhamel integral formula from time to space coordinates. A number of key functions are introduced and the solutions of some basic boundary value problems are expressed in terms of these key functions and in turn the solutions of more general problems in two and three dimensions are then expressed in terms of the solutions of the basic problems. (Received March 1, 1948.)

321. M. Z. Krzywoblocki: *On the so-called lost solutions in adiabatic potential flow equations.* Preliminary report.

The linearization of the quasi-linear equation of an adiabatic potential flow by means of the Molenbrock-Chaplygin transformation permitted to find the well known "boundary lines" of adiabatic flow beyond which the flow has no physical meaning. At the boundary line the adiabatic flow goes from one sheet of the flow plane to another sheet according to the way the Riemannian surface repeatedly overlaps. But there exists another way of the linearization of the mentioned quasi-linear equation, namely, by the application of a generalized Minkowski's function. In the present paper the author tries to find the lines analogous to boundary lines in Molenbrock-Chaplygin transformation. (Received March 4, 1948.)

LOGIC AND FOUNDATIONS

322. A. H. Copeland: *Implicative Boolean algebra.*

In conventional logic implication is a relation which holds between certain propositions and not between others, whereas in implicative Boolean algebra implication is an operation by which any pair of propositions can be combined to form a new proposition which may be regarded as either true or false. Implication is defined in terms of an associative but noncommutative operation called the cross product. This product admits left-hand cancellation and is distributive with respect to both conjunction and disjunction. It is interpreted in terms of certain transformations in the Boolean algebra and implication is interpreted in terms of remainder classes with respect to ideals. Not every Boolean algebra can be implicative but two examples are given, one in which the elements are increasing sequences of natural numbers and the other in which the elements are Lebesgue measurable sets in the unit interval. (Received March 8, 1948.)

STATISTICS AND PROBABILITY

323. Benjamin Epstein: *A modified extreme value problem.*

There may arise situations where one is interested in knowing the distribution of smallest, largest, or more generally the nth smallest or nth largest value in samples taken from a population whose distribution is described by some probability law, and where the sample size is itself a random variable following some other specified probability law. In this paper the problem is solved for the case where the sample size is a random variable following the law of Poisson. The relationship between these results and the known results for the case of fixed sample size is considered and a practical example is given. (Received March 1, 1948.)

324*t*. M. S. Macphail and P. S. Dwyer: *Symbolic matrix derivatives.*

Let X be the matrix $[x_{mn}]$, t a scalar, and let $\partial X/\partial t$, $\partial t/\partial X$ denote the matrices $[\partial x_{mn}/\partial t]$, $[\partial t/\partial x_{mn}]$ respectively. Let $Y=[y_{pq}]$ be any matrix product involving X, X' and independent matrices, for example, $Y=AXBX'C$. Consider the matrix derivatives $\partial Y/\partial x_{mn}$, $\partial y_{pq}/\partial X$. Our purpose is to devise a systematic method for calculating these derivatives. Thus if $Y=AX$, we find that $\partial Y/\partial x_{mn}=AJ_{mn}$, $\partial y_{pq}/\partial X=A'K_{pq}$, where J_{mn} is a matrix of the same dimensions as X, with all elements zero except for a unit in the mth row and nth column, and K_{pq} is similarly defined, with the same dimensions as Y. We consider also the derivatives of sums, differences, powers, the inverse matrix and the function of a function, thus setting up a matrix analogue of elementary differential calculus. This is designed for application to statistics, and gives a concise and suggestive method for treating such topics as multiple regression and canonical correlation. (Received March 15, 1948.)

TOPOLOGY

325. Edwin Hewitt: *A class of topological spaces.*

A topological space X is said to be a K-space if it has a one-to-one continuous image which is a bicompact Hausdorff space. Let $\mathcal{Z}(X)$ denote the family of all sets $Z(f)$ in X, where f is a continuous real-valued function on X and $Z(f)$ is the set of points in X where f vanishes. It is proved that under certain restrictions, a completely regular space X is a K-space if and only if there is a subfamily $\mathcal{A}$ of $\mathcal{Z}(X)$ such that (1) $\prod_{F\in\mathcal{A}} F=0$, (2) any subfamily $\mathcal{B}$ of $\mathcal{A}$ with the finite intersection property has total intersection nonvoid. (Received March 8, 1948.)

326*t*. Edwin Hewitt: *Certain special function rings.*

Certain common rings of functions are examined from the point of view of their being isomorphic to rings of continuous real-valued functions on completely regular spaces. It is shown that the following rings cannot be rings of all continuous real-valued functions on any topological space: (1) the ring of all measurable real functions of a real variable; (2) the ring of all measurable real functions of a real variable modulo the ideal of all functions vanishing almost everywhere; (3) the rings generated by upper (lower) semicontinuous real functions of a real variable. It is shown that the following rings are isomorphic to rings of all continuous real-valued functions on appropriate spaces: (1) the ring of all real functions of a real variable, f, such that as $x\to x_0+$, $\lim f(x)=f(x_0)$ for all real numbers x_0; (2) the ring of all continuous real functions of a real variable which are bounded on the right (left) half-line. (Received March 2, 1948.)

327*t*. E. E. Moise: *A theorem on monotone interior transformations.*

B. Knaster has raised the question whether there is a compact metric continuum M, irreducible between two of its points, and a monotone interior transformation T throwing M into an arc, such that for each x of $T(M)$, $T^{-1}(M)$ is an arc. In the present note, this question is answered in the negative. (Received April 16, 1948.)

328*t*. H. M. Schaerf: *Regular measures.*

A measure defined on a σ-field of subsets of a topological space is called ***regular*** if its value on any measurable set E is the greatest lower bound of its values on open measurable sets containing E. A general ***regularity criterion*** is proved which implies that the following measures are regular: (a) every Radon measure in a locally compact topological space, (b) every measure defined on the σ-field of the Borel sets of a metric space which is the union of a sequence of open sets of finite measure. Theorem (a) generalizes Halmos' recent result that every Haar measure is regular (reported at the Conference on Topological Groups in Chicago, January 22–24, 1948). (Received March 5, 1948.)

329*t*. H. M. Schaerf: *Generalisation of Lusin's theorem to regular measures.*

The following generalization of Lusin's theorem is proved: Let m be a measure defined and regular on a σ-field of subsets of a neighborhood space R which is the union of a sequence of measurable sets of finite measure. Let $f(x)$ be a mapping of a measurable set $E\subset R$ into a topological space T satisfying the second axiom of countability. If the image of any open set of T under f^{-1} is measurable, then for every positive number ϵ there is a measurable closed subset F_ϵ of E such that $m(E-F_\epsilon)<\epsilon$ and such that $f(x)$ is continuous on F. The scope of this generalization is apparent from the above abstract ***Regular measures***. (Received March 5, 1948.)

330*t*. G. S. Young. *On continuous curves irreducible about compact sets.*

In partial answer to a question raised by Zippin (Fund. Math. vol. 20 (1933) p. 197), it is proved that in a connected and locally arcwise connected space every compact finite-dimensional set whose components are all locally connected, and whose nondegenerate components form a null sequence, lies in a compact continuous curve irreducible about it. (Received March 12, 1948.)

331. G. S. Young. *On product manifolds and fiberings.*

It is proved that if an n-dimensional topological manifold, not necessarily compact and possibly with boundary, is the product of two spaces A and B, and if dim $A=1, 2$, then A is itself a manifold. If a connected and locally connected space is a fiber bundle with fiber F, then all components of F are homeomorphic, and the fibering can be factored into a monotone fibering followed by a light fibering. Among other results, it is proved that there is no compact fibering of E^3 and E^4. Many of the methods used are from recent work of Borsuk. (Received March 12, 1948.)

R. H. BRUCK,
Associate Secretary

THE APRIL MEETING IN BERKELEY

The four hundred thirty-sixth meeting of the American Mathematical Society was held at the University of California, Berkeley, California, on Saturday, April 17, 1948. The attendance was approximately 125, including the following 86 members of the Society:

M. T. Aissen, H. L. Alder, T. M. Apostol, H. M. Bacon, E. W. Barankin, H. W. Becker, E. M. Beesley, B. A. Bernstein, M. T. Bird, Gertrude Blanch, H. F. Bohnenblust, J. L. Botsford, J. L. Brenner, H. D. Brunk, Albert Cahn, L. H. Chin, F. G. Creese, P. H. Daus, Harold Davenport, A. C. Davis, E. A. Davis, D. B. Dekker, C. R. DePrima, A. H. Diamond, H. R. C. Dieckmann, L. K. Durst, Arthur Erdélyi, E. A. Fay, A. L. Foster, Marianne Freundlich, J. N. Goodier, Arthur Grad, J. W. Green, Leonard Greenstone, John Gurland, S. M. Hallam, Frank Harary, W. L. Hart, J. G. Herriot, J. L. Hodges, Alfred Horn, Rufus Isaacs, R. C. James, T. A. Jeeves, B. W. Jones, Samuel Karlin, J. L. Kelley, E. L. Lehmann, D. H. Lehmer, R. B. Leipnik, Robert Mann, A. V. Martin, Burnet Meyer, E. D. Miller, C. B. Morrey, A. P. Morse, J. D. Newburgh, Ivan Nivan, A. B. Novikoff, C. D. Olds, Edmund Pinney, R. I. Piper, George Pólya, R. R. Putz, W. C. Randels, J. B. Robinson, R. M. Robinson, R. A. Rosenbaum, Rafael Sánchez-Díaz, S. A. Schaaf, M. M. Schiffer, Abraham Seidenberg, Edward Silverman, Ernst Snapper, D. C. Spencer, Pauline Sperry, R. G. Stoneham, Otto Szász, Gabor Szegö, Alfred Tarski, Olga Taussky-Todd, John Todd, R. K. Wakerling, Morgan Ward, J. G. Wendel, P. A. White, A. R. Williams.

In the morning there was a short session for research papers, followed by the hour address, *On van der Pol's equation*, by Professor H. F. Bohnenblust of the California Institute of Technology. Professor Gabor Szegö presided. In the afternoon, additional papers were presented in two sections, at which Professors B. A. Bernstein and C. B. Morrey, respectively, presided.

Following the meetings, those attending were guests of the department of mathematics of the University of California at a tea at Stevens Union.

Abstracts of all papers presented at the meeting are given below. Papers read by title are indicated by the letter "*t*". Paper number 340 was presented by Mr. Shapley, number 354 by Professor Szegö, and number 361 by Dr. Horn. Mr. Blaney was introduced by Professor Harold Davenport, and Mr. Shapley and Mr. Snow by Professor J. C. C. McKinsey,

ALGEBRA AND THEORY OF NUMBERS

332*t*. Hugh Blaney: *Indefinite quadratic forms in n variables.*

It has been shown recently by Davenport (C. R. Acad. Sci. Paris vol. 224 (1947) 990–991) that Minkowski's theorem on nonhomogeneous binary quadratic forms can be extended to ternary forms. The author has obtained the following results for indefinite forms in n variables. Let $Q(x)=Q(x_1, \cdots, x_n)$ be such a form, with real coefficients and determinant $D\neq 0$. (1) For any $c\geqq 0$ there exists $C=C(c, n)$ such that the inequality $c|D|^{1/n}<Q(x)<C|D|^{1/n}$ has a solution in integers $x_1, \cdots, x_n$ whose highest common factor is 1. (2) For any $\gamma\geqq 0$ there exists $\Gamma=\Gamma(\gamma, n)$ such that for any real $x_1', \cdots, x_n'$ the inequality $\gamma|D|^{1/n}<Q(x+x')<\Gamma|D|^{1/n}$ has an integral solution. In particular, a result of Minkowski's type is valid for indefinite quadratic forms in n variables. (Received February 16, 1948.)

333. J. L. Brenner: *Right multiplication by a unitary matrix.*

A canonical form is obtained for a matrix A under right multiplication by a unitary matrix U. The relation $AU=B$ is an equivalence relation. (Received April 16, 1948.)

334. Harold Davenport: *Indefinite binary quadratic forms and Euclid's algorithm in real quadratic fields.*

Let $f(x, y)=ax^2+bxy+cy^2$ have real coefficients and discriminant $d=b^2-4ac\neq 0$, and suppose $f(x, y)\neq 0$ for integral x, y other than 0, 0. A classical theorem of Minkowski asserts that for any real p, q there are an infinity of integers x, y for which $|f(x+p, y+q)|<(1/4)d^{1/2}$. I have now proved the following result in the opposite direction: there exist real p, q such that $|f(x+p, y+q)|>d^{1/2}/128$ for all integral x, y. This may be regarded as a generalization of a theorem of Khintchine on Diophantine approximation. The construction of p, q is such as to ensure that if a, b, c are integers, then p, q are *rational*. It follows that Euclid's algorithm cannot hold in any field $k(m^{1/2})$ for which $m>(128)^2$. (Received February 16, 1948.)

335. B. W. Jones: *Composition of binary forms.*

An ideal class in a quadratic field is called ambiguous if it is equal to its conjugate class and a binary quadratic form is ambiguous if it is improperly equivalent to itself. Along the lines of Dedekind's theory, it is shown that there is a 1-1 correspondence between ambiguous ideal classes over a given field and ambiguous classes of forms of corresponding determinant, while the correspondence between non-ambiguous ideal classes and form classes is 2-2. This result is used to prove Gauss's theorem on the duplication of binary classes and other classical results. (Received March 8, 1948.)

336. B. W. Jones: *Representations by quadratic forms.*

If A is a symmetric n by n matrix with integral elements and $T=X$ and $T=Y$ are two solutions of $T^TAT=c$ for some integer c, then X and Y are called essentially equal if there is an automorph P of A with integer elements and determinant 1 such that $X=PY$. $G(A, C)$ is the number of essentially distinct solutions of $T^TA_kT=c$ summed over all classes A_k in the genus of A. A formula is obtained for $G(A, C)$ which for the case c prime to $2|A|$ reduces to $\rho(c)\sum s(E)$ summed over all classes of forms E in $n-1$ variables and of a certain genus or two genera; $\rho(c)$ depends solely on the number of odd prime factors of c and $s(E)$ is the reciprocal of the number of automorphs of E whose first rows are distinct (mod c). When $n=3$ this sum may be expressed in terms of the binary class number. (Received March 8, 1948.)

337. D. H. Lehmer: *On a conjecture of Krishnaswami.*

Let $T(N)$ denote the number of right triangles whose perimeters do not exceed

$2N$ and whose sides are relatively prime integers. On the basis of tabular evidence A. A. Krishnaswami has conjectured that $T(N) \sim N/7$. In this paper it is shown that this is not quite the case. In fact it is proved that $T(N) = \pi^{-2}N \log 4 + O(N^{1/2} \log N)$. The constant $\pi^2/\log 4 = 7.1194 \cdots$. Hence Krishnaswami's conjecture is not far wrong. The fact that $T(N)/N$ tends to this constant follows from the theory of "totient points" as developed by D. N. Lehmer in 1900. The above error term is obtained from elementary considerations of Dirichlet series. (Received March 13, 1948.)

338t. D. H. Lehmer: *On the vanishing of Ramanujan's function.* II.

In a previous note (Bull. Amer. Math. Soc. vol. 53 (1947) p. 478) it was shown that Ramanujan's function $\tau(n)$, defined as the coefficient of x^{n-1} in the 24th power of the lacunary power series $(1-x)(1-x^2)(1-x^3) \cdots$, does not vanish for $n < 3316799$. This result has now been extended to $n < 214928639999$. The function $\tau(n)$ vanishes at most 7 times for $n < 10^{15}$. The present results are a consequence of new congruence relations for $\tau(n)$ with respect to the moduli 2^{14}, 3^7, and 5^3, given by Lahiri. (Received March 13, 1948.)

339. Ivan Niven: *Fermat's theorem for matrices*

Let A be any nonsingular matrix of order n with elements in $GF(p^m)$. A minimum value q is specifically determined as a function of p, m, and n such that $A^q = I$ for every A. (Received March 6, 1948.)

340. L. S. Shapley and R. N. Snow: *The solutions of the general two-person zero-sum game with a finite number of strategies.*

It is shown that all solutions of the general two-person zero-sum game can be represented by means of a finite number of "basic solutions," which may be visualized as pairs of vertices from two polyhedra in the spaces of all mixed strategies for the two players. Each solution has associated with it a certain submatrix of the whole game-matrix, called the "kernel" of the solution. The kernels of the basic solutions are nonsingular. Once a basic kernel has been located in the game-matrix, the associated basic solution may be computed directly by a formula. (Received April 4, 1948.)

341. Ernst Snapper: *Completely indecomposable modules.*

Let A be a commutative ring with unit element 1. Let $\mathfrak{B}$ be an abelian group with A as operator domain, where 1 is unit operator and where $\mathfrak{B}$ has a composition series of finite length. $\mathfrak{B}$ is called a *completely indecomposable module* if $\mathfrak{B}$ and all its submodules are indecomposable; that is, if $\mathfrak{B}$ has a unique minimal submodule. It is proved that two completely indecomposable modules are A-isomorphic if and only if they have the same annihilating ideal. In other words, two faithful representations of A whose representation spaces have unique minimal submodules are equivalent. It follows that a completely indecomposable module is cyclic if and only if its annihilating ideal is intersection-irreducible. It is shown how the classical elementary divisor theory can be derived from the theory of completely indecomposable modules. (Received January 29, 1948.)

342. Morgan Ward: *Divisibility sequences associated with the real multiplications of the Jacobian elliptic functions.* Preliminary report.

This paper extends the results of the author on divisibility sequences associated with the real multiplication of the Weirstrass elliptic functions (Amer. J. Math. vol. 70 (1948) pp. 31–74) to the corresponding sequences associated with the Jacobian elliptic functions, obtaining elliptic function analogues of Lucas' function V_n as well as the already treated analogue of U_n. (Received March 13, 1948.)

ANALYSIS

343. H. D. Brunk: *A consistency theorem.*

The concept of asymptotic representation in a strip region of a function by Dirichlet series with a certain logarithmic precision has been introduced by Mandelbrojt (see S. Mandelbrojt, *Sur une inegalite fondamentale*, Ann. École Norm. (3) vol. 63, pp. 351–378). He has proved that if a Dirichlet series represents a function $F(s)$ in a strip region with sufficient logarithmic precision the series converges. The present paper completes the theorem by showing that under the same conditions the series necessarily converges to the function $F(s)$. (Received March 2, 1948.)

344*t*. S. P. Diliberto: *On special properties of measure preserving transformations.*

The author proves: The only point transformations of class C^3 of Euclidean 3-space into itself which preserve area are rigid motions or reflections. A partial generalization is established: Let $T(t)$ be a transformation of Euclidean n-space into itself defined by $dx_i/dt = X_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$; $X_i \in C^3$ (that is, each point x "moves" for fixed time t along the trajectory through it; thus $T(0)$ is the identity map). A necessary and sufficient condition that $T(t)$ preserve p measure, p fixed and not greater than $n-1$, for all t is that $\partial X_i/\partial x_j = -\partial X_j/\partial x_i$ for $i, j = 1, 2, \cdots, n$. This implies that $dx/dt = Ax + a$ where A is skew-symmetric. This can be integrated and shows the transformation is rigid. The first theorem is proved by deriving from the images of the planes $x_i + x_j = 0$ $(i, j = 1, 2, 3)$ conditions which imply the map sends triply orthogonal surfaces into triply orthogonal surfaces. By Liouville's theorem the only possibilities are rigid motions, reflections, and inversions—of which the last is ruled out. The second theorem is established by (a) applying the invariance on the planes $x_i + x_j = 0$, $i, j = 1, 2, \cdots, n$, and their images, (b) requiring conditions (a) to hold for all t. (Received March 12, 1948.)

345. S. P. Diliberto: *On systems of ordinary differential equations.*

For $dx/dt = Ax$ (x a column vector, A an n square matrix with $a_{ij}(t)$ continuous for all t) the author proves there exists an orthogonal matrix B (defined for all t), with $b_{ij}(t) \in C^1$, such that if $By = x$ then $dy/dt = Cy$ with $c_{ij} = 0$ if $i > j$ (c_{ij} bounded if a_{ij} bounded). The proof generalizes the author's result that the variational equations of the system $dx_i/dt = X_i(x_1, x_2)$, $i = 1, 2$, are integrable by quadratures. This quadrature leads to the following (the first results on the 2nd half of Hilbert's 16th problem): If div $X \neq 0$ $(X = (X_1, X_2))$ on any periodic solution (closed trajectory) of $dx_i/dt = X_i(x_1, x_2)$, $i = 1, 2$, where X_i are polynomials of degree not greater than n then the number of periodic solutions is not greater than $(1/2)(n-2)(n-3)+1$; if the periodic solutions are nested the upper bound is sharpened to $[(n-1)/2]$—which is best possible. From the above quadrature it follows that if K, the curvature of the orthogonal trajectories, does not vanish on a periodic solution then that solution is stable if $K < 0$ and unstable if $K > 0$. The author's results are used to give a simplified treatment of

Liapounoff's "characteristic exponents." One difficulty in generalizing the Poincaré-Bendixson theorem is established by proving that there exist flows into a torus which have periodic solutions (inside the torus) which are arbitrarily knotted and linked. (Received March 12, 1948.)

346t. G. E. Forsythe: *An error of Hayashi and Izumi on summability.*

Let $p_0 \neq 0, p_1, p_2, \cdots$ be a given sequence of non-negative numbers, and set $P_n = p_0 + p_1 + \cdots + p_n$. For a sequence $\{s_n\}$ let $\sigma_n = (p_n s_0 + p_{n-1} s_1 + \cdots + p_0 s_n)/P_n$ and let $\rho_n = (p_0 s_0 + p_1 s_1 + \cdots + p_n s_n)/P_n$. If $\sigma_n \to \sigma$ $[\rho_n \to \sigma]$ as $n \to \infty$, $\{s_n\}$ is said to be summable-N_p [summable-R_p] to σ. Hayashi and Izumi (Tôhoku Math. J. vol. 47 (1940) pp. 69–73, Theorem 3, p. 73) stated an incorrect sufficient condition for equivalence of (the Nörlund mean) N_p and (the Riesz typical mean) R_p. For the special weights $p_n^{(r)} = n^{r-1}$ it would follow from Hayashi and Izumi that, for each $r = 1, 2, \cdots$, $R_p^{(r)}$ is equivalent to the Cesàro mean C_r. But Hardy (Quarterly Journal of Pure and Applied Mathematics vol. 38 (1907) pp. 269–288) proved that for each $r \geqq 1$ the Riesz mean $R_p^{(r)}$ is equivalent to the Cesàro mean C_1. Hayashi and Izumi erred in applying the Bloch-Pólya inequality on page 72. (Received March 3, 1948.)

347. Arthur Grad: *The distortion of schlicht functions.*

Let $w = z + a_2 z^2 + a_3 z^3 + \cdots$ be regular and schlicht in the unit circle. The problem of distortion consists in determining all possible values of $w'(z_1)$, where z_1 is any fixed point of $|z| < 1$. In 1909 Koebe proved the existence of bounds for $w'(z_1)$, these bounds depending on $|z_1|$ only. In 1916 Faber, Bieberbach and others found the precise bounds for $|w'(z_1)|$, and in 1936 Golusin found the precise bounds for $|\arg w'(z_1)|$. The distortion problem is now solved completely; the exact region of variability of $w'(z_1)$, z_1 fixed, as $w(z)$ ranges over the whole family of schlicht functions, is determined. The region is more conveniently represented in the logarithmic plane where, for $|z_1| \leqq 1/2^{1/2}$, the boundary of the domain corresponds to functions each of which maps the unit circle on the exterior of an analytic slit extending to infinity. For $1/2^{1/2} < |z_1| < 1$, the boundary consists of four alternating analytic arcs. Two opposite arcs correspond to functions mapping the unit circle on the exterior of slits as above. The other two arcs are straight lines, and correspond to functions for which $\arg w'(z_1)$ attains its maximum and minimum respectively. These functions map the unit circle on the exterior of slits as above, having one finite fork. (Received March 27, 1948.)

348t. Rufus Isaacs: *Linear monodiffric difference equations.*

In the classical sense, the general solutions of many difference equations are obtained by replacing their arbitrary constants in the general solutions of their differential analogues by arbitrary span-periodic functions. If a linear monodiffric difference equation with polynomial coefficients be constructed by utilizing monodiffric multiplication, a set of solutions is obtained similarly by now replacing the arbitrary constants by doubly period functions with a "span square" as period. But aside from these, many such difference equations possess additional solutions which are defined only on the span lattice or on a sub-region of the plane. They are of a more complicated functional nature than the usual solutions and appear to have no classical analogues. (Received February 27, 1948.)

349. Rufus Isaacs: *Monodiffric multiplication.*

The theory of monodiffric functions (here of the first kind only; that is, those polygenic functions which satisfy forward difference equations of the Cauchy-Riemann type) has been handicapped in that the class of them is not closed under multiplication. The following operation appears to be the logical replacement. If $f(z)$ is monodiffric, $z \cdot f(z)$ means $xf(z-1)+iyf(z-i)$ and is monodiffric. "Multiplication" of f by any pseudo-power of z now follows by induction. By requiring distributivity, "multiplication" of f by any monodiffric polynomial has a meaning. The difference quotients of such a "product" is in accordance with the Leibniz rule. For zero span, the "product" becomes the ordinary product. When both factors are pseudo-powers the additive law of exponents is obeyed. Thus the totality of monodiffric polynomials forms a ring which is isomorphic to that of monogenic polynomials. Difference quotients and derivatives correspond under the same isomorphism. Light is thrown on the peculiar role of wedge functions by the fact that they are zero multipliers for this ring. (Received February 27, 1948.)

350. R. C. James: *Unconditional convergence in Banach spaces.*

A series $\sum_{n=1}^{\infty} x_n$ of elements of a Banach space is said to converge unconditionally if it converges for every rearrangement of terms. It is absolutely convergent if $\sum_{n=1}^{\infty}\|x_n\|$ converges. Most known examples of infinite-dimensional Banach spaces are known to have unconditionally convergent sequences of elements which are not absolutely convergent. This class of spaces is extended to include all infinite-dimensional Banach spaces which are uniformly convex, whose adjoint is uniformly convex, or which contain subspaces having a Schauder basis which is unconditionally convergent. (Received March 16, 1948.)

351. Samuel Karlin: *Bases in L^p spaces.* Preliminary report.

A basis in a Banach space E is a set of elements x_n such that each x is uniquely representable in terms of x_n ($x=\sum a_n x_n$). Nina Bary (C. R. Akad. Sci. (Doklody) URSS. vol. 54 (1946)) has introduced the following definitions: A system of elements x_n in L^2 is a Bessel system if there exists a biorthogonal system f_n to x_n with $\sum f_n^2(x)$ convergent for every x. Furthermore, whenever $\sum c_n^2 < \infty$ implies that there exists an x with $f_n(x)=c_n$, then x_n is said to be a Hilbert system. A complete system x_n which is both Hilbert and Bessel constitutes a Riesz basis. A system (x_n, f_n) is said to be bounded if $\|x_n\| \leqq c$ and $\|f_n\| \leqq c$. It is shown in this investigation that a bounded system $(x_n f_n)$ is a Riesz basis if and only if x_n is an absolute basis. (If $\sum a_n x_n$ converges, then $\sum a_{n(i)} x_{n(i)}$ converges for every rearrangement.) In addition, if x_n is a basis and (x_n, f_n) is bounded, then x_n is Bessel if and only if x_n is Hilbert. Let $y_n = \sum a_{nk} x_k$ denote a transformation of a Riesz basis. General conditions are given when y_n remains Riesz. In particular, if (a_{nk}) is a Koch determinant then y_n is Riesz if and only if x_n is Riesz. The above definitions are generalized to L^p spaces. A system x_n in L^p is said to be a Y system if $\sum |f(x_n)|^{p'}$ converges. If $\sum |c_n|^{p'} < \infty$ implies that there exists an x such that $f_n(x)=c_n$, then x_n is said to constitute an (H) system. The cases $p=1$, ∞ have the usual interpretations. Similar results as for the case L^2 are obtained for L^p. (Received March 13, 1948.)

352. H. L. Meyer: *Multiple integral problems in the calculus of variations.* II.

In this paper, general sufficiency theorems established by the author for

$\mathcal{P}_{m,n}$ $(m>1, n>1)$ problems in the calculus of variations are specialized to the $\mathcal{P}_{1,n}$ and $\mathcal{P}_{m,1}$ cases. Weak relative minimum theorems are established under weaker hypotheses: it is only required that the usual Legendre inequality $Q(\pi)>0$ hold for matrices π of rank 1. This hypothesis is the natural one in light of the necessary condition $Q(\pi)\geqq 0$, π of rank 1, for a minimizing surface. The proofs are made by eliminating the rank 1 condition on π through the adjunction of appropriately constructed invariant integrals (cf. Hestenes and McShane, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 501–512), and then applying the previous general theorems. (Received March 12, 1948.)

353. R. A. Rosenbaum: *Sub-additive functions.*

This paper contains a generalization to Euclidean space of n dimensions of the results of Hille (Amer. Math. Soc. Colloquium Publications, vol. 31) on sub-additive functions in 1 dimension. A real-valued function $f(p)$ of a point in E_n is termed "subadditive" if $f(p+p')\leqq f(p)+f(p')$ for all points p, p' in the domain of definition. Among the topics considered are: the construction of sub-additive functions and their relationships to other easily defined classes of functions, among which the convex and concave functions play a large role; infinitary sub-additive functions; boundedness and rate of growth, continuity and differentiability of sub-additive functions; generalizations of the notion of sub-additivity. (Received March 13, 1948.)

354. M. M. Schiffer and Gabor Szegö: *Virtual mass and polarization.*

Let A be a given solid, λ a unit vector. There are defined two harmonic functions g and h both of order r^{-2} at infinity and satisfying the conditions $-\partial g/\partial n = n\lambda$, $h = r\lambda$ on the boundary of A. Their gradient integrals over the exterior of A are two quadratic forms W and P in the projections of λ which represent the *virtual mass* and the *polarization* in the direction λ. The present investigation deals with obtaining bounds for the quantities W and P. Among other inequalities the following is proved: $WP\geqq V^2$ where V is the volume of the given solid A. Moreover it is shown that the quantities $W+V$ and $P+V$ are monotonically increasing set-functions. (Received April 6, 1948.)

355t. Gabor Szegö: *On the virtual mass of nearly spherical solids.*

Let $W=\sum W_{ik}h_ih_k$ $(i, k=1, 2, 3)$ be the virtual mass of a solid immersed in an ideal fluid of density 1 which is at rest at infinity. The solid is moving with uniform speed in the direction h_1, h_2, h_3. In a recent paper (Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 218–221) Professor Pólya dealt with the "average virtual mass W_m defined by $3W_m=W_{11}+W_{22}+W_{33}$ in case of a two-dimensional flow. (The quantity W_m is independent of the coordinate system.) There are numerous evidences that for a wide class of solids $W_m\geqq W_m^0$ holds where W_m^0 is the virtual mass of a *sphere* which has the same volume as the given solid. The purpose of the present investigation is to confirm this inequality for "nearly spherical" solids. The principal result can be formulated as follows. Let $r=1+\rho(\theta, \phi)=1+\sum_{n=0}^{\infty}X_n(\theta, \phi)$ be the equation of the boundary of the solid in polar coordinates, X_n a surface harmonic of degree n with coefficients of order δ. Neglecting terms higher than the second order in δ we have $W_m=2\pi R^3/3$ where $R=1+X_0+\sum_{n=1}^{\infty}\{1+3(n-1)(2n-1)[2(2n+1)]^{-1}\}(4\pi)^{-1}\cdot\iint X_n^2\,d\omega$. (Received March 27, 1948.)

APPLIED MATHEMATICS

356. H. W. Becker: *Circuit algebraic efficiencies.*

A measure of the efficiency of an algebraic derivation is $F=T/(KT+C)$, where T is the number of terms remaining in the formula, after all cancellations. So a result written directly by inspection, or "mental algebra," would be obtained at 100% F. In formulating a complicated circuit function by (1) a chain of $Y\Delta$ transformations, or (2) a mesh determinant, most terms cancel by (1) division, or (2) subtraction. As the number of meshes increases, the F of these methods therefore tends to zero. For example, in the diamond-cross or Shannon bridge $Z=(a+_1A)\|(s+_{1s}S)\|(b+_1B)\perp\alpha\perp\beta=D/N$, $D=\sum 30$ terms obtained at (1) $30/270=11\%$ F, (2) $30/330=9\%$ F. In contrast, the method of fracto components operates at 100% F, even with mutual inductances present, whenever the components are as above self-evident to the operator; thus it is the natural mechanism for amassing an inventory of all possible circuit functions of a given number of branch impedances. The advantage is in transferring jurisdiction of the problem from the theory of equations to the theory of partitions. (Received March 13, 1948.)

357*t*. H. W. Becker: *Semicircuits and semidentities.* Preliminary report.

The most compact storage of all passive circuit functions $Y=N/D$, in loose leaves, punched cards, and so on, is in tabulation of all semicircuit functions N and D, coded for reference from the drawn Y atlas. The economy is in the numerous semidentities $Y=-Y'=N'/D'$, where $N=N'$ or $D=D'$, or $N=D'$. The switching functions $Y\rightarrow Y'$ are called semidentical transformations, s.t. The N s.t. switches one or both source leads to other junctions. The D s.t. repartitions the branches at the source terminals, or switches the source leads between any bi-partition of branches at any other junction. The ND and DN s.t. effect various combinations of N and D s.t. The four s.t. offer the readiest explanation of the cross reciprocities in $Y_s=(rN_\infty+N_0)/(rD_\infty+D_0)\rightarrow=(rN_\infty+D_\infty)/(rN_0+D_0)\leftarrow$, the terms of the latter quotient pertaining to admittances of the circuits looking back from a receiver r, with source s opened and shorted. The semicircuits may be drawn, or put in the notation of graphs, or of a logic with four conjunctions: X, $+$, $\|$, and $\perp$. (Received February 26, 1948.)

358*t*. H. W. Becker: *The duals of nonplanar networks.*

The dual of a pc (passive circuit) $Z=\sum s_1\cdots s_n/\sum s_1\cdots s_{n-1}=\theta_0/\theta_\infty$ is ${}^{-}Z=\sum(s_1\cdots s_{n-1})^{-1}/\sum(s_1\cdots s_n)^{-1}={}^{-}\theta_\infty/{}^{-}\theta_0$, where θ_0 and θ_∞ are combinatorially the characteristic functions of the semicircuits formed by shorting and opening the Z terminals, and topologically the semicircuits themselves. If Z is nonplanar, ${}^{-}Z$ is an imaginary circuit, which nevertheless has real fracto component circuits and physical $Y\Delta$ and ideal transformer equivalent circuits. Also, if Z is (1) planar, (2) global, (3) infraglobal, (4) five to infinite dimensional, then the semiduals ${}^{-}\theta_\infty$ and ${}^{-}\theta_0$ of Z are (1) planar and mutually realizable, (2) one or both planar but source terminal indeterminates, (3) one or both imaginary, (4) both imaginary. The active circuit θ of Z, and its dual ${}^{-}\theta$, are computed from ${}^{\pm}\theta=s\cdot{}^{\pm}\theta_\infty+{}^{\pm}\theta_0$, s the source impedance. The proportion of nonplanar to all pc is 1/717 for 8 branches, about 1/30 for 12 branches, and tends ultimately to a majority. All nonplanar pc are reducible to the abstract multivibrator or Bloch bridge $(a+_1f+_2b)\|(A+_2F+_1B)\perp\phi\perp\Phi$, whose imaginary dual has an equivalent circuit at the terminals and in 5 branches of the

Kelvin bridge $(a+{}_1A)\|\{\phi'+(f+{}_1\Phi)\|b+F'\}\perp B'$, with $F'=FB/S$, $B'=F\phi/S$, $\phi'=\phi B/S$, $S=F+B+\phi$. (Received February 27, 1948.)

359. Edmund Pinney: *Nonlinear external resonance.*

A simple nonlinear oscillatory system subjected to an external harmonic force is studied in its resonance states. Typical systems are the electrical system consisting of a linear inductance, nonlinear resistance, and nonlinear capacitance in series with a sinusoidal generator, and the mechanical system consisting of a mass suspended by a nonlinear spring, acted upon by a nonlinear frictional resistance, and subjected to a nonlinear exciting force. Both systems lead to a differential equation of the form $\ddot{x}+f(\dot{x})+g(x)=a+b\sin(\omega t)$. In addition to the fundamental oscillation, the nonlinearity of the system induces harmonics and subharmonics—oscillations whose frequencies are rational fractions of the fundamental frequency. The present paper develops the general machinery for studying these oscillations, with their attendant phenomena of hard and soft frequency entrainment, asynchronous excitation and quenching. The particular cases $f(\dot{x})=0$ and $g(x)=0$ are treated in greater detail. (Received March 12, 1948.)

360. S. A. Schaaf: *The problem of nozzle design for low pressure supersonic wind tunnels.*

It is shown that the effect of very low pressures in a gas flow is to make viscosity effects of equal importance to compressibility effects, thus invalidating the characteristics method of nozzle design. A generalized Polhausen-Tsien approximate method is developed for calculation of the two-dimensional viscous compressible isothermal flow, and its application to nozzle design is discussed. This work was done under an Office of Naval Research contract. (Received March 6, 1948.)

GEOMETRY

361. Alfred Horn and F. A. Valentine: *Some properties of L sets in the plane.*

A set S is an L set if any two points of S can be joined by an at most two-sided polygon lying in S. If S is a bounded closed L set in the plane, every bounded component of its complement $C(S)$ is an L set. Furthermore through each point of the unbounded component T of $C(S)$ there is an infinite ray lying in T. Hence any two points of T can be joined by a three-sided polygon contained in T. If $C(S)$ has no bounded components, S may be expressed as a sum of convex sets every two of which have a point in common. This statement is proved using the theorem: Given any set of closed intervals on the boundary of a circle such that any two have a common point, then there exists a pair of diametrically opposite points p, q such that every interval of the family contains at least one of the points p, q. Other properties of L sets are investigated. An example of an L set is the set of points of a plane convex body through which there pass two or more lines bisecting the area. (Received March 12, 1948.)

LOGIC AND FOUNDATIONS

362t. J. D. Swift: *On the structure of n-valued logics.*

The set of $n^{(n^m)}$ possible m-ary closed operations on n elements (m-ary n-valued propositional functions) is considered, particular attention being given to the case

$n=2$. The result of Zylinski on the uniqueness of the generating stroke operations for $n=2$ is not true for $n>2$. General classes of such stroke functions are obtained. The operations under which the set is a given algebraic variety are considered by the direct sum technique, and theorems on the number and type of rings, groups, and lattices which are formed by the members and selected operations of the set are obtained. The modular theory of Bernstein is generalized for the case $n=p^k$. (Received March 13, 1948.)

TOPOLOGY

363. P. A. White: *On a certain class of set theoretic properties.*

A study is made of those properties of sets that satisfy the following theorem: If $P\subset S$, both have a property p, and $S-P=X\cup Y$, separate, then $X\cup P$ and $Y\cup P$ have the property p. For example, connectedness satisfies this theorem. A study is also made of the conditions under which a local property satisfies this theorem if the corresponding property in the large satisfies the theorem (for example, local connectedness satisfies it since connectedness does). A total of 53 properties are shown to satisfy the theorem including arcwise and cyclic connectedness, simple-i-connectedness, and the property of being an open generalized-n-manifold. (Received October 30, 1947.)

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BOOK REVIEWS

Foundations of algebraic geometry. By André Weil. (American Mathematical Society Colloquium Publications, vol. 29.) New York, American Mathematical Society, 1946. 20+288 pp. $5.50.

In the words of the author the main purpose of this book is "to present a detailed and connected treatment of the properties of intersection multiplicities, which is to include all that is necessary and sufficient to legitimize the use made of these multiplicities in classical algebraic geometry, especially of the Italian school." There can be no doubt whatsoever that this purpose has been fully achieved by Weil. After a long and careful preparation (Chaps. I–IV) he develops in two central chapters (V and VI) an intersection theory which for completeness and generality leaves little to be desired. It goes far beyond the previous treatments of this foundational topic by Severi and van der Waerden and is presented with that absolute rigor to which we are becoming accustomed in algebraic geometry. In harmony with its title the book is entirely self-contained and the subject matter is developed *ab initio.*

It is a remarkable feature of the book that—with one exception (Chap. III)—no use is made of the higher methods of modern algebra. The author has made up his mind not to assume or use modern algebra "beyond the simplest facts about abstract fields and their extensions and the bare rudiments of the theory of ideals." Weil's faithful realization of this program of strict mathematical economy is an achievement in itself. In some cases this leads to the "best possible" proofs. However, on the whole one may question the wisdom of this self-imposed régime of austerity. The methodical reduction of the theory—which is both difficult and subtle—to the primitives of algebra is bound to be a very laborious process. As a result, the reader finds himself very much in the position of a man who must collect a large amount of cash most of which is in pennies. The author justifies his procedure by an argument of historical continuity, urging a return "to the palaces which are ours by birthright." But it is very unlikely that our predecessors will recognize in Weil's book their own familiar edifice, however improved and completed. If the traditional geometer were invited to choose between "makeshift constructions full of rings, ideals and valuations" on one hand, and constructions full of fields, linearly disjoint fields, regular extensions, independent extensions, generic specializations, finite specializations and specializations of specializations on the other, he most probably would decline the choice and say: "A plague on both your houses!" That being so, we may just as well help ourselves to modern algebra to the fullest possible extent.

To achieve his objectives Weil wages a campaign of the Satz-Beweis type. Most readers will find it difficult to follow the author through the seemingly endless series of propositions, theorems, lemmas and corollaries (their total must be close to 300). A further obstacle to continuous reading of the book are the numerous cross-references in the proofs, such as "by coroll. 2 of th. 2, §1" or "by th. 5, Ch. I, §7" (for instance, the 13 lines of proof of proposition 7, p. 33, contain 8 such references). However, this being said, the fact remains that the scientific value of this surprisingly large collection of theorems and propositions is very high. The specialist, or the advanced student who is interested in some particular phase of the theory, will frequently be able to find in Weil's book just the theorem or lemma he needs.

The first chapter contains algebraic preliminaries. The concept of linearly disjoint fields is introduced and used with good effect for a study of inseparable field extensions. Of particular importance later on are the "regular extensions" of a field k: if (x) is a finite set of quantities, the field $k(x)$ is a regular extension of k if it is separably generated over k and if k is maximally algebraic in $k(x)$.

The "algebraic theory of specializations" is the topic of the second chapter, and it plays a fundamental role in Weil's intersection theory. Here the main result is Theorem 6 which asserts that if (x) and (y) are two (finite) sets of quantities, then any specialization $(x)\rightarrow(x')$ of the set (x) (with reference to a given field k) can be extended to a specialization $(x, y)\rightarrow(x', y')$ of the combined set (x, y). This well known result from the theory of algebraic correspondences is proved here without the use of elimination theory. It must be observed that Weil deals only with varieties in the affine space and never uses homogeneous coördinates. Therefore the symbol ∞ is allowed to occur among his quantities when he discusses specializations. As a matter of fact, projective models are not varieties at all in Weil's sense; they first occur as an afterthought only at the very end of the book (Appendix I).

The principal aim of the next chapter is the concept of the multiplicity of a specialization. Assume that the quantities x are independent variables over k and that the y's are algebraic over $k(x)$. Let V be the algebraic variety having (x, y) as general point over k and let $(\bar{x}, \bar{y})$ be a point of V [whence $(x, y)\rightarrow(\bar{x}, \bar{y})$ is a specialization over k] such that the $\bar{y}$ are algebraic over $k(\bar{x})$. Consider a complete set of conjugates $(y^{(1)}, y^{(2)}, \cdots, y^{(n)})$ of (y) over $k(x)$ and extend in any manner whatsoever the specialization $(x)\rightarrow(\bar{x})$ to a specialization $(x, y^{(1)}, y^{(2)}, \cdots, y^{(n)})\rightarrow(\bar{x}, \bar{y}^{(1)}, \bar{y}^{(2)}, \cdots, \bar{y}^{(n)})$ over k. The main result is as follows: if the intersection $(\bar{x}, \bar{y})$ of the linear space $X=\bar{x}$ with V is *isolated*, then $\bar{y}$ occurs the same number of times in every set $(\bar{y}^{(1)}, \bar{y}^{(2)}, \cdots, \bar{y}^{(n)})$ such as above, and this number is positive (Proposition 7 and Theorem 4). Weil calls this number the multiplicity of $(\bar{y})$ as a specialization of (y) over $(x)\rightarrow(\bar{x})$, with reference to k. Essentially this number is the intersection multiplicity of V with the linear space $L: X=\bar{x}$, at the point $P(\bar{x}, \bar{y})$. This clears the ground completely for the "special case" of the intersection theory, dealt with in Chap. V, where intersection multiplicities of varieties with linear spaces are studied in greater detail. It should be observed that Weil does not assume that *all* the intersections of V and L are of the "right" dimension, that is, of dimension 0 over $k(\bar{x})$, and therefore the result is purely of local character. Therein lies the main difficulty of the proof. But at the same time it is precisely because of its local validity that the theory developed by Weil (as well as the one developed independently by Chevalley, Trans. Amer. Math. Soc. (1945)) is an improvement over the previous treatments of intersection multiplicities. In this chapter the author is forced to use "higher" methods of modern algebra (ideal theory of Noetherian domains, rings of formal power series, and so on). Later on in Chap. IX (*Comments and discussions*) he shows that the main result of the present chapter can also be derived from a theorem on birational correspondences due to the reviewer. Our proof involves a good deal of ideal theory of integrally closed domains. All this illustrates the plain truth that as one advances in the field of algebraic geometry, the plot "thickens" and more powerful tools of modern algebra become indispensable.

In Chap. IV algebraic varieties are introduced, linear varieties and simple points are discussed, and the properties of product varieties are developed. The author restricts his class of varieties V by the following condition: there must exist a field k and a point P of V such that P is a general point of V with respect to k *and the field* $k(P)$ *is a regular extension of* k. Furthermore he restricts the concept of a simple point by defining it in terms of the classical Jacobian criterion. This enables him to state and prove his theorems without any reference to any particular ground field, for it is well known that with the above restrictions nothing much can happen to a variety or to a simple point when the ground field undergoes an arbitrary extension. These limitations are perfectly legitimate and undoubtedly create a fortunate state of affairs, but we could not comprehend Weil's contention that they entail an exclusive point of view that emphasizes "the geometric content of all notions." At any rate it would be desirable to free the intersection theory from all these restrictions. This theory is developed in the next two chapters, V and VI. The main object is of course the introduction and study of the main properties of the intersection multiplicity of two subvarieties A and B of a given variety U, along an irreducible component C of $A\cap B$. This multiplicity, denoted by $i(A\cdot B,\ C;\ U)$, is to be defined only under the following conditions: C is simple for U and has the right dimension ($=\dim A+\dim B-\dim U$). We have already mentioned the "special case": U and one of the varieties A and B are linear. The general case is reduced to the special case by using product varieties. Thus, if U is still linear and C is a point P, say the origin, then the above symbol is defined as $i(A\times B\cdot\Delta,\ P\times P;\ U\times U)$, where Δ is the diagonal of $U\times U$. In the most general case the reduction is quite similar. There follows an exhaustive study of the properties of intersection multiplicities. The basic properties are stated in Theorems 5, 6 and 8, Ch. VI, which express respectively the associative law, the projection formula and the criterion of multiplicity 1.

The long Chap. VII is dedicated entirely to what Weil calls "abstract varieties," or Varieties with a capital V. A Variety is a finite collection $\{V_\alpha-F_\alpha\}$ of pieces of birationally equivalent varieties V_α such that: (1) F_α is a subvariety of V_α: (2) whenever two pieces $V_\alpha-F_\alpha$, $V_\beta-F_\beta$ contain corresponding points of V_α and V_β, the local rings of these points coincide. A Variety is *complete* if it satisfies a suitable condition of closure. Thus projective models are complete Varieties. The chapter deals primarily with the extension of the results of the preceding chapters to Varieties.

We find it very difficult to estimate the necessity or the permanence value of this new concept. The book contains no examples of complete Varieties other than projective models. It is true that the realization of an Abelian variety by means of a *nonsingular* complete Variety follows immediately from the definition of Abelian varieties, while the existence of a corresponding nonsingular projective model still has to be proved (in the classical case this has been proved by Lefschetz). But we point out that there exists no proof that every function field can be realized by a nonsingular complete Variety. Moreover, we believe that, as to degree of difficulty, such a proof would differ only by an ϵ from a proof of the stronger statement that every function field possesses a nonsingular projective model. Indeed, it may be conjectured that every complete Variety is itself a projective model. In view of all this, one wonders whether Chap. VII is absolutely necessary for the *general* theory of algebraic varieties, or even for the theory of Abelian varieties.

Weil's book is the first purely arithmetic exposition of an important sector of algebraic geometry, and is therefore a landmark in the literature of this field. This, and the competence of the author, give the book added significance.

In the remainder of the book we wish to recommend especially the interesting chapter entitled *Comments and discussions*, where various unsolved problems are discussed and possible directions of future research are indicated. The book has an excellent list of definitions and table of notations.

OSCAR ZARISKI

Transformations on lattices and structures of logic. By S. A. Kiss. New York, 1947. 10+322 pp. $5.00.

This volume by Dr. Kiss is an instance of a rare phenomenon—a significant contribution to mathematics by one who is not a professional mathematician. Dr. Kiss is a patent lawyer by profession, with an advanced degree in chemistry, who has here enriched the algebra of logic by a new idea, and developed the idea in full detail. Moreover enough elementary material has been adapted from standard sources so that the book is self-contained as regards both algebra and logic. The book was published by the author, and can be obtained from him at 11 E. 92nd Street, New York City.

Boolean algebra may be described as the algebra of true and false. Numerous equivalent postulate systems for it are known, involving from one to three undefined operations or relations, in terms of which all 2^{2^n} possible n-ary operations on a two-element system can be defined. Thus *every finitary operation on a two-element system is a Boolean operation.*

But the corresponding result for the four-element Boolean algebra B^2 does not hold. Only 2^{2^3} of the 4^{4^2} possible binary operations on B^2 are "Boolean," in the usual sense. Dr. Kiss proposes the following ingenious extension. Consider the self-dual ternary operation $(x, y, z) = (x \cup y) \cap (y \cup z) \cap (z \cup x)$. In terms of this, joins and meets can be defined by fixing y as one of the two logical constants 0 and I: $x \cup z = (x, I, z)$ and $x \cap z = (x, 0, z)$. If we use the other two constants e and e' of B^2 in place of 0 and I, we get two further binary Boolean-like operations (x, e, z) and (x, e', z). And in terms of these, all binary operations can be defined.

It is still too early to appraise the ultimate importance of these and related algebraic operations discussed by Dr. Kiss. In the reviewer's opinion, they meet various esthetic requirements of symmetry and harmony, which it is hard to formulate exactly. Thus all operations are mutually distributive; in the case of the Boolean algebra B^n of order 2^n, Dr. Kiss's algebra admits $2^n(n!)$ "link"-automorphisms, in place of the $2(n!)$ automorphisms and dual automorphisms usually admitted. Mr. Kiss suggests applications to the algebra of classes of classes which, if they can be developed more precisely and fully, may be of great value.

But for the present, the book's chief merit consists in its orderly and detailed development of an intriguing and highly symmetrical calculus, which contains ordinary Boolean algebra as a special case. The author deserves the sincere thanks of persons interested in this field, for his work.

GARRETT BIRKHOFF

NOTES

The Unione Matematica Italiana will hold its Third National Congress in Pisa, Italy, on September 23–26, 1948. The Unione cordially invites all members of the American Mathematical Society to participate and asks that those who are interested communicate with Professor Luigi Berzolari, President of the Unione, at the University of Bologna.

The Domus Galilaeana was founded in Pisa in 1941 for the purpose of coordinating studies and research on the life and works of Galileo. It gathers books, manuscripts and historical documents, issues appropriate publications and maintains a library open to the public.

Associate Professor Maurice Ewing of Columbia University and Professor E. J. McShane of the University of Virginia have been elected to membership in the National Academy of Sciences.

Professor C. C. Chevalley of Princeton University, Assistant Professor Irving Kaplansky of the University of Chicago, and Associate Professor Norman Levinson of Massachusetts Institute of Technology have been awarded Guggenheim fellowships.

President W. M. Whyburn of Texas Technological College has been awarded an honorary degree by Texas Technological College. He has been appointed to a professorship at the University of North Carolina.

Dr. Churchill Eisenhart, Chief of the Statistical Engineering Laboratory, National Bureau of Standards, has been elected a fellow of the Royal Statistical Society of Great Britain.

Professor E. W. Sheldon of the University of Alberta has retired with the title emeritus.

Professor A. S. Besicovitch of Trinity College, Cambridge University, will be a visiting professor at the University of Pennsylvania during the academic year 1948–1949.

Assistant Professor I. S. Cohen of the University of Pennsylvania has been appointed to an assistant professorship at Massachusetts Institute of Technology.

Assistant Professor Lester Dawson of Michigan College of Mining and Technology has been appointed to an associate professorship at Adams State College, Alamosa, Colorado.

Professor G. A. Hedlund of the University of Virginia has been appointed to a professorship at Yale University.

Assistant Professor D. M. Hester, formerly of the New Mexico School of Mines, has accepted a position at the Northeast Missouri State College, Maryville, Missouri.

Dr. H. D. Huskey has been appointed Chief of the Machine Development Laboratory of the National Bureau of Standards.

Mr. Eric Kristensen has been appointed lecturer at Washington University.

Professor D. C. Lewis of the University of Maryland has been appointed to a professorship at Johns Hopkins University.

Associate Professor A. K. Mitchell of the University of Maryland has accepted a position as aerodynamicist at the Applied Physics Laboratory of the Johns Hopkins University.

Dr. Zeev Nehari of Hebrew University has been appointed to an associate professorship at Washington University.

Professor Rufus Oldenburger of Illinois Institute of Technology has been appointed to a professorship at De Paul University.

Mr. Edgar Phibbs has been appointed to an assistant professorship at the University of Alberta.

The retirement of Dean R. G. D. Richardson of Brown University has been announced.

Dr. R. D. Schafer has been appointed to an assistant professorship at the University of Pennsylvania.

Mr. E. T. Sheffield has been appointed to an assistant professorship at the University of Alberta.

Assistant Professor Ruth G. Simond of Morningside College, Sioux City, Iowa, has been appointed to an assistant professorship at the University of Vermont.

Dr. C. V. L. Smith has been appointed Head of the Computing Machine Section of the Mathematics Branch of the Office of Naval Research.

Dr. E. H. Spanier of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor D. W. Western of Brown University has been appointed to an associate professorship at Franklin and Marshall College.

Dr. Albert Wilansky of Brown University has been appointed to an assistant professorship at Lehigh University.

Professor Antoni Zygmund of the University of Chicago is now at the University of Buenos Aires, as visiting professor.

The following promotions have been announced.

Herbert Federer, Brown University, to an associate professorship.

S. G. Hacker, State College of Washington, to a professorship.

Franklin Haimo, Washington University, to an assistant professorship.

Witold Hurewicz, Massachusetts Institute of Technology, to a professorship.

R. E. Huston, Rensselaer Polytechnic Institute, to a professorship.

Bjarni Jónsson, Brown University, to an assistant professorship.

Wouter van der Kulk, Brown University, to an assistant professorship.

C. T. McCormick, Illinois State Normal University, to a professorship.

H. M. Schaerf, Washington University, to an associate professorship.

K. C. Schraut, University of Dayton, to a professorship.

E. R. Stabler, Hofstra College, to an associate professorship.

Mary V. Sunseri, Stanford University, to an acting assistant professorship.

G. B. Van Schaack, Washington University, to an associate professorship.

M. S. Webster, Purdue University, to an associate professorship.

G. W. Whitehead, Brown University, to an associate professorship.

The appointment of Mr. Charles Saltzer to an instructorship at Case Institute of Applied Sciences is announced.

Dr. H. W. Richmond of Cambridge University died April 22, 1948, at the age of eighty-four years. He had been a member of the Society for forty-seven years.

Professor Emeritus Clara L. Bacon of Goucher College died April 14, 1948, at the age of eighty-one years. She had been a member of the Society for thirty-seven years.

Associate Professor Emeritus J. K. Whittemore of Yale University died March 22, 1948, at the age of seventy-two years. He had been a member of the Society for over fifty years.

Correction: On page 496 of the May 1948 issue of this Bulletin (vol. 54, no. 5) the title of the dissertation of R. M. Cohn should be *Manifolds of difference polynomials.*

NEW PUBLICATIONS

CAMBI, E. *Eleven and fifteen-place tables of Bessel functions of the first kind, to all significant orders*. New York, Dover, 1948. 5+154 pp. $3.95.

HAHN, H., and ROSENTHAL, A. *Set functions*. Albuquerque, University of New Mexico Press, 1948. 324 pp. $12.50.

HOPF, L. *Introduction to the differential equations of physics*. New York, Dover, 1948. 5+154 pp. $1.95.

KATTSOFF, L. O. *A philosophy of mathematics*. Ames, Iowa State Press, 1948. 9+263 pp. $5.00.

LAMBERT, J. H. *Mathematische Werke*. Vol. 2. Ed. by A. Speiser. Zurich, Fussli, 1947. 342 pp.

McLACHLAN, N. W. *Modern operational calculus*. New York, Macmillan, 1948. 14+218 pp. 21s net.

ROSENTHAL, A. See HAHN, H.

SAMPLING INSPECTION. Statistical research group, Columbia University. New York, McGraw-Hill, 1948. 20+395 pp. $5.25.

TABLES OF THE BESSEL FUNCTIONS OF FRACTIONAL ORDER. Vol. 1. Prepared by the Computation Laboratory of the National Bureau of Standards. New York, Columbia University Press, 1948. 42+413 pp. $7.50.

GENERALIZATION OF AN INEQUALITY OF HEILBRONN AND ROHRBACH

F. A. BEHREND

Let $a_1, \cdots, a_m$ be positive integers and

$$(1)\quad T(a_1, \cdots, a_m) = \begin{cases} 1 - \sum\limits_{\mu_1=1}^{m} \dfrac{1}{a_{\mu_1}} + \sum\limits_{\mu_1=2}^{m} \sum\limits_{\mu_2=1}^{\mu_1-1} \dfrac{1}{\{a_{\mu_1}, a_{\mu_2}\}} - \cdots \\ \qquad\qquad + \dfrac{(-1)^m}{\{a_1, \cdots, a_m\}} & \text{for } m > 0, \\ 1 & \text{for } m = 0, \end{cases}$$

where $\{u_1, \cdots, u_r\}$ denotes the least common multiple of $u_1, \cdots, u_r$. H. A. Heilbronn[1] and H. Rohrbach[2] proved that

$$(2)\quad \begin{aligned} T(a_1, \cdots, a_m) &\geq \left(1 - \frac{1}{a_1}\right) \cdot \cdots \cdot \left(1 - \frac{1}{a_m}\right) \\ &= T(a_1) \cdot \cdots \cdot T(a_m). \end{aligned}$$

The object of this paper is to prove the following generalization of (2):

$$(3)\quad T(a_1, \cdots, a_m, b_1, \cdots, b_n) \geqq T(a_1, \cdots, a_m) T(b_1, \cdots, b_n) \qquad \text{for } m \geqq 0,\ n \geqq 0.$$

$T(a_1, \cdots, a_m)$ may be interpreted as the density of the set S of all positive integers not divisible by any a_μ, that is,

$$T(a_1, \cdots, a_m) = \lim_{z \to \infty} z^{-1} M(z),$$

where $M(z)$ is the number of elements of S not exceeding z.

For the proof of (3) we require the following lemma.

LEMMA. *If* $k \geq 0$, $l \geqq 0$, *and* $(d, v_\lambda) = 1$ *for* $\lambda = 1, \cdots, l$, *then*

$$T(du_1, \cdots, du_k, v_1, \cdots, v_l) = \frac{1}{d} T(u_1, \cdots, u_k, v_1, \cdots, v_l) + \left(1 - \frac{1}{d}\right) T(v_1, \cdots, v_l).$$

Received by the editors March 14, 1947.

[1] *On an inequality in the elementary theory of numbers*, Proc. Cambridge Philos. Soc. vol. 33 (1937) pp. 207–209.

[2] *Beweis einer zahlentheoretischen Ungleichung*, J. Reine Angew. Math. vol. 177 (1937) pp. 193–196.

PROOF. $T(du_1, \cdots, du_k, v_1, \cdots, v_l)$ is the density of the set S of all x not divisible by any of $du_1, \cdots, du_k, v_1, \cdots, v_l$. We divide S into two sets S_1 and S_2. S_1 contains all elements of S which are divisible by d; they are of the form $x_1 = dy$ subject to the condition that none of $du_1, \cdots, du_k, v_1, \cdots, v_l$ divides dy, which is equivalent to the condition that none of $u_1, \cdots, u_k, v_1, \cdots, v_l$ divides y; the density of the set of integers y is thus $T(u_1, \cdots, u_k, v_1, \cdots, v_l)$ and the density of S_1 becomes $(1/d)T(u_1, \cdots, u_k, v_1, \cdots, v_l)$. S_2 contains all elements of S which are not divisible by d; as $du_1, \cdots, du_k$ do not divide these elements, S_2 consists of all positive integers x_2 not divisible by any of $d, v_1, \cdots, v_l$, and its density is $T(d, v_1, \cdots, v_l)$. Thus we have

$$
\begin{aligned}
&T(du_1, \cdots, du_k, v_1, \cdots, v_l) \\
(4) \qquad &\qquad = \frac{1}{d} T(u_1, \cdots, u_k, v_1, \cdots, v_l) + T(d, v_1, \cdots, v_l).
\end{aligned}
$$

Note that this proof still holds when one or both of $k, l = 0$; for $k = 0$, (4) reduces to

$$
(5) \qquad T(v_1, \cdots, v_l) = \frac{1}{d} T(v_1, \cdots, v_l) + T(d, v_1, \cdots, v_l),
$$

whence

$$
(6) \qquad T(d, v_1, \cdots, v_l) = \left(1 - \frac{1}{d}\right) T(v_1, \cdots, v_l).
$$

Substituting (6) in (4) we obtain the lemma.

PROOF OF (3): by induction with respect to $N = a_1 + \cdots + a_m + b_1 + \cdots + b_n$. For $N = 0$, $m = n = 0$ and the three T's in (3) reduce to 1. Assume that (3) holds for $N' < N$.

First case: Any two of $a_1, \cdots, a_m, b_1, \cdots, b_l$ relatively prime. In this case

$$
\begin{aligned}
&T(a_1, \cdots, a_m, b_1, \cdots, b_n) \\
&\quad = \left(1 - \frac{1}{a_1}\right) \cdot \cdots \cdot \left(1 - \frac{1}{a_m}\right) \cdot \left(1 - \frac{1}{b_1}\right) \cdot \cdots \cdot \left(1 - \frac{1}{b_n}\right) \\
&\quad = T(a_1, \cdots, a_m) T(b_1, \cdots, b_n).
\end{aligned}
$$

Second case: s exists such that $2 \leq s \leq m+n$ and (i) certain s of the $a_1, \cdots, a_m, b_1, \cdots, b_n$ have a common divisor $d > 1$, (ii) any $s+1$ of them have the greatest common divisor 1 (this condition being vacuous for $s=m+n$). Rearranging the $a_1, \cdots, a_m, b_1, \cdots, b_n$ we may assume that $a_1, \cdots, a_\mu, b_1, \cdots, b_\nu$ have the common divisor $d>0$ where $\mu+\nu=s$ (μ or ν may be 0); then

$$\begin{aligned} a_\rho &= d\bar{a}_\rho \quad \text{for} \quad \rho \leqq \mu; & (a_\rho, d) &= 1 \quad \text{for} \quad \rho > \mu; \\ b_\sigma &= d\bar{b}_\sigma \quad \text{for} \quad \sigma \leqq \nu; & (b_\sigma, d) &= 1 \quad \text{for} \quad \sigma > \nu. \end{aligned}$$

By the lemma

$$\begin{aligned} &T(a_1, \cdots, a_m)T(b_1, \cdots, b_n) \\ &= T(d\bar{a}_1, \cdots, d\bar{a}_\mu, a_{\mu+1}, \cdots, a_m)T(d\bar{b}_1, \cdots, d\bar{b}_\nu, b_{\nu+1}, \cdots, b_n) \\ &= \left(\frac{1}{d}T(\bar{a}_1, \cdots, \bar{a}_\mu, a_{\mu+1}, \cdots, a_m) + \left(1-\frac{1}{d}\right)T(a_{\mu+1}, \cdots, a_m)\right) \\ &\quad\cdot\left(\frac{1}{d}T(\bar{b}_1, \cdots, \bar{b}_\nu, b_{\nu+1}, \cdots, b_n) + \left(1-\frac{1}{d}\right)T(b_{\nu+1}, \cdots, b_n)\right) \\ &= \frac{1}{d}T(\bar{a}_1, \cdots, \bar{a}_\mu, a_{\mu+1}, \cdots, a_m)T(\bar{b}_1, \cdots, \bar{b}_\nu, b_{\nu+1}, \cdots, b_n) \qquad (7) \\ &\quad + \left(1-\frac{1}{d}\right)T(a_{\mu+1}, \cdots, a_m)T(b_{\nu+1}, \cdots, b_n) \\ &\quad - \frac{1}{d}\left(1-\frac{1}{d}\right)\left(T(a_{\mu+1}, \cdots, a_m) - T(\bar{a}_1, \cdots, \bar{a}_\mu, a_{\mu+1}, \cdots, a_m)\right) \\ &\quad\cdot\left(T(b_{\nu+1}, \cdots, b_n) - T(\bar{b}_1, \cdots, \bar{b}_\nu, b_{\nu+1}, \cdots, b_n)\right). \end{aligned}$$

Observing that the first two terms may be estimated by the induction hypothesis and that the factors of the third term are not less than 0, we get

$$\begin{aligned} &T(a_1, \cdots, a_m)T(b_1, \cdots, b_n) \\ &\quad \leqq \frac{1}{d}T(\bar{a}_1, \cdots, \bar{a}_\mu, a_{\mu+1}, \cdots, a_m, \bar{b}_1, \cdots, \bar{b}_\nu, b_{\nu+1}, \cdots, b_n) \\ (8) &\quad\quad + \left(1-\frac{1}{d}\right)T(a_{\mu+1}, \cdots, a_m, b_{\nu+1}, \cdots, b_n) \\ &\quad = T(a_1, \cdots, a_m, b_1, \cdots, b_n) \end{aligned}$$

by the lemma. Hence (3) is proved.

It is easy to decide when equality holds in (3). Equality will certainly hold if $(a_\rho, b_\sigma)=1$ for $\rho=1, \cdots, m$, $\sigma=1, \cdots, n$; this can be seen on the lines of the above proof, or, directly, by substituting the explicit value (1) of T into (3) and observing that $\{a_{\rho_1}, \cdots, a_{\rho_p}\} \cdot \{b_{\sigma_1}, \cdots, b_{\sigma_q}\} = \{a_{\rho_1}, \cdots, a_{\rho_p}, b_{\sigma_1}, \cdots, b_{\sigma_q}\}$. The converse is obviously not true as, for example, $T(2, 4)T(3, 6) = T(2, 4, 3, 6)$; the reason is that, in this example, the numbers 4, 6 are redundant; the example may be written simpler $T(2)T(3) = T(2, 3)$. In general u_k will be redundant in $T(u_1, \cdots, u_k)$ if it is a multiple of another u_λ. If redundant elements in $T(a_1, \cdots, a_m)$ and $T(b_1, \cdots, b_n)$ are removed, the converse of the above statement can be proved: If for some ρ, σ $(a_\rho, b_\sigma) > 1$, inequality holds in (3). We may assume that $(a_1, b_1) > 1$ and can apply (7) with $\mu \geqq 1$, $\nu \geqq 1$. Now, if u is not divisible by any of $v_1, \cdots, v_l$, then

$$T(v_1, \cdots, v_l) > T(u, v_1, \cdots, v_l),$$

for the set of positive integers not divisible by $v_1, \cdots, v_l$ contains the numbers $u(v_1 \cdot \cdots \cdot v_l s + 1)$, $s = 0, 1, 2, \cdots$, which possess a positive density and are not contained in the set of numbers not divisible by $u, v_1, \cdots, v_l$. As a_1, and hence $\bar{a}_1$, is not a multiple of any of $a_{\mu+1}, \cdots, a_m$ and $\bar{b}_1$ not a multiple of any of $b_{\nu+1}, \cdots, b_n$, it follows that the factors of the last term of (7) are positive, and the inequality sign will hold in (8).

THE UNIVERSITY OF MELBOURNE

ON THE DENSITY OF SOME SEQUENCES OF INTEGERS

P. ERDÖS

Let $a_1 < a_2 < \cdots$ be any sequence of integers such that no one divides any other, and let $b_1 < b_2 < \cdots$ be the sequence composed of those integers which are divisible by at least one a. It was once conjectured that the sequence of b's necessarily possesses a density. Besicovitch[1] showed that this is not the case. Later Davenport and I[2] showed that the sequence of b's always has a logarithmic density, in other words that $\lim_{n\to\infty} (1/\log n) \sum_{b_i \leq n} 1/b_i$ exists, and that this logarithmic density is also the lower density of the b's.

It is very easy to see that if $\sum 1/a_i$ converges, then the sequence of b's possesses a density. Also it is easy to see that if every pair of a's is relatively prime, the density of the b's equals $\prod(1-1/a_i)$, that is, is 0 if and only if $\sum 1/a_i$ diverges. In the present paper I investigate what weaker conditions will insure that the b's have a density. Let $f(n)$ denote the number of a's not exceeding n. I prove that if $f(n) < cn/\log n$, where c is a constant, then the b's have a density. This result is best possible, since we show that if $\psi(n)$ is any function which tends to infinity with n, then there exists a sequence a_n with $f(n) < n \cdot \psi(n)/\log n$, for which the density of the b's does not exist. The former result will be obtained as a consequence of a slightly more precise theorem. Let $\phi(n; x; y_1, y_2, \cdots, y_n)$ denote generally the number of integers not exceeding n which are divisible by x but not divisible by $y_1, \cdots, y_n$. Then a necessary and sufficient condition for the b's to have a density is that

$$\text{(1)} \qquad \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \sum_{n^{1-\epsilon} < a_i \leq n} \phi(n; a_i; a_1, a_2, \cdots, a_{i-1}) = 0.$$

The condition (1) is certainly satisfied if $f(n) < cn/\log n$, since

$$\frac{1}{n} \sum_{n^{1-\epsilon} < a_i \leq n} \phi(n; a_i; a_1 \cdots a_{i-1}) < \frac{1}{n} \sum_{n^{1-\epsilon} < a_i \leq n} \left[\frac{n}{a_i}\right]$$
$$< \sum_{n^{1-\epsilon} < m \log m < n} \frac{c'}{m \log m} = O(\epsilon) + O\left(\frac{1}{n}\right).$$

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[1] Math. Ann. vol. 110 (1934–1935) pp. 336–341.

[2] Acta Arithmetica vol. 2.

As an application of the condition (1) we shall prove that the set of all integers m which have two divisors d_1, d_2 satisfying $d_1 < d_2 \leqq 2d_1$ exists. I have long conjectured that this density exists, and has value 1, but have still not been able to prove the latter statement.

At the end of the paper I state some unsolved problems connected with the density of a sequence of positive integers.

THEOREM 1. *Let* $\psi(n) \to \infty$ *as* $n \to \infty$. *Then there exists a sequence* $a_1 < a_2 < \cdots$ *of positive integers such that no one of them divides any other, with* $f(n) < n\psi(n)/\log n$, *and such that the sequence of* b*'s does not have a density.*

PROOF. We observe first that the condition that one a does not divide another is inessential here, since we can always select a subsequence having this property, such that every a is divisible by at least one a of the subsequence. The condition on $f(n)$ will remain valid, and the sequence of b's will not be affected.

Let $\epsilon_1, \epsilon_2, \cdots$ be a decreasing sequence of positive numbers, tending to 0 sufficiently rapidly, and let $n_r = n_r(\epsilon_r)$ be a positive integer which we shall suppose later to tend to infinity sufficiently rapidly. We suppose that $n_r^{1-\epsilon_r} > n_{r-1}$ for all r. We define the a's to consist of all integers in the interval $(n_r^{1-\epsilon_r}, n_r)$ which have all their prime factors greater than $n_r^{\epsilon_r^2}$, for $r = 1, 2, \cdots$.

We have first to estimate $f(m)$, the number of a's not exceeding m. Let r be the largest suffix for which $n_r^{1-\epsilon_r} \leqq m$. If $m \geqq n_r^2$, then clearly

$$f(m) < n_r \leqq m^{1/2} < \frac{m}{\log m} \cdot$$

Suppose, then, that $m < n_r^2$. We have

$$f(m) < n_{r-1} + M_\epsilon(m),$$

where $M_\epsilon(m)$ denotes the number of integers not exceeding m which have all their prime factors greater than $m^{\epsilon_r^2/2}$. By Brun's[3] method we obtain

$$M_\epsilon(m) < c_1 m \sum_{p \leqq m^{\epsilon_r^2/2}} (1 - p^{-1}) < c_2 \frac{m}{\epsilon_r^2 \log m},$$

where c_1, c_2, denote positive absolute constants. Hence

$$f(m) < n_{r-1} + c_2 \frac{m}{\epsilon_r^2 \log m} < \frac{m\psi(m)}{\log m}$$

---

[3] P. Erdös and M. Kac, Amer. J. Math. vol. 62 (1940) pp. 738–742.

provided $n_r(\epsilon_r)$ is sufficiently large. It will suffice if

$$\frac{c_2}{\epsilon_r^2} < \frac{1}{2}\psi(n_r^{1-\epsilon_r}).$$

We have now to prove that the sequence of b's (the multiples of the a's) have no density. Denote by $A(\epsilon, n)$ the density of the sequence of all integers which have at least one divisor in the interval $(n^{1-\epsilon}, n)$. In a previous paper[4] I proved that $A(\epsilon, n) \to 0$ if $\epsilon \to 0$ and $n \to \infty$ independently. Thus if $\epsilon \to 0$ and $n \to \infty$ sufficiently fast, we have

$$\sum_{r=1}^{\infty} A(\epsilon_r, n_r) < \frac{1}{2}\cdot \tag{2}$$

Denote the number of b's not exceeding m by $B(m)$. It follows from (2) that if $n_r \to \infty$ sufficiently rapidly, and $m = n_r^{1-\epsilon_r}$, then

$$B(m) < m/2. \tag{3}$$

This proves that the lower density of the b's is at most $1/2$.

Next we show that the upper density of the b's is 1, and this will complete the proof of Theorem 1. It suffices to prove that

$$n_r - B(n_r) = o(n_r), \tag{4}$$

in other words that the number of integers up to n_r which are not divisible by any a is $o(n_r)$. Consider any integer t satisfying $n_r^{1-\epsilon_r/2} < t \leq n_r$, and define

$$(g_{\epsilon_r}(t)) = g_r(t) = \prod_p{}' p^\alpha,$$

where the dash indicates that the product is extended over all primes p with $p \leq n_r^{\epsilon_r^2}$, and p^α is the exact power of p dividing t.

If $g_r(t) < n^{\epsilon_r/2}$, then t is divisible by an a, since $t/g_r(t) > n^{1-\epsilon_r}$ and $t/g_r(t)$ has all its prime factors greater than $n_r^{\epsilon_r^2}$, and so is an a. Hence

$$n_r - B(n_r) < n_r^{1-\epsilon_r/2} + C(n_r), \tag{5}$$

where $C(n_r)$ denotes the number of integers $t \leq n_r$ for which $g_r(t) \geq n_r^{\epsilon_r/2}$. We recall that the exact power of a prime p dividing $N!$ is

$$\sum_{r=1}^{\infty}\left[\frac{N}{p^r}\right] < \sum_{r=1}^{\infty}\frac{N}{p^r} = \frac{N}{p-1}\cdot$$

Hence

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[4] J. London Math. Soc. vol. 11 (1936) pp. 92–96.

$$\prod_{t=1}^{n_r} g_r(t) \leqq \prod_{p \leqq n\epsilon_r^2} p^{n_r/p-1} = \exp\left(n_r \sum_{p \leqq n\epsilon_r^2} \frac{\log p}{p-1}\right.$$
$$< \exp\,(c_3\epsilon_r^2 n_r \log n_r) = n_r^{c_3\epsilon_r^2 n_r}.$$

Hence $(n_r^{\epsilon_r/2})^{C(n_r)} < n_r^{c_3\epsilon_r^2 n_r}$, whence

(6) $\quad C(n_r) < 2c_3\epsilon_r n_r$.

Substituted in (5), this proves (4), provided that $n_r^{\epsilon_r} \to \infty$, which we may suppose to be the case. This completes the proof of Theorem 1.

THEOREM 2. *A necessary and sufficient condition that the b's shall have a density is that* (1) *shall hold.*

PROOF. The necessity is easily deduced from an old result. Davenport and I[3] proved that the logarithmic density of the b's exists and has the value

$$\lim_{i\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{j\leqq i} \phi(n; a_j; a_1, \cdots, a_{j-1}).$$

Thus if the density of the b's exists, we obtain

$$\lim_{i\to\infty} \lim_{n\to\infty} \frac{1}{n} \sum_{j>i} \phi(n; a_j; a_1, \cdots, a_{j-1}) = 0.$$

This proves the necessity of (1).

The proof of the sufficiency is much more difficult. We have

$$B(n) = \sum_{a_i \leqq n} \phi(n; a_i; a_1, \cdots, a_{i-1}) = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3,$$

where $\sum_1$ is extended over $a_i \leqq A$, $\sum_2$ over $A < a_i \leqq n^{1-\epsilon}$, $\sum_3$ over $n^{1-\epsilon} < a_i \leqq n$. Here $A = A(n)$ will be chosen later to tend to infinity with n. By the hypothesis (1) we have

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \sum\nolimits_3 = 0. \tag{7}$$

It follows from the earlier work[3] that if $A = A(n)$ tends to infinity sufficiently slowly, then $(1/n)\sum_1$ has a limit, this limit being the logarithmic density of the b's, and also

$$\lim_{j\to\infty}\left(\sum_{i\leqq j} \frac{1}{a_i} - \sum_{i_1<i_2\leqq j} \frac{1}{[a_{i_1}, a_{i_2}]} + \cdots\right).$$

Thus the proof of Theorem 2 will be complete if we are able to prove that

$$(8)\qquad \frac{1}{n}\sum\nolimits_2 = \lim_{\epsilon\to 0}\ \limsup_{n\to\infty}\frac{1}{n}\sum_{A<a_i\leqq n^{1-\epsilon}}\phi(n;\, a_i;\, a_1, \cdots, a_{i-1}) = 0.$$

We have

$$\phi(n;\, a_i;\, a_1, \cdots, a_{i-1}) = \phi\left(\frac{n}{a_i},\ 1;\, d_1^{(i)} \cdots\right),$$

where

$$d_j^{(i)} = \frac{a_j}{(a_i,\, a_j)}\,.$$

We shall prove that

$$(9)\qquad \lim_{\epsilon\to 0}\ \limsup_{n\to\infty}\frac{1}{n}\sum_{A<a_i\leqq n^{1-\epsilon}}\phi'\left(\frac{n}{a_i};\, 1;\, d_1^{(i)} \cdots\right) = 0$$

where the dash indicates that we retain only those $d_j^{(i)}$ which satisfy $d_j^{(i)} < n^{\epsilon^2}$. Clearly (8) follows from (9). (Since $n^{\epsilon^2}\to\infty$, not all the $d_j^{(i)}$ are greater than or equal to n^{ϵ^2}.)

We define $g_\epsilon(t)$ as before, with n in place of n_r and ϵ in place of ϵ_r. It follows from (5) and (6) that it will suffice to prove that

$$(10)\qquad \lim_{\epsilon\to 0}\ \limsup_{n\to\infty}\frac{1}{n}\sum_{A<a_i\leqq n^{1-\epsilon}}\phi''\left(\frac{n}{a_i};\, 1;\, d_1^{(i)} \cdots\right) = 0,$$

where $\phi''(n/a_i;\, 1;\, d_1^{(i)} \cdots)$ denotes the number of integers m satisfying

$$(11)\qquad m \leqq \frac{n}{a_i};\qquad m \not\equiv 0 \pmod{d_j^{(i)}},\quad d_j^{(i)} < n^{\epsilon^2};\qquad g_\epsilon(m) < n^{\epsilon/2}.$$

Consider the integers satisfying (11). They are of the form $u\cdot v$ where $u<n^{\epsilon/2}$ and all prime factors of u are less than n^{ϵ^2}, $u\not\equiv 0 \pmod{d_j^{(i)}}$ for $d_j^{(i)}<n^{\epsilon^2}$, and all prime factors of v are greater than n^{ϵ^2}. We obtain by Brun's method[3] that the number of integers $m\leqq n/a_i$ with fixed u does not exceed $(n/u\cdot a_i>n^{\epsilon/2})$

$$(12)\qquad c_4\frac{n}{a_i u}\prod_{p<n^{\epsilon^2}}(1-p^{-1}).$$

Thus the number N_i of integers satisfying (11) does not exceed

$$(13)\qquad c_4\frac{n}{a_i}\sum{}'\frac{1}{u}\prod_{p<n^{\epsilon^2}}(1-p^{-1})\geqq\phi''\left(\frac{n}{a_i};\, 1;\, d_1^{(i)} \cdots\right),$$

where the dash indicates that the summation is extended over the $u<n^{\epsilon/2}$, $u\not\equiv 0 \pmod{d_j^{(i)}}$, $d_j^{(i)}<n^{\epsilon^2}$ and all prime factors of u are less than n^{ϵ^2}.

We have to estimate $\sum N_i$. Put

$$\lim_{m\to\infty}\frac{1}{m}\,\phi\left(\frac{m}{a_i}\,;\,1;\,d_1^{(i)}\,\cdots\right)=t_i,\tag{14}$$

where in (14) all the $d_j^{(i)}$ are considered. (It follows from the definition of the $d_j^{(i)}$ that they are all less than n. Thus the limit (14) exists.) It follows from our earlier work[2] that

$$\sum_{a_i>A} t_i = o(1).\tag{15}$$

Next we estimate t_i' where

$$t_i' = \lim_{m\to\infty}\frac{1}{m}\,\phi\,(\frac{m}{a_i};\,1;\,d_j^{(i)}),\qquad d_j^{(i)} < n^{\epsilon^2}.$$

Here we use the following result of Behrend[3]

$$\lim_{n\to\infty}\frac{1}{n}\,\phi(n;\,1;\,a_1,\,\cdots,\,a_i,\,b_1,\,\cdots,\,b_s)$$
$$\geqq \lim_{n\to\infty}\frac{1}{n^2}\,\phi(n;\,1;\,a_1,\,\cdots,\,a_s)\cdot\phi(n;\,1;\,b_1,\,\cdots,\,b_s)\,\cdots.$$

Thus clearly

$$t_i' \leqq t_i\left(\lim_{m\to\infty}\frac{1}{m}\,\phi(m;\,1;\,x_r)\right)^{-1} = t_i/t_i'',\tag{16}$$

where x_r runs through the integers from n^{ϵ^2} to n. It follows from the Sieve of Eratosthenes that the density of integers with $g_\epsilon(m)=k$ equals

$$\frac{1}{k}\prod_{p<n^{\epsilon^2}}(1-p^{-1}).$$

Thus clearly

$$t_i'' \geqq \sum_{k<n^{\epsilon^2}}\ \frac{1}{k}\prod_{p\leqq n}(1-p^{-1}) > c_8\epsilon^2$$

or

$$t_i' \leqq t_i/c_8\epsilon^2.\tag{17}$$

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[3] Bull. Amer. Math. Soc. vol. 54 (1948) pp. 681–684.

Thus from (15) and (17),

$$\sum_{a_i > A} t_i = o(1). \tag{18}$$

We have by the Sieve of Eratosthenes

$$t_i' = \frac{1}{a_i} \sum{}' \frac{1}{x} \prod_{p < n^{\epsilon^2}} (1 - p^{-1}) \tag{19}$$

where the dash indicates that $x \not\equiv 0 \pmod{d_j^{(i)}}$ $d_j^{(i)} < n^{\epsilon^2}$ and all prime factors of x are less than n^{ϵ^2}. Comparing (13) and (19) we obtain

$$N_i < c_4 t_i' n. \tag{20}$$

Thus finally from (10) and (18) we obtain $\sum_{a_i > A} N_i = o(n)$ which proves (10) and completes the proof of Theorem 2.

THEOREM 3. *The density of integers having two divisors d_1 and d_2 with $d_1 < d_2 < 2d_1$ exists.*

PROOF. Define a sequence $a_1, a_2, \cdots$ of integers as follows: An integer m is an a if m has two divisors d_1 and d_2 with $d_1 < d_2 < 2d_1$, but no divisor of m has this property. To prove Theorem 3 it will be sufficient to show that the multiples of the a's have a density. Thus by Theorem 2 we only have to show that (1) is satisfied. We shall only sketch the proof.

Clearly the a's are of the form xy, where $x < y < 2x$. Thus it will be sufficient to show that the number of integers $m \leqq n$ having a divisor in the interval $(n^{1/2-\epsilon}, n^{1/2})$ is less than ηn where $\eta \to 0$ as $\epsilon \to 0$. But I proved that the density $c_{\epsilon,t}$ of integers having a divisor in $(t, t^{1+\epsilon})$ satisfies

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} c_{\epsilon,t} = 0.$$

A similar argument will prove the above result, and so complete the proof of Theorem 3.

It can be shown that the density of integers having two divisors d_1 and d_2 with $d_1 < d_2 \leqq 2d_1$ and either d_1 or d_2 a prime exists and is less than 1. This result is not quite trivial, since if we denote by $a_1 < a_2 < \cdots$ the sequence of those integers having this property and such that no divisor of any a has this property, then $\sum 1/a_i$ diverges.

We now state a few unsolved problems.

I. Besicovitch[1] constructed a sequence $a_1 < a_2 < \cdots$ of integers such that no a divides any other, and the upper density of the a's is positive. A result of Behrend[6] states that

$$\lim \frac{1}{\log n} \sum_{a_i \leqq n} \frac{1}{a_i} = 0 \tag{21}$$

and I[7] proved that

$$\sum \frac{1}{a_i \log a_i} < A \tag{22}$$

where A is an absolute constant. It follows from the last two results that the lower density of the a's must be 0. In fact Davenport and I[8] proved the following stronger result: Let $d_1 < d_2 < \cdots$ be a sequence of integers of positive logarithmic density, then there exists an infinite subsequence $d_{i_1} < d_{i_2} < \cdots$ such that $d_{i_j} \mid d_{i_{j+1}}$. Let now $f_1 < f_2 < \cdots$ be a sequence of positive lower density. Can we always find two numbers f_i and f_j with $-f_i \nmid f_j$ and so that $[f_i | f_j]$ also belongs to the sequence? This would follow if the answer to the following purely combinatorial conjecture is in the affirmative: Let c be any constant and n large enough. Consider $c2^n$ subsets of n elements. Then there exist three of these subsets B_1, B_2, B_3 such that B_3 is the union of B_1 and B_2.

II. Let $a_1 < a_2 < \cdots$ be a sequence of real numbers such that for all integers k, i, j we have $|ka_i - a_j| \geqq 1$. Is it then true that $\sum 1/a_i \log a_i$ converges and that $\lim (1/\log n) \sum_{a_i < n} 1/a_i = 0$? If the a's are all integers the condition $|ka_j - a_i| \geqq 1$ means that no a divides any other, and in this case our conjectures are proved by (21) and (22).

III. Let $a_1 < a_2 < \cdots \leqq n$ be any sequence of integers such that no one divides any other, and let $m > n$. Denote by $B(m)$ the number of b's not exceeding m. Is it true that

$$\frac{B(m)}{m} > \frac{1}{2} \frac{B(n)}{n} ?$$

It is easy to see that the constant 2 can not be replaced by any smaller one. (Let the a's consist of a_1 and $n = a_1$, $m = 2a_1 - 1$.)

I was unable to prove or disprove any of these results.

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[6] J. London Math. Soc. vol. 10 (1935) pp. 42–44.

[7] Ibid. vol. 10 (1935) pp. 126–128.

THE CRITICAL NUMBERS FOR UNSYMMETRICAL APPROXIMATION

RAPHAEL M. ROBINSON

1. Statement of the main theorem. If ξ is an irrational number, then the modulus of approximability from the right, $M^+(\xi)$, is defined as the least upper bound of the values of μ for which the inequality

$$0 < \frac{A}{B} - \xi < \frac{1}{\mu B^2}$$

has infinitely many solutions. In a similar way, $M^-(\xi)$ is defined measuring the approximability of ξ from the left.

The number ξ is called *critical* if there is no other irrational number ξ' for which

$$M^+(\xi') < M^+(\xi), \qquad M^-(\xi') < M^-(\xi).$$

That is, roughly speaking, ξ is called critical if there is no other number which is harder to approximate both from the right and from the left. The purpose of this paper is to give a necessary and sufficient condition that ξ be critical.

A few definitions are necessary before stating the main theorem. A sequence of non-negative integers $r_1, r_2, r_3, r_4, \cdots$ will be called *derivable* if

$$\liminf r_n + 1 = \limsup r_n < \infty,$$

that is, if ultimately just two different numbers occur in the sequence, and these are consecutive integers k and $k+1$. In this case, the sequence has the form

$$r_1, r_2, \cdots, r_\nu, k, (k+1)^{s_1}, k, (k+1)^{s_2}, k, (k+1)^{s_3}, k, \cdots,$$

the exponents denoting repetition of the term $k+1$. Here the s_n are non-negative integers; if there are consecutive k's in the sequence, then we must take some of the s_n equal to zero.

We shall call $s_1, s_2, s_3, \cdots$ the derived sequence. Together with k, this derived sequence determines the end of the primitive sequence:

$$k, (k+1)^{s_1}, k, (k+1)^{s_2}, k, (k+1)^{s_3}, k, \cdots.$$

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It will be convenient in the proof to know that the preceding term r_v is also either k or $k+1$. For definiteness, we may suppose that v is chosen as small as possible so that this is true.

If the derived sequence is derivable, its derivative will be called the second derivative of the primitive sequence, and likewise for higher derivatives. The primitive sequence itself will be called the zeroth derivative.

MAIN THEOREM. *The number ξ is critical if and only if its expansion as a continued fraction has the form*

$$\xi = [q_0, q_1, \cdots, q_{u-1}, 1, r_1, 1, r_2, 1, r_3, 1, \cdots],$$

where the sequence $r_1, r_2, r_3, \cdots$ either has infinitely many derivatives, or its highest derivative is a sequence

$$t_1, t_2, t_3, \cdots$$

such that $t_n \to k$, where k is a positive integer, or $t_n \to \infty$.

2. **Discussion of the theorem.** By $M(\xi)$ is meant the least upper bound of numbers μ such that the inequality

$$\left| \frac{A}{B} - \xi \right| < \frac{1}{\mu B^2}$$

has infinitely many solutions. Thus

$$M(\xi) = \max \{ M^+(\xi), M^-(\xi) \}.$$

The condition that ξ be critical is strikingly similar to the known condition that $M(\xi) \leqq 3$. We have indeed $M(\xi) \leqq 3$ *if and only if the continued fraction expansion of ξ has the form*

$$\xi = [q_0, q_1, \cdots, q_{u-1}, r_1, r_1, r_2, r_2, r_3, r_3, \cdots],$$

where each r_n is either 1 or 2, and the sequence $r_1, r_2, r_3, \cdots$ satisfies the same conditions as in the theorem.[1] Despite the similarity of the results, no connection is apparent between the problems, and the methods used to establish the two results are different.

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[1] This result does not appear in the literature in exactly this form. However, it is closely related to results obtained by A. Markoff, *Sur les formes quadratiques indéfinies*, Math. Ann. vol. 15 (1879) pp. 381–406 and vol. 17 (1880) pp. 379–399. While the problem he considered was different, the methods and results can be carried over to the problem concerning $M(\xi) \leqq 3$. (Conversely, the main theorem of this paper could be carried over into a theorem concerning the numerically smallest positive and negative values of an indefinite quadratic form.) See also P. J. Heawood, *The classification of rational approximations*, Proc. London Math. Soc. (2) vol. 20 (1921) pp. 233–250.

In a previous paper,[2] which will be referred to as I, we showed how to determine the values of $M^+(\xi)$ and $M^-(\xi)$ from the continued fraction for ξ. Indeed, if

$$\xi = [q_0, q_1, q_2, q_3, \cdots],$$

and we set

$$\lambda_n = [q_{n+1}, q_{n+2}, q_{n+3}, \cdots] + [0, q_n, q_{n-1}, \cdots, q_1],$$

then

$$M^+(\xi) = \limsup \lambda_{2n+1}, \qquad M^-(\xi) = \limsup \lambda_{2n}.$$

The proof of the theorem will depend entirely on a study of the λ_n.

A special notation was introduced for a sum of two continued fractions, such as λ_n. We write

$$\lambda_n = [q_1, q_2, \cdots, q_n, [q_{n+1}], q_{n+2}, q_{n+3}, \cdots].$$

This expression will be called a *double continued fraction*, and will be said to be centered at q_{n+1}, or q_{n+1} will be called the marked term.

A knowledge of I, §§1–5, is required for the present paper. However, §§3–5 below are independent of I.

3. The average of two continued fractions. Suppose that the two real numbers ρ and σ can be expanded in continued fractions

$$\rho = [r_0, r_1, \cdots, r_{v-1}, \alpha],$$
$$\sigma = [r_0, r_1, \cdots, r_{v-1}, \beta],$$

which agree except for the last term, and in which all the terms are positive. In what form can we write the continued fraction for the average of ρ and σ? We shall put

$$(\rho + \sigma)/2 = [r_0, r_1, \cdots, r_{v-1}, \gamma],$$

and investigate the value of γ. It is clear that γ is between α and β. But we shall prove more, namely that γ *is between the harmonic and arithmetic means of* α *and* β:

$$\frac{2\alpha\beta}{\alpha+\beta} \leqq \gamma \leqq \frac{\alpha+\beta}{2}.$$

We see that for $v=0$, γ is equal to the arithmetic mean, and for $v=1$

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[2] R. M. Robinson, *Unsymmetrical approximation of irrational numbers*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 351–361. References to the work of Segre and others will be found there.

to the harmonic mean. Now suppose that the conclusion is known for some v, and prove it for $v+1$. We are given

$$\rho = [r_0, \cdots, r_{v-1}, r_v, \alpha] = [r_0, \cdots, r_{v-1}, r_v + \alpha^{-1}],$$
$$\sigma = [r_0, \cdots, r_{v-1}, r_v, \beta] = [r_0, \cdots, r_{v-1}, r_v + \beta^{-1}],$$

and are to express their average in the form

$$(\rho + \sigma)/2 = [r_0, \cdots, r_{v-1}, r_v, \gamma] = [r_0, \cdots, r_{v-1}, r_v + \gamma^{-1}].$$

We have, by the hypothesis of the induction,

$$\frac{2(r_v + \alpha^{-1})(r_v + \beta^{-1})}{(r_v + \alpha^{-1}) + (r_v + \beta^{-1})} \leqq r_v + \gamma^{-1} \leqq \frac{(r_v + \alpha^{-1}) + (r_v + \beta^{-1})}{2}.$$

The right side of the inequality is equivalent to $\gamma \geqq 2\alpha\beta/(\alpha+\beta)$, and the left side when simplified gives

$$\gamma \leqq \frac{\alpha + \beta + r_v\, 2\alpha\beta}{2 + r_v(\alpha + \beta)}.$$

Since the right side here is between the arithmetic and harmonic means of α and β, the desired conclusion follows.

REMARK. If we wish to "symmetrize" the double continued fraction

$$[\alpha, r_{v-1}, \cdots, r_1, [r_0], r_1, \cdots, r_{v-1}, \beta]$$

by replacing α and β by one and the same number γ, then the required value of γ is between the arithmetic and harmonic means of α and β.

4. Comparison of two double continued fractions.

There is no simple rule for deciding in general which of two double continued fractions is the larger. We shall however give a rule which applies in certain cases where the central parts of the continued fractions are symmetrical.

THEOREM. *Suppose that* $r_0, r_1, \cdots, r_{v-1}$, *and* s *are positive integers, and that* α, β, γ, *and* δ *are real numbers greater than* 1. *Let* R *stand for the sequence, including a marked term,*

$$r_{v-1}, \cdots, r_2, r_1, [r_0], r_1, r_2, \cdots, r_{v-1}.$$

Then if v *is even, we shall have*

$$[\gamma, 1, s, R, s - 1, 1, \delta] < [\alpha, 1, s, R, s, 1, \beta],$$

and provided $\alpha < 4s+3$ *and* $\beta < 4s+3$, *also*

$$[\gamma, 1, s, R, s + 1, 1, \delta] > [\alpha, 1, s, R, s, 1, \beta].$$

If v is odd, the inequalities are reversed.

Proof. It will be sufficient to consider the case in which v is even. To prove each inequality, we start with the left side, and apply the results of §3 to symmetrize.

The proof of the first inequality proceeds as follows:

$$[\gamma, 1, s, R, s - 1, 1, \delta] < [s + 1, R, s]$$
$$\leq [s + 1/2, R, s + 1/2] < [\alpha, 1, s, R, s, 1, \beta].$$

Here we symmetrized using the arithmetic mean. For the second inequality we have

$$\begin{aligned}[\gamma, 1, s, R, s + 1, 1, \delta] &> [s + 1/2, R, s + 3/2] \\ &\geqq [4s + 3, 1, s, R, s, 1, 4s + 3] \\ &> [\alpha, 1, s, R, s, 1, \beta].\end{aligned}$$

Here we symmetrized by introducing the harmonic mean; a simple calculation shows that the harmonic mean of $s+1/2$ and $s+3/2$ is $[s, 1, 4s+3]$.

We shall now prove a generalization of this theorem. The following lemma is needed.

Lemma. *If*

$$[\cdots, r_{n-1}, [r_n], r_{n+1}, r_{n+2}, \cdots] < [\cdots, r'_{n-1}, [r'_n], r'_{n+1}, r'_{n+2}, \cdots]$$

and

$$[r_{n+1}, r_{n+2}, \cdots] > [r'_{n+1}, r'_{n+2}, \cdots],$$

then

$$[\cdots, r_{n-1}, r_n, [r_{n+1}], r_{n+2}, \cdots] > [\cdots, r'_{n-1}, r'_n, [r'_{n+1}], r'_{n+2}, \cdots].$$

Proof. Let $\rho = [r_{n+1}, r_{n+2}, \cdots]$ and $\sigma = [r_n, r_{n-1}, \cdots]$, and similarly for ρ' and σ'. By hypothesis, $\rho > \rho'$; thus if $\sigma \leqq \sigma'$, the conclusion

$$\rho + 1/\sigma > \rho' + 1/\sigma'$$

is clear. On the other hand, if $\sigma > \sigma'$, we have

$$\rho + 1/\rho + \sigma + 1/\sigma > \rho' + 1/\rho' + \sigma' + 1/\sigma'.$$

Subtracting the inequality

$$\sigma + 1/\rho < \sigma' + 1/\rho',$$

which is true by hypothesis, we again have the desired conclusion.

Remark. Thus if the order of two double continued fractions is what would be expected by looking only at the part to the right of the marked term, then when the marked term is shifted one place to the right in each continued fraction, the order is reversed.

Using this lemma, we may generalize the preceding theorem. A remark is necessary concerning a notation used in the extended theorem. If $b_1, \cdots, b_s$ is a sequence of numbers, then for any x with $1 \leqq x \leqq s$ we may form the marked block

$$b_1, \cdots, b_{x-1}, [b_x], b_{x+1}, \cdots, b_s.$$

But B_x may also be interepreted for other values of x. For example, B_{s+1} has such a meaning that for any b, the block B_{s+1}, b is

$$b_1, b_2, \cdots, b_s, [b].$$

EXTENDED THEOREM. *Suppose that $r_0, r_1, \cdots, r_{v-1}$, and s are positive integers, and that α, β, γ, and δ are real numbers greater than* 1. *For $0 \leqq n \leqq v$, let R_n stand for the sequence, with a marked term at or to the right of the middle,*

$$r_{v-1}, \cdots, r_1, r_0, r_1, \cdots, [r_n], \cdots, r_{v-1}.$$

Then if $v-n$ is even, we shall have

$$[\gamma, 1, s, R_n, s-1, 1, \delta] < [\alpha, 1, s, R_n, s, 1, \beta],$$

and provided $\alpha < 4s+3$ and $\beta < 4s+3$, also

$$[\gamma, 1, s, R_n, s+1, 1, \delta] > [\alpha, 1, s, R_n, s, 1, \beta].$$

If $v-n$ is odd, the inequalities are reversed.

PROOF. For $n=0$, we have the preceding theorem. If the theorem is true for any n with $0 \leqq n \leqq v-1$, then it is also true for $n+1$, by the lemma.

5. Lemmas on derivable sequences.

Given a derivable sequence whose terms are ultimately a and $a+1$. If

$$b_1, b_2, \cdots, b_n$$

is a block of the derived sequence, then by the corresponding block of the primitive sequence we shall mean

$$a, (a+1)^{b_1}, a, (a+1)^{b_2}, a, \cdots, a, (a+1)^{b_n}, a.$$

The block in the primitive corresponding to a block in the mth derivative is found by repeated use of this rule. Notice that abutting blocks of a derivative correspond to overlapping blocks of the primitive. It is also clear that a symmetrical block in the mth derivative corresponds to a symmetrical block in the primitive.

Consider now a family of sequences, all having m derivatives ($m \geqq 1$), and all ultimately containing the same two numbers in the pth derivative, for each p with $0 \leqq p < m$. We wish to study the blocks in the primitives corresponding to certain types of blocks in the mth derivatives. It is understood that the primitive sequences occurring in the following two lemmas belong to a family of the type mentioned. The letters A and B will stand for blocks of terms.

LEMMA 1. *Suppose that the block b, B, b in an mth derivative corresponds to a block a, A, a in the primitive. Then if $c > b$ and $d > b$, blocks*

$$b,\ B,\ d; \qquad c,\ B,\ d; \qquad c,\ B,\ b$$

in mth derivatives will correspond respectively to blocks in the primitives including

$$a,\ A,\ a+1; \qquad a+1,\ A,\ a+1; \qquad a+1,\ A,\ a.$$

PROOF. Suppose first that $m = 1$. Then to the block b, B, b in the derivative corresponds the block

$$a,\ (a+1)^b,\ \cdots,\ (a+1)^b,\ a$$

in the primitive. To the block b, B, d in the derivative corresponds the block

$$a,\ (a+1)^b,\ \cdots,\ (a+1)^d,\ a$$

in the primitive, which includes the block

$$a,\ (a+1)^b,\ \cdots,\ (a+1)^b,\ a+1,$$

as required. The other two cases are similar. If $m > 1$, the proof follows easily by induction, using the result for $m = 1$.

LEMMA 2. *If a term k in an mth derivative corresponds to a block a, A, a in the primitive, then corresponding to any term greater than k in an mth derivative will be a block in the primitive including a, A, $a+1$.*

PROOF. If $m = 1$, then corresponding to a term k in the derivative is a block

$$a,\ (a+1)^k,\ a$$

in the primitive, and corresponding to a term greater than k is a block including

$$a,\ (a+1)^k,\ a+1.$$

If $m>1$, the result follows by induction, using Lemma 1.

6. **Terminology.** It will be sufficient to prove the main theorem for numbers ξ such that $M^+(\xi)\geqq M^-(\xi)$. From the results of I, §5, it is clear that if such a number is critical, then it has the form

$$\xi=[q_0, q_1, \cdots, q_{2u}, 1, r_1, 1, r_2, 1, r_3, 1, \cdots],$$

where the sequence $r_1, r_2, r_3, \cdots$ is either derivable or tends to a limit, finite or infinite. It follows furthermore that a number ξ of the above form is critical if the sequence of r's tends to a limit. Thus to complete the proof of the main theorem, we need consider only numbers ξ of the above form, with a derivable sequence of r's, that is, a sequence which ultimately contains just two numbers, k and $k+1$. Furthermore, a number of this type is critical unless there is a number ξ' also of this type (and indeed with the same k), such that $M^+(\xi')<M^+(\xi)$ and $M^-(\xi')<M^-(\xi)$.

Corresponding to the number ξ, we have a sequence of λ's (see §2). Any λ which is centered at a term r_n will be called a *major* λ, whereas a λ which is centered at a unit adjacent to a term r_n will be called a *minor* λ. Notice that $M^+(\xi)$ is the limit superior of the major λ's, and $M^-(\xi)$ is the limit superior of the minor λ's. If the sequence of r's has an mth derivative, then any major λ centered at a term r_n included in the block corresponding to a given term of the mth derivative will be considered as belonging to this term, and likewise for a minor λ centered at a unit adjacent to such a term r_n. Notice that the same λ may belong to more than one term of the mth derivative.

We shall now classify numbers ξ of the above form according to properties of the sequence $r_1, r_2, r_3, \cdots$. If this sequence has m derivatives ($m\geqq 1$), and for $0\leqq p<m$ the pth derivative contains the numbers a_p and a_p+1, then ξ will be said to be of class

$$(a_0, a_1, \cdots, a_{m-1}),$$

or of the class S, if S stands for this m-term sequence. If in addition the mth derivative is $t_1, t_2, t_3, \cdots$, where

$$\liminf t_n = k, \qquad \limsup t_n = l,$$

then ξ will be said to be of class

$$(a_0, a_1, \cdots, a_{m-1}; k, l),$$

or of class $(S; k, l)$. Here k and l are integers or ∞, and may have any values in the intervals $0\leqq k\leqq\infty$, $1\leqq l\leqq\infty$. Notice that if $0<p<m$, a number of class S is certainly also of class

$$(a_0, a_1, \cdots, a_{p-1}; a_p, a_p + 1).$$

7. **Fundamental inequality for $M^+(\xi)$.** Considering numbers of a fixed class S, where S is an m-term sequence, we ask how the major λ's depend on the terms of the mth derived sequence.

THEOREM. *Let S be an m-term sequence $(m \geqq 1)$, and consider numbers ξ of class S. Then every major λ belonging to a term l in an mth derived sequence is smaller than $L_1(S; l)$, whereas to any term larger than l in an mth derived sequence a major λ greater than $L_2(S; l)$ can be found, where $L_1(S; l)$ and $L_2(S; l)$ depend only on S and l, and*

$$L_2(S; l) > L_1(S; l).$$

PROOF. Corresponding to a term l in an mth derived sequence is a block a, B, a in the primitive, where B is symmetrical. Also, the block B has the form

$$b_1, b_2, \cdots, b_{s-1}, b_s,$$

where each b_x is either a or $a+1$. For $0 \leqq x \leqq s+1$, let B_x stand for the marked sequence

$$1, b_1, 1, b_2, 1, \cdots, 1, [b_x], 1, \cdots, 1, b_s, 1.$$

If $x=0$ or $x=s+1$, this is interpreted as in §4. It is clear that we may take

$$L_1(S; l) = [a + 2, 1, a, B_x, a, 1, a + 2],$$

for a suitable value of x in the interval $0 \leqq x \leqq s+1$. Since the block B is symmetrical, we may suppose $x \geqq (s+1)/2$.

On the hand, to a term larger than l in an mth derived sequence corresponds a block in the primitive including $a, B, a+1$ (§5, Lemma 2). Thus we may take

$$L_2(S; l) = [a, 1, a, B_x, a + 1, 1, a],$$

for any x which we like; we shall use the same of x as in defining $L_1(S; l)$. By the extended theorem of §4, we see that

$$L_2(S; l) > L_1(S; l).$$

COROLLARY. *If ξ is of class $(S; k, l)$ and ξ' is of class $(S; k', l')$ with $l' > l$, then $M^+(\xi') > M^+(\xi)$.*

8. **Fundamental inequality for $M^-(\xi)$.** Considering numbers of a fixed class S, where S is an m-term sequence, we ask how the minor λ's depend on the terms of the mth derived sequence.

THEOREM 1. *Let S be an m-term sequence ($m \geqq 1$), and consider numbers ξ of class S. Then every minor λ belonging to two consecutive terms, both greater than k, in an mth derived sequence is smaller than $K_1(S; k)$, whereas to any pair of consecutive terms in an mth derived sequence, of which the smaller is equal to k, a minor λ belonging to both and greater than $K_2(S; k)$ can be found, where $K_1(S; k)$ and $K_2(S; k)$ depend only on S and k, and*

$$K_2(S; k) > K_1(S; k).$$

PROOF. Corresponding to a hypothetical block k, k in an mth derivative, is a block a, C, a in the primitive, where C is symmetrical. The block C has the form

$$c_1, c_2, \cdots, c_{t-1}, c_t,$$

each c_x being either a or $a+1$. We may choose w so that just those terms c_x with $w+1 \leqq x \leqq t-w$ correspond to both of the k's. For $0 \leqq x \leqq t$, let C_x stand for the marked sequence

$$1, c_1, 1, c_2, 1, \cdots, 1, c_x, [1], c_{x+1}, 1, \cdots, 1, c_t, 1.$$

Now by §5, Lemma 1, to a block k_1, k_2 in an mth derivative, where $k_1 > k$ and $k_2 > k$, will correspond a block in the primitive including $a+1$, C, $a+1$. The terms of the primitive corresponding to both k_1 and k_2 are exactly $c_{w+1}, \cdots, c_{t-w}$. Hence we may take

$$K_1(S; k) = [a, 1, a+1, C_x, a+1, 1, a],$$

for some value of x with $w \leqq x \leqq t-w$. Because of the symmetry of C, we may suppose $x \geqq t/2$.

Now suppose that a pair of consecutive terms, k_0 and k, are given in an mth derivative, with $k_0 \geqq k$. We may suppose that k_0 precedes k. Then to the block k_0, k in the mth derivative there will correspond a block in the primitive including either $a+1$, C, a or a, C, a (§5, Lemma 1). The terms corresponding to both k_0 and k will again be $c_{w+1}, \cdots, c_{t-w}$. Hence we may take

$$K_2(S; k) = [a+2, 1, a+1, C_x, a, 1, a+2],$$

with the same x used before. The inequality

$$K_2(S; k) > K_1(S; k)$$

follows from the extended theorem of §4.

THEOREM 2. *Let S be an m-term sequence ($m \geqq 1$), and consider numbers ξ of class S. Then every minor λ belonging to a single term (but* *not the first term) of the mth derivative is smaller than* $K_1^*(S)$, *whereas to any pair of consecutive terms a corresponding minor* λ *can be found larger than* $K_2^*(S)$, *where* $K_1^*(S)$ *and* $K_2^*(S)$ *depend only on* S, *and*

$$K_2^*(S) > K_1^*(S).$$

Proof. Consider first the case $m=1$, and let $S=(a)$. Notice that a minor λ belonging to some term of the first derivative beyond the first term will belong to two consecutive terms of the first derivative if and only if it is centered at a unit adjacent to an a. Hence we may take

$$K_1^*(a) = [a, 1, a+1, [1], a+1, 1, a]$$

and

$$K_2^*(a) = [a+2, 1, a+1, [1], a, 1, a+2].$$

The inequality $K_2^*(a) > K_1^*(a)$ follows from the theorem of §4.

Now assume the theorem for numbers of class S, and prove for numbers of class (S, k), where (S, k) denotes the $(m+1)$-term sequence obtained by adjoining k to S. We shall show that we may take

$$K_1^*(S, k) = K_1(S; k), \qquad K_2^*(S, k) = K_2(S; k),$$

where $K_1(S; k)$ and $K_2(S; k)$ are quantities satisfying Theorem 1. Indeed, any λ belonging to a single term (not the first) of the $(m+1)$st derivative can belong only to terms $k+1$ of the mth derivative. By the inductive hypothesis, the largest such minor λ's belong to two consecutive such terms; and by Theorem 1, such λ's satisfy

$$\lambda < K_1(S; k).$$

On the other hand, any pair of consecutive terms of the $(m+1)$th derivative have in common all λ's belonging to a certain term k of the mth derivative, and hence by Theorem 1 there is certainly some minor λ belonging to both which satisfies

$$\lambda > K_2(S; k).$$

From Theorems 1 and 2, the following result is clear.

Corollary. *If* ξ *is of class* $(S; k, l)$ *and* ξ' *of class* $(S; k', l')$ *with* $k' < k$, *then* $M^-(\xi') > M^-(\xi)$.

9. **Proof of the main theorem.** Two more lemmas will be needed for the proof of the main theorem.

Lemma 1. *Let* S *be an* m*-term sequence* $(m \geqq 1)$. *If* ξ_1 *is of class*

$(S; k_1, l_1)$ and ξ_2 of class $(S; k_2, l_2)$, where $k_1 \leqq l_1 \leqq k_2 \leqq l_2$, then

$$M^+(\xi_1) \leqq M^+(\xi_2), \qquad M^-(\xi_1) \geqq M^-(\xi_2).$$

PROOF. If $k_1 = l_1 = k_2 = l_2$, then evidently

$$M^+(\xi_1) = M^+(\xi_2), \qquad M^-(\xi_1) = M^-(\xi_2).$$

Hence we may suppose that $k_1 < l_2$. It will be sufficient to show that if we are given terms j_1 and j_2, sufficiently far out in the mth derived sequences for ξ_1 and ξ_2, respectively, then the largest major λ belonging to j_2 is greater than any major λ belonging to j_1, whereas the largest minor λ belonging to j_1 is greater than any minor λ belonging to j_2. If $j_1 < j_2$, this follows for the major λ's from the theorem of §7, and for minor λ's from §8, Theorems 1 and 2.

Suppose now that $j_1 = j_2$. Match up the two mth derived sequences so that these terms are paired, and continue the pairing to the left and right until unequal terms are encountered. This will give us blocks g_1, J, h_1 and g_2, J, h_2 in the two mth derivatives, with j_1 and j_2 appearing at the same position in the block J, and with $g_1 < g_2$, $h_1 < h_2$. To the block g_1, J, h_1 in the mth derivative will correspond a block a, D, a in the primitive. By §5, Lemma 1, the block in the primitive corresponding to g_2, J, h_2 will include $a+1, D, a+1$. It is then clear that of a pair of major λ's, one for ξ_1 and the other for ξ_2, centered at the same position relative to the block D, the second will be the larger. For a similar pair of minor λ's, the first will be the larger. Thus, in particular, the desired relation is established between the λ's belonging to j_1 and j_2.

LEMMA 2. *Let S be an m-term sequence ($m \geqq 1$). If ξ is of class S, and*

$$M^+(\xi') < M^+(\xi), \qquad M^-(\xi') < M^-(\xi),$$

then ξ' is also of class S.

PROOF. For $S = (a)$, the result follows from I, §5. Now assume the theorem for numbers of class S, and prove for numbers of class (S, k), where (S, k) denotes the $(m+1)$-term sequence obtained by adjoining k to S. We are given a number ξ of class $(S, k) = (S; k, k+1)$. The inductive hypothesis shows that ξ' is of class S. The corollaries of §§ 7–8 show that ξ' must be of class $(S; k, k)$, $(S; k+1, k+1)$, or $(S; k, k+1)$. The first two possibilities are eliminated by Lemma 1. Hence ξ' must be of class $(S; k, k+1) = (S, k)$.

PROOF OF MAIN THEOREM. Suppose first that ξ has exactly m derived sequences ($m \geqq 1$), and is of class S, where S is an m-term sequence. To test whether ξ is critical, according to Lemma 2 we need only find whether there is a number ξ' of class S such that

$$M^+(\xi') < M^+(\xi), \qquad M^-(\xi') < M^-(\xi).$$

Now ξ must be of class $(S; k, l)$ for some k and l, and indeed $l=k$ or $l>k+1$, since ξ has exactly m derived sequences. If ξ' is of class $(S; k', l')$, then by the corollaries of §§7–8, we must have $k \leqq k' \leqq l' \leqq l$.

If $l=k\leqq \infty$, we must have $k=k'=l'=l$, whence

$$M^+(\xi') = M^+(\xi), \qquad M^-(\xi') = M^-(\xi).$$

In other words, no suitable ξ' can be chosen, hence ξ is critical. If $l>k+1$, any ξ' of class $(S; j, j)$ with $k<j<l$ serves as a comparison number to show that ξ is not critical. Thus when ξ has a last derived sequence, it is critical only when this sequence has a limit, finite or infinite.

Suppose on the other hand that ξ has infinitely many derived sequences. Let the numbers appearing in the pth derivative be a_p and a_p+1. Let ξ_m be any number of class

$$(a_0, a_1, \cdots, a_{m-1}; 1, 1).$$

Then ξ_m is critical. It is easily seen that

$$M^+(\xi_m) \to M^+(\xi), \qquad M^-(\xi_m) \to M^-(\xi).$$

Hence ξ is also critical.

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ON A CLASS OF PERFECT SETS

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Let $\{a_n\}$ be a sequence of positive numbers such that

$$\sum_1^\infty a_n = 1, \tag{1}$$

$$a_n \geqq R_n = \sum_{n+1}^\infty a_k \qquad (n = 1, 2, 3, \cdots). \tag{2}$$

Also let S be the set of numbers α defined by

$$\begin{gathered} \alpha = \sum_1^\infty a_n' \\ a_n' = 0 \quad \text{or} \quad a_n \qquad (n = 1, 2, 3, \cdots). \end{gathered} \tag{3}$$

Then we have the following:

THEOREM 1. *The set S is perfect; two series of the form* (3) *have the same value if and only if they are of the forms*

$$z_1 + z_2 + \cdots + z_{k-1} + 0 + a_{k+1} + a_{k+2} + \cdots, \tag{4}$$

$$z_1 + z_2 + \cdots + z_{k-1} + a_k + 0 + 0 + \cdots \tag{5}$$

where $z_i = 0$ or a_i $(i = 1, 2, \cdots, k-1)$ and $a_k = R_k$; if $a_k > R_k$ then no number of the set S lies between two numbers (4) *and* (5); *every number of the closed interval* (0, 1) *other than those between pairs of numbers of the forms* (4) *and* (5) *belongs to S; if among the relations* (2) *there are an infinity of strict inequalities then the set S is totally disconnected; if all but a finite number of the relations* (2) *are equalities then the set S consists of a finite number of closed intervals; and, finally, the measure of the set S is $\lim_{n\to\infty} 2^n R_n$.*

It is obvious that all numbers of the set S lie in the interval (0, 1). It is also obvious that when $a_k = R_k$ the series (4) and (5) have the same value. Let us now suppose that the kth terms are the first which differ in two series of the form (3) having the same value. It is clear that the kth term in one of the series is 0, and in the other a_k. Subtracting the former series from the latter we get

$$a_k - \sum \pm a_{i_n} = 0 \tag{6}$$

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where the suffixes i_n run through the whole or a subset of the set of numbers $k+1, k+2, \cdots$. But from (2) we have

$$a_k \geqq \sum_{k+1}^{\infty} a_n. \tag{7}$$

This can be compatible with (6) only if the suffixes i_n run exactly through the whole set of numbers $k+1, k+2, \cdots$, all the signs under $\sum$ in (6) are positive, and (7) is an equality. This implies that the two series having the same value are of the forms (4) and (5) with $a_k = R_k$.

If $a_k > R_k$ then the series (5) is obviously greater than (4). Also, of all series of the form (3) whose first k terms are the same as those in (4) the latter is the greatest, and of all series whose first k terms are the same as those in (5) the latter is the least. Hence if the value α of a series lies between (4) and (5) then the first $k-1$ terms of the series cannot all be the same as those of (4) and (5). Let, therefore, the lth term, $l \leqq k-1$, be the first in α to differ from those in (4) and (5). Then α will be greater than (5) if $s_l = 0$ and less than (4) if $s_l = a_l$, which contradicts the assumption that α lies between (4) and (5).

Let us suppose that α is any number in the interval $(0, 1)$ which does not lie between two series of the forms (4) and (5). If α is the sum of a finite number of distinct a_n's then it is obviously a number of the set S. If that is not the case, then let, in the first instance, α lie between 1 and a_1. From (1) it follows that there is a number ν_1 greater than 1 such that

$$a_1 + a_2 + \cdots + a_{\nu_1 - 1} < \alpha < a_1 + a_2 + \cdots + a_{\nu_1}; \tag{8}$$

but since α does not lie between

$$a_1 + a_2 + \cdots + a_{\nu_1} + 0 + 0 + \cdots$$

and

$$a_1 + a_2 + \cdots + a_{\nu_1 - 1} + 0 + a_{\nu_1 + 1} + a_{\nu_1 + 2} + \cdots$$

the right-hand inequality in (8) can be sharpened into

$$\alpha \leqq a_1 + a_2 + \cdots + a_{\nu_1 - 1} + 0 + a_{\nu_1 + 1} + a_{\nu_1 + 2} + \cdots. \tag{9}$$

If (9) is an equality then α is a number of the set S; if it is an inequality then there exists a suffix ν_2 (>2) such that

$$\begin{aligned} &a_1 + \cdots + a_{\nu_1 - 1} + 0 + a_{\nu_1 + 1} + \cdots + a_{\nu_2 - 1} \\ &\qquad < \alpha < a_1 + \cdots + a_{\nu_1 - 1} + 0 + a_{\nu_1 + 1} + \cdots + a_{\nu_2}. \end{aligned} \tag{10}$$

It is, of course, supposed that if $\nu_2=\nu_1+1$, then the first member of the inequalities (10) is just $a_1+\cdots+a_{\nu_1-1}$. As before we may sharpen the right-hand inequality in (10) into

$$\begin{aligned}\alpha \leqq a_1+\cdots+a_{\nu_1-1}+0+a_{\nu_1+1}+\cdots+a_{\nu_2-1}+0 \\ +a_{\nu_2+1}+a_{\nu_2+2}+\cdots.\end{aligned} \tag{11}$$

If (11) is an equality it follows that α is a number of the set S; otherwise, we can find another number ν_3 $(>\nu_2)$ such that

$$\begin{aligned}a_1+\cdots+a_{\nu_1-1}+0+a_{\nu_1+1}+\cdots+a_{\nu_2-1}+0+a_{\nu_2+1}+\cdots+a_{\nu_3-1} \\ <\alpha<a_1+\cdots+a_{\nu_1-1}+0+a_{\nu_1+1}+\cdots+a_{\nu_2-1}+0 \\ +a_{\nu_2+1}+\cdots+a_{\nu_3}.\end{aligned}$$

Proceeding in this manner we either arrive at an expression for α of the form (3) after a finite number of steps, or enclose α between narrower and narrower bounds tending to the common limit α. This common limit is, by the nature of the construction of the bounds, a series of the form (3) so that α is a number of the set S. If, on the other hand, $a_1>\alpha$, then there exists a suffix i (>1) such that

$$a_{i+1}<\alpha<a_i$$

since a_n decreases to zero as n tends to infinity. We proceed by sharpening the right-hand inequality into

$$\alpha \leqq 0+a_{i+1}+a_{i+2}+\cdots$$

and complete the proof that α belongs to S exactly as above.

Since for a given k the number of pairs of numbers of the forms (4) and (5) is finite, the set of all such pairs of numbers is enumerable and therefore the numbers of the interval (0, 1) not contained in S are those of an enumerable set of disjoint open intervals. This completes the proof that S is perfect.

If all but a finite number of the relations (2) are equalities it is obvious that S consists of a finite number of closed intervals. If, on the other hand, an infinity of the relations (2) are inequalities, then the set is totally disconnected. For, let α be any number of the set S, say,

$$\alpha=b_1+b_2+b_3+\cdots,\quad b_n=0 \quad\text{or}\quad a_n \quad (n=1,2,3,\cdots).$$

Given $\epsilon>0$, arbitrarily small, choose n so large that

$$R_n=a_{n+1}+a_{n+2}+\cdots<\epsilon.$$

By hypothesis there exists a number $k>n$ such that $a_k>R_k$. The open interval I whose end points are

$$b_1 + \cdots + b_{k-1} + 0 + a_{k+1} + a_{k+2} + \cdots ,$$
$$b_1 + \cdots + b_{k-1} + a_k + 0 + 0 + \cdots$$

is contained in the interval $(\alpha-\epsilon, \alpha)$ or in the interval $(\alpha, \alpha+\epsilon)$ according as $b_k = a_k$ or 0. It follows that there are numbers arbitrarily close to α and not belonging to S which proves that S is totally disconnected.

We may write the relations (2) in the form

$$R_{n-1} \geqq 2R_n \qquad (n = 1, 2, \cdots)$$

from which it follows that

$$1 \geqq 2R_1 \geqq 2^2R_2 \geqq 2^3R_3 \geqq \cdots \tag{12}$$

and hence that $\lim_{n\to\infty} 2^nR_n$ exists. We shall now show that this limit is the measure of the set S.

In fact the length of the open interval whose end points are (4) and (5) is obviously

$$a_k - \sum_{k+1}^{\infty} a_n = R_{k-1} - 2R_k;$$

also, for a given k, the number of pairs of numbers of the forms (4) and (5) is 2^{k-1}; therefore the total length of such intervals is

$$2^{k-1}(R_{k-1} - 2R_k);$$

summing over all k from 1 to n we get

$$\sum_{1}^{n} (2^{k-1}R_{k-1} - 2^kR_k) = 1 - 2^nR_n;$$

finally, letting n tend to infinity we get the measure of the complement of the set S from which it follows that the measure of S is $\lim_{n\to\infty} 2^nR_n$.

If the relations (2) are all equalities then the relations (12) are also all equalities and we get

$$a_n = R_n = 1/2^n \qquad (n = 1, 2, \cdots).$$

In this case the set S consists of the whole of the interval (0, 1) and the representation of the numbers of S by series of the form (4) is the ordinary binary representation. If the relations (2) are not all equalities then the relations (12) are also not all equalities and so $\lim_{n\to\infty} 2^nR_n < 1$.

It is easy to construct sets of the type S having any given measure α, $0 \leqq \alpha < 1$. In fact, let $\{s_n\}$ be a monotonic decreasing sequence tending to α with $s_0 = 1$, and let the sequence $\{a_n\}$ be defined by the relations

$$a_n = \frac{s_{n-1}}{2^{n-1}} - \frac{s_n}{2^n} \qquad (n = 1, 2, \cdots).$$

It is clear that the a_n's satisfy conditions (1) and (2) and that the set S constructed with these a_n's has the measure α.

Kakeya has proved[1] that if $\{a_n\}$ is a sequence of positive numbers satisfying the relations

$$\sum_1^\infty a_n = s, \tag{13}$$

$$a_n \leqq R_n, \qquad a_n \geqq a_{n+1} \qquad (n = 1, 2, 3, \cdots) \tag{14}$$

then the set S of all numbers of the form (3) consists of the whole interval $(0, s)$.

Let us now suppose that $\{a_n\}$ is a sequence of positive numbers satisfying the relations (1) and

$$a_n \geqq R_n \qquad (n = 1, 2, \cdots, k-1), \tag{15}$$

$$a_n \leqq R_n, \qquad a_n \geqq a_{n+1} \qquad (n = k, k+1, \cdots), \tag{16}$$

and consider the set S of all numbers of the form (3). If all the relations (15) are equalities then (15) and (16) together reduce to a particular case of Kakeya's relations (14) and the set S will be the whole of the interval $(0, 1)$. Let us therefore suppose that the relations (15) are not all equalities. It can then be shown that S consists of a finite number of disjoint closed intervals.

To prove this let us observe that by Kakeya's Theorem the numbers of the form

$$\sum_k^\infty a_n', \qquad a_n' = 0 \text{ or } a_n,$$

fill the whole interval $I(0, R_k)$. The set S will therefore consist of I and the intervals obtained by moving I towards the right through the distances

$$a_1' + a_2' + \cdots + a_{k-1}', \quad a_i' = 0 \text{ or } a_i \qquad (i = 1, 2, \cdots, k-1).$$

The interval I and those obtained by the translations have no points in common except perhaps end points, and they do not cover the entire interval (0, 1).

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[1] G. Pólya and G. Szegö, *Aufgaben*, Part 1, no. 131.

In fact, if one of the relations (15) is an inequality, say, $a_l > R_l$, then, as in our previous discussion, it can be seen that there is no number of the set within the interval whose end points are

$$z_1 + \cdots + z_{l-1} + 0 + a_{l+1} + a_{l+2} + \cdots,$$
$$z_1 + \cdots + z_{l-1} + a_l + 0 + 0 + \cdots.$$

It is also not difficult to see that the only gaps in the set S are those which arise in this manner.

For, if we arrange the numbers

$$a_1' + a_2' + \cdots + a_{k-1}', \qquad a_i' = 0 \text{ or } a_i \, (i = 1, 2, \cdots, k-1)$$

in increasing order of magnitude, then any consecutive pair will be of the form

$$\alpha = z_1 + z_2 + \cdots + z_{l-1} + 0 + a_{l+1} + a_{l+2} + \cdots + a_{k-1},$$
$$\beta = z_1 + z_2 + \cdots + z_{l-1} + a_l + 0 + 0 + \cdots + 0$$

where $z_i = 0$ or a_i $(i = 1, 2, \cdots, l+1)$, $l \leqq k-1$. Moving the interval I through the distances α and β we get the two intervals $(\alpha, \alpha + R_k)$ and $(\beta, \beta + R_k)$. The right-hand end point of the former is clearly less then or equal to the left-hand end point of the latter according as a_l is greater than or equal to R_l.

As an illustration we may prove the following.

THEOREM 2. *Let* $S(t)$ *be the set of numbers of the form*

$$t^{i_1} - t^{i_2} + t^{i_3} - \cdots \tag{17}$$

where $\{i_n\}$ *is a finite or infinite sequence of increasing positive integers, zero included, and* $0 < t < 1$. *Then* $S(t)$ *will consist of the whole of the interval* (0, 1) *if* $1/2 \leqq t < 1$ *and will be a totally disconnected perfect set of measure zero if* $0 < t < 1/2$. *Also, in the latter case, for a given* t, *no two series of the form* (17) *represent the same number of the set* $S(t)$.

To prove this we have only to take $a_n = t^{n-1} - t^n$ $(n = 1, 2, \cdots)$. Then for $1/2 \leqq t < 1$ we get relations of the form (14) (Kakeya's type) and, for $0 < t < 1/2$, relations of the form (2). In the latter case it is moreover clear that all the relations are strict inequalities. Hence the first part of the Theorem follows from Kakeya's Theorem and the rest from Theorem 1.

MADRAS CHRISTIAN COLLEGE

THE NONEXISTENCE OF CERTAIN IDENTITIES IN THE THEORY OF PARTITIONS AND COMPOSITIONS

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1. **Introduction.** Let $q_{d,m}(n)$ be the number of partitions of n into parts differing by at least d, each part being greater than or equal to m. We discuss here the question of the existence of identities involving $q_{d,m}(n)$ analogous to the tautology:

$$\sum_{n=0}^{\infty} q_{1,1}(n)x^n = \prod_{\nu=1}^{\infty}(1+x^\nu),$$

the Euler identity:

$$\sum_{n=0}^{\infty} q_{1,1}(n)x^n = \prod_{\nu=1}^{\infty}\frac{1}{1-x^{2\nu-1}},$$

and the Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty} q_{2,1}(n)x^n = \prod_{\nu=0}^{\infty}\frac{1}{(1-x^{5\nu+1})(1-x^{5\nu+4})},$$

$$\sum_{n=0}^{\infty} q_{2,2}(n)x^n = \prod_{\nu=0}^{\infty}\frac{1}{(1-x^{5\nu+2})(1-x^{5\nu+3})}.$$

We shall in fact show that, aside from the following simple extensions of the first two:

$$\sum_{n=0}^{\infty} q_{1,m}(n)x^n = \prod_{\nu=m}^{\infty}(1+x^\nu),$$

$$\sum_{n=0}^{\infty} q_{1,m}(n)x^n = \frac{\prod_{\nu=m}^{\infty}(1-x^{2\nu})}{\prod_{\nu=m}^{\infty}(1-x^\nu)}$$

$$= \frac{1}{(1-x^m)(1-x^{m+1})\cdots(1-x^{2m-1})}\prod_{\nu=m}^{\infty}\frac{1}{1-x^{2\nu+1}},$$

no other such identities exist. More specifically we shall prove the following two theorems, both of which were proved for the case

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$m=1$ by D. H. Lehmer.[1]

THEOREM 1. *The number $q_{d,m}(n)$ of partitions of n into parts differing by at least d, each part being greater than or equal to m, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever unless $d=1$ or $d=2$, $m=1$, 2.*

THEOREM 2. *The number $q_{d,m}(n)$ of partitions of n into parts differing by at least d, each part being greater than or equal to m, is not equal to the number of partitions of n into distinct parts taken from any set of integers whatsoever unless $d=1$.*

If we now define $Q_d(n)$ as the number of partitions of n into parts congruent to 1 or $d+2$ modulo $d+3$, then we have $q_{1,1}(n)=Q_1(n)$ by the Euler identity and $q_{2,1}(n)=Q_2(n)$ by the first Rogers-Ramanujan identity. A theorem of I. Schur states that $Q_3(n)$ is equal to the number of partitions of n into parts differing by 3 or more among which no two consecutive multiples of 3 appear. Thus $q_{3,1}(n)\geqq Q_3(n)$, and in fact $q_{3,1}(n)-Q_3(n)$ has a rather simple interpretation. Numerical evidence for $d=4$, 5 seems to indicate the plausibility of the conjecture that $q_{d,1}(n)\geqq Q_d(n)$ for all positive integers d and n. For $d\geqq 4$, however, there seems to be no simple interpretation of the difference $q_{d,1}(n)-Q_d(n)$, even if we could prove it to be non-negative. The following theorem shows in particular that there cannot be an interpretation exactly like that for $d=3$.

THEOREM 3. *The number of partitions of n into parts differing by at least d and where parts divisible by d differ by at least $2d$ is not equal to the number of partitions of n into parts taken from any set of integers whatsoever if $d>3$.*

Finally we have for compositions a theorem analogous to Theorem 1.

THEOREM 4. *The number $c_{d,m}(n)$ of compositions of n into parts differing by at least d, each part being greater than or equal to m, is not equal to the number of compositions of n into parts taken from any set of integers whatsoever.*

We shall make use of the following two important formulae, which give the generating functions of $q_{d,m}(n)$ and $c_{d,m}(n)$ in a closed form:

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[1] D. H. Lehmer, *Two nonexistence theorems on partitions*, Bull. Amer. Math. Soc vol. 52 (1946) pp. 538–544. See this paper also for references to the sources of the facts quoted here.

$$\sum_{n=0}^{\infty} q_{d,m}(n)x^n = \sum_{s=0}^{\infty} \frac{x^{ms+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)}, \tag{1}$$

$$\sum_{n=0}^{\infty} c_{d,m}(n)x^n = \sum_{s=0}^{\infty} \frac{s!x^{ms+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)}. \tag{2}$$

The first is proved exactly as Lehmer proved his Theorem 1 (which is the case $m=1$), namely by using the equality

$$m + (m+d) + (m+2d) + \cdots + (m+(s-1)d) = ms + ds(s-1)/2.$$

The second follows from the observation that each partition into s parts differing by at least d, each part being greater than or equal to m, gives rise to $s!$ compositions.

2. Proof of Theorem 1. We suppose that the theorem is false and that there exists such a set of integers $a_1 < a_2 < a_3 < \cdots$. Then making use of (1) we would have

$$\begin{aligned}\prod_{\nu=1}^{\infty} (1-x^{a_\nu})^{-1} &= \sum_{s=0}^{\infty} \frac{x^{ms+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)} \\ &= 1 + \frac{x^m}{1-x} + \frac{x^{2m+d}}{(1-x)(1-x^2)} + \cdots.\end{aligned} \tag{3}$$

Therefore $a_1 = m$. Multiplying (3) by $1-x^m$, we have:

$$\begin{aligned}\prod_{\nu=2}^{\infty} (1-x^{a_\nu})^{-1} &= 1 + x^{m+1} + x^{m+2} + \cdots + x^{2m-1} \\ &\quad + \frac{x^{2m+d}(1-x^m)}{(1-x)(1-x^2)} + \cdots.\end{aligned} \tag{4}$$

Now $a_2 = m+1$. Multiplying (4) by $1-x^{m+1}$ we have:

$$\begin{aligned}\prod_{\nu=3}^{\infty} (1-x^{a_\nu})^{-1} &= 1 + x^{m+2} + x^{m+3} + \cdots \\ &\quad + x^{2m-1} - x^{2m+2} - x^{2m+3} - x^{2m+4} - \cdots \\ &\quad + \frac{x^{2m+d}(1-x^m)(1-x^{m+1})}{(1-x)(1-x^2)} + \cdots.\end{aligned} \tag{5}$$

Noting that for $d \geqq 3$ the coefficient of x^{2m+2} on the right-hand side of (5) is negative we have a contradiction, as the coefficients of the terms on the left-hand side are all non-negative. It remains to prove the theorem for $d=2$. Here we can suppose that $m \geqq 3$, as we know the cases $m=1$ and $m=2$ from the Rogers-Ramanujan identities. If $d=2$, equation (5) can be written for odd values of m in the form:

$$
\begin{aligned}
\prod_{\nu=3}^{\infty}(1-x^{a_\nu})^{-1} = 1 &+ x^{m+2} + x^{m+3} + \cdots \\
&+ x^{2m-1} + x^{2m+4} + x^{2m+5} + 2x^{2m+6} + 2x^{2m+7} \\
&+ 3x^{2m+8} + 3x^{2m+9} + \cdots \\
&+ \left(\left[\frac{c}{2}\right]-1\right)x^{2m+c} + \cdots \\
&+ \left(\frac{m-3}{2}\right)x^{3m} + \left(\frac{m+1}{2}\right)x^{3m+1} + \cdots + x^{4m} \\
&+ \frac{(x^{3m+6}+x^{3m+9}+x^{3m+12}+\cdots)(1+x^2+x^4+\cdots+x^{m-1})}{1-x} \\
&+ \frac{x^{4m+12}+\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \cdots .
\end{aligned} \tag{6}
$$

Since $a_3=m+2$, $a_4=m+3, \cdots, a_m=2m-1$, the left-hand side of (6) can be written as

$$
\begin{aligned}
&(1+x^{m+2}+x^{2m+4}+\cdots)(1+x^{m+3}+x^{2m+6}+\cdots)\cdots \\
&\qquad \cdot(1+x^{2m-1}+x^{4m-2}+\cdots)\prod_{\nu=m+1}^{\infty}(1-x^{a_\nu})^{-1}.
\end{aligned} \tag{7}
$$

Considering only that part of (7) which precedes the product sign, we see that in that part the coefficient of x^{2m+c} for $4\leqq c\leqq m$ is equal to $[c/2]-1$, that is, the same as the coefficient of the corresponding term on the right-hand side of (6). The coefficient of x^{3m+1} in the part of (7) before the product sign is seen to equal $(m-1)/2$, whereas the corresponding term on the right-hand side of (6) has as coefficient $(m+1)/2$. Therefore $a_{m+1}=3m+1$, and the left-hand side of (6) can now be written as:

$$
\begin{aligned}
&(1-x^{m+2})^{-1}(1-x^{m+3})^{-1}\cdots \\
&\qquad \cdot(1-x^{2m-1})^{-1}(1-x^{3m+1})^{-1}\prod_{\nu=m+2}^{\infty}(1-x^{a_\nu})^{-1}.
\end{aligned} \tag{8}
$$

We shall now determine the coefficient of x^{4m+3} on both sides of (6), remembering that the left-hand side of (6) can be written as (8). Denoting the coefficient of x^{4m+3} in

$$(9)\qquad (1 - x^{m+2})^{-1}(1 - x^{m+3})^{-1}(1 - x^{m+4})^{-1} \cdots (1 - x^{2m-1})^{-1}$$

by T_m, we see that x^{4m+3} is obtained—except for the contribution from (9)—only as $x^{m+2} \cdot x^{3m+1}$, so that the coefficient of x^{4m+3} on the left-hand side of (6) is T_m+1 or T_m+2, depending on whether one of the a's is equal to $4m+3$ or not. To determine T_m we note that (9) can be written as

$$\begin{aligned}(1 + x^{m+2} &+ x^{2m+4} + x^{3m+6} + x^{4m+8} + \cdots) \\ &\cdot (1 + x^{m+3} + x^{2m+6} + x^{3m+9} + \cdots) \\ &\cdot (1 + x^{m+4} + x^{2m+8} + \cdots) \cdots (1 + x^{2m-1} + x^{4m-2} + \cdots).\end{aligned}$$

Now x^{4m+3} can be obtained in three ways:

(a) As the product of x^{2m+2k} and $x^{2m-(2k-3)}$ with $4 \leqq 2k \leqq 2m-2$ and $1 \leqq 2k-3 \leqq m-2$ and $2m+2k \neq 2(2m-2k+3)$. This can happen in $(m-1)/2$ ways if $m \equiv 0 \pmod 3$ and in $(m-3)/2$ ways if $m \not\equiv 0 \pmod 3$.

(b) As a third order term $x^{3(m+m/3+1)}$. This can only happen if $m \equiv 0 \pmod 3$ exactly once. Therefore cases (a) and (b) together occur $(m-1)/2$ times whether $m \equiv 0 \pmod 3$ or not.

(c) As the product of x^{m+k_1}, x^{m+k_2} and x^{m+k_3} with $2 \leqq k_1 < k_2 < k_3 \leqq m-1$. This can happen in

$$\begin{aligned}\sum_{2 \leqq k_1 < k_2 < k_3 \leqq m-1,\, k_1+k_2+k_3=m+3} 1 &= \sum_{k_1,\, 2 \leqq k_1 \leqq m/3} \;\; \sum_{k_2,\, k_3,\, k_1 < k_2 < k_3,\, k_2+k_3=m+3-k_1} 1 \\ &= \sum_{k_1=2}^{[m/3]} \;\; \sum_{0<i<j,\, i+j=m+3-3k_1} 1 \\ &= \sum_{k_1=2}^{[m/3]} \left[\frac{m+2-3k_1}{2} \right]\end{aligned}$$

ways. (We have used $k_2 = i + k_1$, $k_3 = j + k_1$.) Consequently:

$$T_m = \sum_{k_1=1}^{[m/3]} \left[\frac{m+2-3k_1}{2} \right]$$

and the coefficient of x^{4m+3} on the left-hand side of (6) is equal to:

$$(10)\qquad 1 + \sum_{k=1}^{[m/3]} \left[\frac{m+2-3k}{2} \right] \quad \text{or} \quad 2 + \sum_{k=1}^{[m/3]} \left[\frac{m+2-3k}{2} \right].$$

On the other hand the coefficient of x^{4m+3} on the right-hand side of (6) appearing only due to the contribution from the term

$$\frac{(x^{3m+6} + x^{3m+9} + x^{3m+12} + \cdots)(1 + x^2 + x^4 + \cdots + x^{m-1})}{1 - x}$$

is equal to

$$
(11)\quad
\begin{aligned}
&\sum_{0\leqq 2j\leqq m-1,\,6\leqq k\leqq m+3,\,2j+(3m+k)=4m+3}\left(\left[\frac{k}{3}\right]-1\right)\\
&=\sum_{0\leqq 2j\leqq m-3}\left(\left[\frac{m+3-2j}{3}\right]-1\right)=\sum_{0\leqq j\leqq (m-3)/2}\left[\frac{m-2j}{3}\right]\\
&=\sum_{0\leqq j\leqq (m-3)/2}\ \sum_{1\leqq l\leqq (m-2j)/3} 1=\sum_{1\leqq k\leqq m/3}\ \sum_{0\leqq j\leqq (m-3k)/2} 1\\
&=\sum_{k=1}^{[m/3]}\left[\frac{m+2-3k}{2}\right].
\end{aligned}
$$

Therefore the coefficient of x^{4m+3} on the left-hand side of (6) is larger by at least 1 than the corresponding coefficient on the right-hand side of (6), which is a contradiction.

For even values of m the proof proceeds analogously. Again we have in this case: $a_1=m$, $a_2=m+1$, $a_3=m+2, \cdots, a_m=2m-1$. Comparing again the coefficient of x^{4m+3} we are led to exactly the same expressions as in (10) and (11).

3. **Proof of Theorem 2.** Again we suppose that the theorem is false, in which case a set consisting of integers $a_1<a_2<a_3<\cdots$ would exist such that

$$
(12)\quad
\begin{aligned}
\prod_{\nu=1}^{\infty}(1+x^{a_\nu}) = 1+\frac{x^m}{1-x}+\frac{x^{2m+d}}{(1-x)(1-x^2)}&\\
+\frac{x^{3m+3d}}{(1-x)(1-x^2)(1-x^3)}&+\cdots.
\end{aligned}
$$

Therefore $a_1=m$. Multiplying both sides of (12) by $1-x^m$ we have:

$$
(13)\quad
\begin{aligned}
(1-x^{2m})\prod_{\nu=2}^{\infty}(1+x^{a_\nu}) &= 1+x^{m+1}+x^{m+2}+\cdots\\
&+x^{2m-1}+\frac{x^{2m+d}(1-x^m)}{(1-x)(1-x^2)}\\
&+\frac{x^{3m+3d}(1-x^m)}{(1-x)(1-x^2)(1-x^3)}+\cdots.
\end{aligned}
$$

We note that $a_2=m+1$, $a_3=m+2, \cdots, a_m=2m-1$. Now, since $a_k+a_l\geqq 2m+3$ for $2\leqq k<l\leqq m$, we have $a_{m+1}=2m$, so that (13) can be written:

$$(14)\quad \begin{aligned}(1 - x^{4m})(1 + x^{m+1})(1 + x^{m+2}) \cdots (1 + x^{2m-1}) \prod_{\nu=m+2}^{\infty} (1 + x^{a_\nu})\\ = 1 + x^{m+1} + x^{m+2} + \cdots + x^{2m-1}\\ + \frac{x^{3m+d}(1 - x^m)}{(1 - x)(1 - x^2)} + \frac{x^{3m+3d}(1 - x^m)}{(1 - x)(1 - x^2)(1 - x^3)} + \cdots .\end{aligned}$$

Case 1. $m \geqq 3$, $d \geqq 4$.

In this case the coefficient of x^{3m+3} on the right-hand side of (14) is zero, whereas the coefficient of the same term on the left-hand side is greater than or equal to one, so that we have a contradiction.

Case 2. $m \geqq 3$, $d = 3$.

In this case equation (14) reduces to:

$$(15)\quad \begin{aligned}(1 - x^{4m})(1 + x^{m+1})(1 + x^{m+2}) \cdots (1 + x^{2m-1}) \prod_{\nu=m+2}^{\infty} (1 + x^{a_\nu})\\ = 1 + x^{m+1} + x^{m+2} + \cdots + x^{2m-1}\\ + \frac{x^{3m+3}(1 - x^m)}{(1 - x)(1 - x^2)} + \frac{x^{3m+9}(1 - x^m)}{(1 - x)(1 - x^2)(1 - x^3)} + \cdots .\end{aligned}$$

Comparing now the coefficient of x^{3m+2} on both sides of (15) we note that on the left-hand side the term x^{3m+2} is obtained only as the product $x^k \cdot x^l$ with $m+1 \leqq k < l \leqq 2m-1$, since $a_{m+2} > 2m+3$ and $x^k \cdot x^l\, x^n$ does not equal x^{3m+2} for $m+1 \leqq k < l < n \leqq 2m-1$. Therefore the coefficient of x^{3m+2} on the left-hand side of (15) is $[(m+1)/2] - 1$ or $[(m+1)/2] - 2$, according as one of the a's equals $3m+2$ or not. The coefficient of x^{3m+2} on the right-hand side of (15), appearing solely as the contribution from the term

$$\frac{x^{3m+3}(1 - x^m)}{(1 - x)(1 - x^2)},$$

can easily be seen to be equal to $[(m+1)/2]$, again a contradiction.

Case 3. $m \geqq 3$, $d = 2$.

We substitute $d = 2$ in (14) and compare, if m is even, just as in case 2 the coefficients of x^{3m+2} on both sides and find the one on the left-hand side to be $m/2 - 1$ or $m/2 - 2$ and on the right-hand side $m/2$. For odd values of m, however, we compare the coefficients of x^{3m+1} and calculate it to be $(m+1)/2 - 1$ or $(m+1)/2 - 2$ on the left and $(m+1)/2$ on the right-hand side.

Case 4. $m = 2$, $d \geqq 2$.

Here equation (14) can be written:

$$(16)\qquad \begin{aligned}(1 - x^{8})(1 + x^{3})\prod_{\nu=4}^{\infty}(1 + x^{a_\nu})\\ = 1 + x^{3} + \frac{x^{4+d}}{1 - x} + \frac{x^{6+3d}}{(1 - x)(1 - x^{3})} + \cdots .\end{aligned}$$

Now let λ be defined by

$$(17)\qquad 2^{\lambda-1} < d + 4 \leqq 2^{\lambda}.$$

Since the coefficient of x^{k} on the right of (16) vanishes for $3<k<4+d$, we need: $a_4=a^3$, $a_5=2^4$, $\cdots$, $a_\lambda=2^{\lambda-1}$, so we can rewrite (16):

$$(18)\qquad \begin{aligned}(1 - x^{2^\lambda})(1 + x^{3})\prod_{\nu=\lambda+1}^{\infty}(1 + x^{a_\nu})\\ = 1 + x^{3} + \frac{x^{4+d}}{1 - x} + \frac{x^{6+3d}}{(1 - x)(1 - x^{3})} + \cdots .\end{aligned}$$

Therefore $a_{\lambda+1}=4+d$. The coefficient of x^{2^λ} on the right-hand side of (18) is $+1$, since $6+3d>2^\lambda$ by (17). If we suppose that $a_{\lambda+j}+3\neq 2^\lambda$ for every $j\geqq 1$, the coefficient of x^{2^λ} on the left-hand side of (18) is seen to be 0 or -1, according as some $a_{\lambda+i}$ is equal to 2^λ or not.

For the case where $a_{\lambda+j}+3=2^\lambda$ for some $j\geqq 1$, we note that: $a_{\lambda+2}=5+d$, $a_{\lambda+3}=6+d$. Considering first now the case where $j=1$, that is, $d+7=2^\lambda$, we notice that for the smallest value of d satisfying this relation, that is, $d=9$, we have: $a_6=14$, $a_7=15$, $a_8=16$, $a_9=19$, $a_{10}=20$, $a_{11}=21$, $a_{12}=25$, so that (18) becomes in this case:

$$\begin{aligned}(1 - x^{32})(1 + x^{3})(1 + x^{13})(1 + x^{14})(1 + x^{15})(1 + x^{19})(1 + x^{20})(1 + x^{21})\\ \cdot(1 + x^{25})\prod_{\nu=13}^{\infty}(1 + x^{a_\nu}) = 1 + x^{3} + \frac{x^{13}}{1 - x} + \frac{x^{33}}{(1 - x)(1 - x^{3})} + \cdots ,\end{aligned}$$

which is impossible, as the coefficient of x^{28} on the left-hand side is greater than or equal to 2, whereas it is 1 on the right-hand side. We therefore can suppose $d>9$. In that case, as is seen from (18): $a_{\lambda+2}=d+5$, $a_{\lambda+3}=d+6$, $a_{\lambda+4}=2^\lambda=d+7$, $a_{\lambda+5}=d+10$, $a_{\lambda+6}=d+11$, so that (18) now becomes:

$$(19)\qquad \begin{aligned}(1 - x^{2^{\lambda+1}})(1 + x^{3})(1 + x^{4+d})(1 + x^{5+d})(1 + x^{6+d})(1 + x^{10+d})\\ \cdot(1 + x^{11+d})\prod_{\nu=\lambda+7}^{\infty}(1 + x^{a_\nu})\\ = 1 + x^{3} + \frac{x^{4+d}}{1 - x} + \frac{x^{6+3d}}{(1 - x)(1 - x^{3})} + \cdots .\end{aligned}$$

We note that the coefficient of x^{2d+15} on the left-hand side of (19) is greater than or equal to 2, since the term x^{2d+15} is obtained both as $x^{4+d}\cdot x^{11+d}$ and as $x^{5+d}\cdot x^{10+d}$ and $2^{\lambda+1}\neq 2d+15$. The coefficient of the same term on the right-hand side, however, is 1, since $3d+6>2d+15$ for $d>9$, which again is a contradiction.

For $j\geqq 2$, let us first suppose $d>8$. Again $a_{\lambda+2}=d+5$, $a_{\lambda+3}=d+6$. If $j=2$, we have $a_{\lambda+4}=2^{\lambda}=d+8$, $a_{\lambda+5}=d+10$ and comparing the coefficients of x^{2d+14} we find it to be 2 or 3 on the left-hand side, since the term x^{2d+14} is obtained as $x^{3}\cdot x^{5+d}\cdot x^{6+d}$ and as $x^{4+d}\cdot x^{10+d}$ and $2^{\lambda+1}\neq 2d+14$, while the coefficient of the corresponding term on the right-hand side is 1 in view of the fact that $3d+6>2d+14$ for $d>8$. If $j=3$: $a_{\lambda+4}=2^{\lambda}=d+9$, $a_{\lambda+5}=d+10$, so that again we obtain a contradiction comparing the coefficient of x^{2d+14}. For $j\geqq 4$, we have $a_{\lambda+4}=d+10$, so that again comparing the coefficient of x^{2d+14} gives us the desired conclusion.

For $d\leqq 8$ contradictions are easily obtained by considering the cases individually: for $d=5$ and 6 by comparing the coefficient of x^{16}, for $d=7$ by comparing the coefficient of x^{30} and for $d=8$ by comparing the coefficient of x^{27}.

4. Proof of Theorem 3. Suppose again that the theorem is false and that there exists such a set of integers $a_1<a_2<a_3<\cdots$, then

$$\begin{aligned}\prod_{\nu=1}^{\infty}(1-x^{a_\nu})^{-1} = \sum_{s=0}^{\infty}&\frac{x^{s+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)}\\ &-\sum_{t=0}^{\infty}A(t)x^t,\end{aligned}\tag{20}$$

where $A(t)$ denotes the number of partitions of t into parts differing by at least d and each partition is such that it contains at least two consecutive multiples of d, that is,

$$A(t)=0$$

for $t=0, 1, 2, \cdots, 3d-1$, $A(3d)=1$. Since $a_1=1$, we multiply (20) by $1-x$ and obtain:

$$\begin{aligned}\prod_{\nu=2}^{\infty}(1-x^{a_\nu})^{-1} = 1+\frac{x^{d+2}}{1-x^2}&+\frac{x^{3d+3}}{(1-x^2)(1-x^3)}+\cdots\\ &-\sum_{t=3d}^{\infty}A(t)x^t+\sum_{t=3d}^{\infty}A(t)x^{t+1}.\end{aligned}\tag*{2 1)}$$

Now $a_2=d+2$ and we multiply (21) by $1-x^{d+2}$:

$$
\begin{aligned}
\prod_{\nu=1}^{\infty}(1-x^{a_\nu})^{-1} = 1 - x^{d+2} + \frac{x^{d+2}}{1-x^2} - \frac{x^{2d+4}}{1-x^2} \\
+ \frac{x^{3d+3}}{(1-x^2)(1-x^3)} + \cdots \\
- \sum_{t=3d}^{\infty} A(t)x^t(1-x^{d+2}) \\
+ \sum_{t=3d}^{\infty} A(t)x^{t+1}(1-x^{d+2}).
\end{aligned} \tag{22}
$$

If d is odd, the coefficient of x^{2d+4} on the right-hand side of (22) is equal to -1 as $3d+1>2d+4$ for $d>3$, which is a contradiction of the fact that the corresponding term on the left-hand side has a non-negative coefficient. If d is even, the coefficient of x^{3d} on the right-hand side is -1 and non-negative on the left-hand side, which completes the proof of the theorem.

The above can be extended to more general cases by similar procedure. It can, for example, be shown that there cannot exist a dual to Schur's theorem in the sense that the second of the Rogers-Ramanujan identities is a dual to the first one, that is, that the number of partitions of n into parts differing by at least 3, no part being equal to 1, where parts divisible by 3 differ by at least 6, is not equal to the number of partitions of n into parts taken from any set of integers whatsoever.

5. Proof of Theorem 4. In case the theorem were false we would have, recalling (2):

$$
\begin{aligned}
\sum_{t=0}^{\infty}(x^{a_1}+x^{a_2}+\cdots)^t = \sum_{s=0}^{\infty} \frac{s!\,x^{ms+ds(s-1)/2}}{(1-x)(1-x^2)\cdots(1-x^s)} \\
= 1 + \frac{x^m}{1-x} + \frac{2x^{2m+d}}{(1-x)(1-x^2)} + \cdots .
\end{aligned} \tag{23}
$$

Therefore $a_1=m$; then (23) becomes

$$
\begin{aligned}
\frac{1}{1-x^m} + x^{a_2} + x^{a_3} + \cdots \\
= 1 + \frac{x^m}{1-x} + \frac{2x^{2m+d}}{(1-x)(1-x^2)} + \cdots .
\end{aligned} \tag{24}
$$

If $m=1$, we obtain:

$$x^{a_2} + x^{a_3} + \cdots = \frac{2x^{2+d}}{(1-x)(1-x^2)} + \cdots, \tag{25}$$

which is a contradiction, as the coefficient of the term with smallest exponent on the left-hand side of (25) is equal to 1, whereas it is equal to 2 on the right-hand side.

If $m > 1$, we multiply (24) by $1 - x^m$ and obtain:

$$\begin{aligned} &1 + (x^{a_2} + x^{a_3} + \cdots)(1 - x^m) \\ &= 1 + x^{m+1} + x^{m+2} + \cdots + x^{2m-1} + \frac{2x^{2m+d}(1-x^m)}{(1-x)(1-x^2)} + \cdots, \end{aligned} \tag{26}$$

so that $a_2 = m + 1$.

Now if $d > 1$, the coefficient of x^{2m+1} on the right-hand side of (26) equals 0, while we find the coefficient of the same term on the left-hand side to be greater than or equal to 1, which is evident, if we recall (23). For $d = 1$ a contradiction is easily obtained by considering more terms in (24).

The preceding shows in particular that there is no analog in the theory of compositions to the Rogers-Ramanujan identities and to Euler's Theorem.

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ON ISOMETRIES OF PRODUCT SETS

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1. Introduction. If A is a metric set, and A^2 its cartesian product with itself, it is possible, in terms of the metric A, to metrize A^2 in many different ways. Thus, the problem of determining when A^2 isometric to B^2 implies A isometric to B can be attacked by placing limitations on the sets A and B or on the product metrics. The present paper proves that the isometry is implied when A and B are bounded and linear and the metric in A^2 and B^2 is determined by the use of a modified Minkowski gauge.

Also stated, without proof, are certain conditions under which the isometry of two product sets implies the isometry of the factors. The proofs for these results, given in the author's doctoral thesis,[1] are long and differ only in detail from the one demonstrated.

2. Definitions.

(A) *Metric set.* A is a metric set if to each pair of its element a_i, a_j there corresponds a real, non-negative number $\rho(a_i, a_j)$ which is independent of the order of the elements, zero if and only if $a_i = a_j$, and which satisfies $\rho(a_i, a_j) + \rho(a_j, a_k) \geqq \rho(a_i, a_k)$.

(B) *Isometry.* Metric sets A and B are isometric (indicated by $A \cong B$) if there exists between them a 1-1 correspondence of elements, T, such that $\rho(a_i, a_j) = \rho(T(a_i), T(a_j))$. The metric in A and B need not be the same, but in writing $\rho(a_i, a_j)$ we shall understand that ρ is the metric for the set containing a_i and a_j.

(C) *Linear set.* Linear, here, will mean a set isometric to a subset of the euclidean line.

(D) *Bounded set.* A metric set A is bounded if $\rho(a_i, a_j)$ is equal to or less than some real number R for all couples in A. If there exists a couple a_i, a_j, such that $\rho(a_i, a_j) = R$, the set assumes its bound.

(E) *Modified Minkowski gauge.* In the first quadrant of the cartesian plane let C' be a curve having nonzero intercepts. A curve C consisting of C' together with the segments of the x and y axes which C' intercepts will be called a modified Minkowski gauge if the following properties hold: (1) C is a simply connected, closed, convex curve; (2) if $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are any two points in the first quadrant and the lines joining the origin to P_1 and P_2 cut C in P_1' and P_2', respectively, then the relations $x_1 \leqq x_2$, and $y_1 \leqq y_2$ imply OP_1/OP_1'

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[1] *On isometric transformations*, University of Wisconsin, 1942.

$\leqq OP_2/OP_2'$. It is easily shown, and we shall need to use, the fact that (1) and (2) imply that: (3) when $x_1 < x_2$ and $y_1 < y_2$ then $OP_1/OP_1' < OP_2/OP_2'$. In addition to (1) and (2) the gauge will be called symmetric if: (4) C is symmetric about the line $y = x$.

(F) *Minkowski metrizations.* Let A, with elements a_i, be any metric set. We use C to metrize A^2 as follows. To a pair of elements (a_i, a_j), (a_k, a_l) in A^2 there corresponds a metrizing point P in the plane with coordinates $[\rho(a_i, a_k), \rho(a_j, a_l)]$. Let the line determined by O and P cut C in the point P'. Then $\rho[(a_i, a_j), (a_k, a_l)]$ is defined as the ratio OP/OP'. We shall also use the notation $\|P\|$ for OP/OP'. Thus condition (2) under (E) could be stated that when $x_1 \leqq x_2$ and $y_1 \leqq y_2$ then $\|(x_1, y_1)\| \leqq \|(x_2, y_2)\|$.

It can be shown that if C has properties (1) and (2) under (E) then A^2 will satisfy definition (A) of a metric set.[3] Some condition similar to (2) is essential, for the metric defined this way with an ordinary Minkowski gauge permits examples in which A^2 is not a metric set. Symmetry about $y = x$ is not a necessary but a natural condition to assume in dealing with square sets. Since $\|P\| = 1$ if and only if P lies on C, C is often referred to as a unit gauge. Various familiar metrics derive in a natural way from different choices of the gauge. If C' is the arc of a unit circle then $\|(x, y)\| = (x^2 + y^2)^{1/2}$. If C is a unit square, then $\|(x, y)\| = \text{Max}\,(x, y)$. If C' is the line joining (1, 0) and (0, 1) then $\|(x, y)\| = x + y$.

THEOREM 1. *If A and B are bounded linear sets which assume their bounds and if A^2, B^2 are metrized by the same symmetric gauge C, then $A^2 \equiv B^2$ implies $A \equiv B$.*

PROOF. (1) Let T be the correspondence establishing $A^2 \equiv B^2$. Let M_1 and M_2 be the bounds of A and B respectively. From linearity these bounds are assumed at one and only one pair of elements in each set. Let $\rho(\bar{a}, a^*) = M_1$, $\rho(\bar{b}, b^*) = M_2$.

(2) Let P_1, P_2, P_3, P_4 in A^2 be the points $(\bar{a}, \bar{a})$, $(a^*, \bar{a})$ (a^*, a^*), $(\bar{a}, a^*)$ and Q_1, Q_2, Q_3, Q_4 in B^2 have similar coordinates with "a" replaced by "b." Let $T(P_i) = S_i = (u_i, v_i)$, $i = 1, 2, 3, 4$.

(3) The set $\{S_i\}$ is in some order the set $\{Q_i\}$, $i = 1, 2, 3, 4$. We prove this by showing each u_i and each v_i is $\bar{b}$ or b^* as follows.

(3.1) From the coordinates of P_i and the definition of the metric in A^2 we have $\rho(P_1, P_3) = \rho(P_2, P_4) = \|(M_1, M_1)\|$ and $\rho(Q_1, Q_3) = \rho(Q_2, Q_4) = \|(M_2, M_2)\|$. From property (2) under (E) these must be the largest distances which occur in A^2 and B^2 respectively. Then from $A^2 \cong B^2$, we have $\|(M_1, M_1)\| = \|(M_2, M_2)\|$. Since (M_1, M_1) and (M_2, M_2) both lie on $y=x$ and this line cuts C' in only one point it follows that $M_1 = M_2$. Let $M_1 = M_2 = M$. From symmetry of C the x and y intercepts of C' are equal and for simplicity we may take them as $(M, 0)$ and $(0, M)$. (This could always be done by a radial expansion or contraction of C that would change all distances in A^2 and B^2 by the same multiple.) With this condition it follows from (1) and (2) that $\rho(P_1, P_2) = \rho(P_2, P_3) = \rho(P_3, P_4) = \rho(P_4, P_1) = 1$.

---

[3] Ibid.

From (3.1) and $A^2 \cong B^2$ we have

$$\rho(S_1, S_2) = \rho(S_2, S_3) = \rho(S_3, S_4) = \rho(S_4, S_1) = 1, \tag{3.2}$$

and

$$\rho(S_1, S_3) = \rho(S_2, S_4) = \|(M, M)\|. \tag{3.3}$$

Now $\|(M, M)\| \geqq 1$ and we treat these two possibilities separately.

(3.4) *Case* 1. $\|(M, M)\| > 1$. Then if $\bar{P}(\bar{x}, \bar{x})$ is the gauge point on $y=x$ (that is, intersection of C' and $y=x$), we have $0 < \bar{x} < M$. From (1) we have

$$\rho(u_1, u_2) \leqq M, \qquad \rho(v_1, v_2) \leqq M \tag{3.5}$$

and

$$\rho(u_3, u_4) \leqq M, \qquad \rho(v_3, v_4) \leqq M. \tag{3.6}$$

From (3.3) and property (3) of the gauge C, one of the equalities in (3.5) and one in (3.6) must hold. To show that all four equalities must hold, suppose:

(3.7) $\rho(u_1, u_2) = M$, $\rho(v_1, v_2) < M$. Let R be the point $[\rho(u_1, u_2), \rho(v_1, v_2)]$. From $\|\rho(u_1, u_2), \rho(v_1, v_2)\| = \|(M, M)\|$ it follows, using similar triangles, that the gauge point R' on the line joining O and R lies on the line $x = \bar{x}$. Since the points $\bar{P}$ and (M, O) are on C, the chord joining them must lie on or interior to C. Hence, R'', the intersection of this chord with the radial line through R, must lie on or interior to C, which is impossible since $OR'' > OR'$. This contradiction shows that if $\rho(u_1, u_2) = M$ then $\rho(v_1, v_2) = M$. The four possibilities of one inequality holding in (3.5) and (3.6) lead to a similar contradiction, hence all four equalities hold. But from (1) that implies each u_i and each v_i, $i = 1, 2, 3, 4$, is either $\bar{b}$ or b^* and establishes (3).

(3.8) *Case* 2. $\|(M, M)\| = 1$. Then (M, M) as well as (M, O) and (O, M) are gauge points. From the definition of convexity then there can be no gauge point interior to the square (O, O), (M, O), (M, M), (O, M). But now from (3.2) and (3.3) all six distances in the set $\{T(P_i)\}$, $i = 1, 2, 3, 4$, are 1, therefore the metrizing point corresponding to any pair of these elements must lie on C, and hence must have either its 1st or 2nd coordinate equal to M. Then either $\rho(u_i, u_j) = M$ or $\rho(v_i, v_j) = M$, $i, j = 1, 2, 3, 4$, $i \neq j$. Now assume $v_1 \neq \bar{b}$ or b^*. Then $\rho(v_1, v_2) \neq M$, $\rho(v_1, v_3) \neq M$, $\rho(v_1, v_4) \neq M$. Therefore, $\rho(u_1, u_2) = \rho(u_1, u_3) = \rho(u_1, u_4) = M$. Take $u_1 = \bar{b}$, then $u_2 = u_3 = u_4 = b^*$. Therefore $\rho(u_2, u_3) = \rho(u_2, u_4) = \rho(u_3, u_4) = 0$. Then $\rho(v_2, v_4) = \rho(v_2, v_3) = \rho(v_3, v_4) = M$. Let $v_2 = \bar{b}$ then $v_3 = v_4 = b^*$, and $\rho(v_3, v_4) = 0$ contradicting $\rho(v_3, v_4) = M$. Therefore v_1 must equal $\bar{b}$ or b^* and by the same argument each u_i and each v_i, $i = 1, 2, 3, 4$, must be $\bar{b}$ ar b^*, which establishes (3).

(4) If we think of P_1, P_3 and P_2, P_4 as diagonal pairs in A^2 and Q_1, Q_3 and Q_2, Q_4 as diagonal pairs in B^2 it is clear, since there are six pairs in each set, that T must map at least one nondiagonal pair of the P_i's onto a nondiagonal pair of the Q_i's. The symmetry of C makes the argument the same in any case and we may suppose for definiteness that $T(P_1) = Q_1$, $T(P_2) = Q_2$.

We define a mapping, S, of A onto B as follows. To any element a_i in A there corresponds in A^2 the point $P_i(a_i, \bar{a})$. Let $T(P_i) = Q_i = (b_i, b_k)$, that is, label the first coordinate of the transformed point b_i. Define S by $S(a_i) = b_i$, $S(\bar{a}) = \bar{b}$, $S(a^*) = b^*$.

(5) S establishes $A \equiv B$.

(5.1) From the coordinates of P_1, P_i, P_2 we have $\rho(P_1, P_i) = \rho(\bar{a}, a_i)/M$, $\rho(P_i, P_2) = \rho(a_i, a^*)/M$, $\rho(P_1, P_2) = \rho(\bar{a}, a^*)/M$.

(5.2) Linearity in A gives $\rho(\bar{a}, a_i) + \rho(a_i, a^*) = \rho(\bar{a}, a^*)$.

(5.3) (5.1) and (5.2) imply $\rho(P_1, P_i) + \rho(P_i, P_2) = \rho(P_1, P_2)$.

(5.4) From (5.3) and $A^2 \equiv B^2$ we have $\rho(Q_1, Q_i) + \rho(Q_i, Q_2) = \rho(Q_1, Q_2)$.

(5.5) Consider the point $\bar{Q}_i(b_i, \bar{b})$. From the coordinates of Q_1, $\bar{Q}_i$, Q_2 and linearity in B it follows that $\rho(Q_1, \bar{Q}_i) + \rho(\bar{Q}_i, Q_2) = \rho(Q_1, Q_2)$.

(5.6) (5.4) and (5.5) give $\rho(Q_1, Q_i) + \rho(Q_i, Q_2) = \rho(Q_1, \bar{Q}_i) + \rho(\bar{Q}_i, Q_2)$.

(5.7) The metrizing point corresponding to the couple Q_1, Q_i has its coordinates respectively equal to or greater than those for the point corresponding to the couple Q_1, $\bar{Q}_i$. Hence, $\rho(Q_1, Q_i) \geqq \rho(Q_1, \bar{Q}_i)$. Similarly $\rho(Q_i, Q_2) \geqq \rho(\bar{Q}_i, Q_2)$. These relations together with (5.6) imply $\rho(Q_1, Q_i) = \rho(Q_1, \bar{Q}_i)$ and $\rho(Q_i, Q_2) = \rho(\bar{Q}_i, Q_2)$.

(5.8) Since $\rho(Q_1, \bar{Q}_i) = \rho(\bar{b}, b_i)/M$, from (5.1), $A^2 \equiv B^2$ and (5.7) we obtain $\rho(\bar{a}, a_i) = M\rho(P_1, P_i) = M\rho(Q_1, Q_i) = M\rho(Q_1, \bar{Q}_i) = \rho(\bar{b}, b_i)$.

(5.9) Let a_i, a_j be any two elements in A with b_i, b_j their transforms under S. From (5.8) we have $\rho(\bar{a}, a_j) = \rho(\bar{b}, b_j)$ and $\rho(\bar{a}, a_i) = \rho(\bar{b}, b_i)$, and from the linearity of A and B it follows that $\rho(a_i, a_j) = \rho(b_i, b_j)$. This last implies that if $a_i \neq a_j$ then $b_i \neq b_j$ so the mapping cannot be many-to-one. Also S is not a mapping of A onto a part of B

because, using the same method, we could define an isometric transform S' of B onto A, then SS' would be an isometry of A onto a subset of itself which is impossible. Thus S is 1-1 and (5) is established.

3. **Other results.** If $A_1, A_2, \cdots, A_n$ are bounded linear sets which assume their bounds, they do so only on one pair of elements which might be called end points. In the product set $A_1\times A_2\times \cdots \times A_n$ there are 2^n n-tuples each coordinate of which is an end point of a base set. Call this collection of points in the product the vertex set. In the following theorems, as in the proof just given, the method employed was to show that the mapping giving the isometry of two product sets had to map the vertex set of one into the vertex set of the other. It was then possible, as in the proof of Theorem 1, to use the isometry of the products to establish the isometry of the base sets.

If $P_1(a_{11}, a_{12}, \cdots, a_{1n})$, $P_2(a_{21}, a_{22}, \cdots, a_{2n})$ are two elements in $A_1\times A_2\times \cdots \times A_n$, then by the euclidean, maximum, and sum metrics in the product we shall mean that $\rho(P_1, P_2)$ is given respectively by $[\sum\rho(a_{1i}, a_{2i})^2]^{1/2}$, Max $[\rho(a_{11}, a_{21}), \rho(a_{12}, a_{22}), \cdots.$ $\rho(a_{1n}, a_{2n})]$, or by $\sum\rho(a_{1i}, a_{2i})$.

THEOREM 2. *If A_1, A_2, B_1, B_2 are bounded, linear sets assuming nonzero bounds and if $A_1\times A_2$ and $B_1\times B_2$ are both metrized under* (1) *the sum metric or* (2) *the maximum metric, then $A_1\times A_2\cong B_1\times B_2$ implies $A_1\cong B_1$, $A_2\cong B_2$ or $A_1\cong B_2$, $A_2\cong B_1$.*

THEOREM 3. *If A is a bounded linear set, assuming its bound, and B is a metric set, then if A^n and B^n are metrized by the maximum metric $A^n\cong B^n$ implies $A\cong B$.*

THEOREM 4. *If $A_1, A_2, \cdots, A_n$ are linear, metric sets, each assuming a nonzero bound and if $B_1, B_2, \cdots, B_n$ are metric sets bounded from zero (not consisting of a single point) and if $A=A_1\times A_2\times \cdots \times A_n$ and $B=B_1\times B_2\times \cdots \times B_n$ are metrized by the euclidean metric then $A\cong B$ implies that the sets A_i are isometric to the sets B_i in some order.*

THEOREM 5. *Let A_1, A_2, B_1, B_2 be euclidean plane sets, each of which contains the vertices of some rectangle bounding it, and let $A_1\times A_2$ and $B_1\times B_2$ be metrized by the euclidean metric. If one of the sets A_1, A_2 is not isometric to a product of two linear sets, then $A_1\times A_2\cong B_1\times B_2$ implies the sets A_i are isometric in some order to the sets B_i, $i=1, 2$.*

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BOUNDS ON CHARACTERISTIC VALUES

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In a previous paper [1][1] the author obtained several upper bounds for the characteristic roots of a finite matrix. The purpose of this note is to establish the corresponding results for characteristic values of certain linear transformations in certain infinite spaces.

1. **Infinite matrices.** We consider a non-null infinite matrix, $A=(a_{st})$, s, $t=1, 2, \cdots$, of complex numbers. (The present considerations pertain also to the finite case when $a_{st}=0$ for $s>n$ and $t>n$.) We shall say that A is *absolutely summable* (abbreviated a.s.) when the double series $\sum_{s,t}|a_{st}|$ converges. In general, if $\sum_{s,t}|a_{st}|^p$ converges for a positive number p, A will be said to be *absolutely summable* (p). If A is a.s.(p), then $\sum_t|a_{st}|^p$ converges for all s, $\sum_s|a_{st}|^p$ converges for all t, and

$$\sum_{s,t}|a_{st}|^p=\sum_s\sum_t|a_{st}|^p=\sum_t\sum_s|a_{st}|^p.$$

We define

$$R_s^{(p)}=\sum_t|a_{st}|^p,\qquad T_t^{(p)}=\sum_s|a_{st}|^p,$$

and, for brevity, $R_s=R_s^{(1)}$, $T_t=T_t^{(1)}$. Clearly if A is a.s.(p), it is a.s.(q) for every $q>p$. Observe also that $R_s^{(p)}=0$ or $T_t^{(p)}=0$ imply, respectively, the vanishing of all elements in the sth row or of all elements in the tth column of A.

Let A be a.s.(p), $p\leqq 1$, and l denote the space of bounded sequences of complex numbers $\{x_r\}$. Throughout, we shall view A as a transformation in l. As such, A carries l into a proper sub-manifold. For, let $\{x_r\}\in l$, and $|x_r|<M$, $r=1, 2, \cdots$. Then the transform of $\{x_r\}$ has the components

$$y_s=\sum_t a_{st}x_t,\qquad s=1, 2, \cdots.$$

Since $p\leqq 1$, we have

$$|y_s|^p\leqq\left(\sum_t|a_{st}|\cdot|x_t|\right)^p\leqq M^p\left(\sum_t|a_{st}|\right)^p\leqq M^p\sum_t|a_{st}|^p,$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

and therefore $\sum_s |y_s|^p$ is convergent. An immediate consequence of this is the following.

LEMMA. *If A is a.s.(p), $p \leqq 1$, and $\{x_r\} \in l$ is a characteristic vector of A belonging to a nonzero characteristic value, then $\sum |x_r|^p$ is convergent.*

It will be assumed in the details of all proofs to follow that the characteristic value $\lambda \neq 0$; the truth of the theorems is evident for $\lambda = 0$.

THEOREM 1. *If A is a.s., and has a characteristic value λ, then:*

$$\text{(i)}\quad |\lambda|^2 \leqq \max_{(s)} R_s T_s.$$

$$\text{(ii)}\quad |\lambda| \leqq \max_{(s)} R_s.$$

$$\text{(iii)}\quad |\lambda| \leqq \max_{(s)} T_s.$$

The bound (i) is stated separately because of its simplicity; it is a particular case of the next theorem. The proofs of (ii) and (iii) are completed by the addition to the proof for the finite case (cf. [1]) of the remark that, by virtue of the lemma above, the l.u.b. of the components of a characteristic vector is attained.

THEOREM 2. *If A is a.s.(p), $p \leqq 1$, and has a characteristic value λ, then for any number q in the interval $p \leqq q \leqq 2-p$,*

$$|\lambda|^2 \leqq \max_{(s)} R_s^{(q)} T_s^{(2-q)}.$$

Let $\{x_s\}$ be a characteristic vector of A belonging to λ, so that $\lambda x_s = \sum_t a_{st} x_t$, $s = 1, 2, \cdots$. From these equations we derive the inequalities

$$|\lambda|\,|x_s| \leqq \sum_t |a_{st}|\,|x_t|, \qquad s = 1, 2, \cdots, \tag{1}$$

which we write in the form:

$$|\lambda|\,|x_s| \leqq \sum_t |a_{st}|^{q/2} \cdot |a_{st}|^{1-q/2} |x_t|.$$

Application of the Schwarz inequality to the right-hand member, as indicated by the decomposition, yields

$$\begin{aligned} |\lambda|\,|x_s| &\leqq \left\{\left(\sum_t |a_{st}|^q\right)\left(\sum_t |a_{st}|^{2-q} |x_t|^2\right)\right\}^{1/2} \\ &= \left(R_s^{(q)}\right)^{1/2}\left(\sum_t |a_{st}|^{2-q} |x_t|^2\right)^{1/2}; \end{aligned}$$

or, squaring this,

$$(2) \qquad |\lambda|^2 |x_s|^2 \leqq R_s^{(q)} \sum_t |a_{st}|^{2-q} |x_t|^2, \qquad s = 1, 2, \cdots .$$

Since q lies in the interval $[p, 2-p]$, the numbers q and $2-q$ are not less than p, and therefore $R_s^{(q)}$ and

$$\sum_t |a_{st}|^{2-q} |x_t|^2 \qquad (\leqq M^2 R_s^{(2-q)})$$

exist for each s.

Now if $R_k^{(q)} = 0$, the inequality (2) for $s = k$ gives $x_k = 0$. Hence, in *each* of the inequalities (2), the term $|a_{sk}|^{2-q} \cdot |x_k|^2$, in the sum on the right, vanishes. That is, if all the matrix elements of the kth row vanish, the elements of the kth *column* do not appear in the inequalities (2). Let us denote by $\bar{A} = (\bar{a}_{mn})$ the matrix obtained from A by deleting the null-rows and the corresponding columns, and re-indexing the elements that remain, in the order in which they remain. Also, let $\{\bar{x}_n\}$ be the re-indexed characteristic vector after dropping those zero-components which correspond to null-rows. We can depict the correspondence of the elements of $\bar{A}$ with those of A by the notation $\bar{a}_{mn} = a_{s_m s_n}$. We remark that

$$\sum_m |\bar{a}_{mn}|^{2-q} = T_{s_n}^{(2-q)}.$$

This is so since the elements of the s_nth column of A which are deleted to leave the nth column of $\bar{A}$ are those which lie in deleted rows, and such elements are all 0. Thus

$$\sum_m |\bar{a}_{mn}|^{2-q} \text{ differs from } \sum_r |a_{rs_n}|^{2-q} = T_{s_n}^{(2-q)}$$

only by 0-terms.

With the new notation, (2) is completely represented by the system

$$(3) \qquad |\lambda|^2 |\bar{x}_m|^2 \leqq R_{s_m}^{(q)} \sum_n |\bar{a}_{mn}|^{2-q} |\bar{x}_n|^2, \qquad m = 1, 2, \cdots ,$$

where now $R_{s_m}^{(q)} \neq 0$ for each m.

We may therefore divide the mth inequality by $R_{s_m}^{(q)}$, and sum over m, to obtain

$$(4) \qquad \sum_m \frac{|\lambda|^2}{R_{s_m}^{(q)}} |\bar{x}_m|^2 \leqq \sum_{m,n} |\bar{a}_{mn}|^{2-q} |\bar{x}_n|^2 = \sum_n T_{s_n}^{(2-q)} |\bar{x}_n|^2 .$$

This last summation is valid since

$$\sum_{n} T_{s_n}^{(2-q)} |x_n|^2 \leqq M^2 \sum_{n} T_{s_n}^{(2-q)} \leqq M^2 \sum_{t} T_t^{(2-q)},$$

the last member here being convergent for $2-q \leqq p$. The inequality (4) may be put in the form

$$\sum_{m} \left(\frac{|\lambda|^2}{R_{s_m}^{(q)}} - T_{s_m}^{(2-q)} \right) |x_m|^2 \leqq 0.$$

Hence it appears that for some s_0, since $\{x_s\} \neq \{0\}$,

$$\frac{|\lambda|^2}{R_{s_0}^{(q)}} - T_{s_0}^{(2-q)} \leqq 0,$$

or

$$|\lambda|^2 \leqq R_{s_0}^{(q)} T_{s_0}^{(2-q)} \leqq \max_{(s)} R_s^{(q)} T_s^{(2-q)}.$$

This proves the theorem.

We state without detailed proof the following generalization of the last result.

THEOREM 3. *Let A, λ be as in the preceding theorem. Let (q_{st}), $s, t = 1, 2, \cdots$, be a matrix of real numbers, only a finite number of which lie outside the interval $(p, 2-p)$. Then*

$$|\lambda|^2 \leqq \max_{(s)} \left\{ \left(\sum_{t} |a_{st}|^{q_{st}} \right) \left(\sum_{r} |a_{rs}|^{2-q_{rs}} \right) \right\}.$$

The method of proof is clear. One starts with the inequalities (1) decomposed as follows: $|\lambda| |x_s| \leqq \sum_t |a_{st}|^{q_{st}/2} \cdot |a_{st}|^{1-q_{st}/2} |x_t|$.

In the direction of a still more comprehensive class of bounds, we indicate the possibility of substituting the general Hölder inequality for the Schwarz inequality above.

2. **Integral equations.** Analogues of all the results of §1 hold for characteristic values of integral equations. Here we shall state and prove only the correspondent of Theorem 1, to fix the procedure. We consider the integral equation

$$\lambda f(s) = \int_{G_t} K(s, t) f(t) dt, \tag{5}$$

where G_t is a fixed set on the t-axis, and the kernel $K(s, t)$ is summable on $G_s \times G_t$, G_s being the set on the s-axis corresponding to G_t on the t-axis (such corresponding sets will be designated in this way throughout). We are concerned with characteristic functions $f(s)$ of (5) in the space of bounded measurable functions on G_s. In this case we have, corresponding to the lemma of §1, that *a characteristic function of K, belonging to a nonzero characteristic value, is integrable on G_s.*

If, when these integrals exist, we set

$$R(s) = \int_{G_t} |K(s, t)|\, dt, \qquad T(s) = \int_{G_t} |K(t, s)|\, dt,$$

we have the following theorem.

THEOREM 4. *Let the kernel K be as above, and λ a characteristic value of K. Then*:

$$\text{(i)}\ \ |\lambda|^2 \leq \sup_{(s\in G_s)} R(s)T(s),$$

$$\text{(ii)}\ \ |\lambda| \leq \sup_{(s\in G_s)} R(s),$$

$$\text{(iii)}\ \ |\lambda| \leq \sup_{(s\in G_s)} T(s).$$

From the summability of K it follows that $R(s)$ and $T(s)$ exist almost everywhere on G_s; on the set E_s, say. Then, from (5),

$$\begin{aligned} |\lambda|\cdot|f(s)| &\leq \int_{E_t} |K(s, t)|\cdot|f(t)|\, dt \\ &= \int_{E_t} |K(s, t)|^{1/2}\cdot|K(s, t)|^{1/2}|f(t)|\, dt. \end{aligned}$$

For $s\in E_s$ we may apply the Schwarz inequality to the extreme right member, and on squaring the result obtain

$$|\lambda|^2|f(s)|^2 \leq R(s)\int_{E_t} |K(s, t)|\,|f(t)|^2 dt.$$

The integral on the right is finite since $f(t)$ is bounded. Let H_s be the subset of E_s on which $R(s) > 0$. We observe that $f(t) = 0$ for $t\in E_t - H_t$. Therefore, we may write, for $s\in H_s$,

$$\frac{|\lambda|^2}{R(s)}|f(s)|^2 \leq \int_{H_t} |K(s, t)|\,|f(t)|^2 dt.$$

We integrate both sides of this inequality over H_s:

$$\int_{H_s} \frac{|\lambda|^2}{R(s)}|f(s)|^2 ds \leq \int_{H_s}\int_{H_t} |K(s, t)|\cdot|f(t)|^2 dt\cdot ds.$$

By Fubini's theorem the order of integration in the right-hand member can be reversed, the two repeated integrals being in fact equal to

$$\iint_{H_s\times H_t} |K(s,t)|\,|f(t)|^2 ds dt \leqq (\sup |f(t)|)^2 \iint_{G_s\times G_t} |K(s,t)|\,ds dt,$$

which is evidently finite. Hence, we have, on further interchanging the letters s and t on the right,

$$\int_{H_s} \frac{|\lambda|^2}{R(s)} |f(s)|^2 ds \leqq \int_{H_s} |f(s)|^2 \left\{ \int_{H_t} |K(t,s)|\,dt \right\} ds$$
$$\leqq \int_{H_s} T(s)\,|f(s)|^2 ds.$$

This may be put into the form

$$\int_{H_s} \left(\frac{|\lambda|^2}{R(s)} - T(s) \right) |f(s)|^2 ds \leqq 0.$$

$f(s)$ is non-null on H_s, and therefore, for some s_0,

$$\frac{|\lambda|^2}{R(s_0)} - T(s_0) \leqq 0,$$

or

$$|\lambda|^2 \leqq R(s_0)\cdot T(s_0) \leqq \sup_{s\in G_s} R(s)T(s),$$

which is the desired result.

To establish part (ii), we let $h = \sup_{s\in E_s}|f(s)|$. Then, for $s\in E_s$,

$$|\lambda| \frac{|f(s)|}{h} \leqq \int_{E_t} |K(s,t)|\,\frac{|f(t)|}{h}\,dt$$
$$\leqq \int_{E_t} |K(s,t)|\,dt = R(s) \leqq \sup_{s\in E_s} R(s).$$

A sequence of values $\{s_n\}$ can be found such that $\lim_{n\to\infty}|f(s_n)| = h$. Then, clearly,

$$\lim_{n\to\infty} |\lambda|\,\frac{|f(s_n)|}{h} = |\lambda| \leqq \sup_{s\in G_s} R(s).$$

Finally, consider

$$|\lambda|\cdot|f(s)| \leqq \int_{E_t} |K(s,t)|\cdot|f(t)|\,dt.$$

We integrate both sides of this inequality over E_s. The order of integration may be reversed, and we obtain, on interchanging the letters s and t,

$$\int_{E_s} |\lambda| \cdot |f(s)|\, ds \leq \int_{E_s} |f(s)| \left\{ \int_{E_t} |K(t, s)|\, dt \right\} ds$$
$$= \int_{E_s} T(s)\, |f(s)|\, ds$$

or

$$\int_{E_s} (|\lambda| - T(s))\, |f(s)|\, ds \leq 0.$$

Thus, for some s_0, $|\lambda| \leq T(s_0) \leq \sup_{s \in G_s} T(s)$, which is the third inequality of Theorem 4.

3. **Concerning bounds.** The literature dealing with estimation of characteristic values of finite matrices begins in 1900 with the work of Bendixson [2]. A well detailed account of results in this field up to 1939 is given by Browne [4]. In the bibliography below are listed most of the papers that have appeared since that time. One of the most effective results is the following, due to Schur [8]: *if* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *are the* n *characteristic roots of the matrix* $A = (a_{st})$, $s, t = 1, 2, \cdots, n$, *then*

$$\sum_1^n |\lambda_i|^2 \leq \sum_{s,t=1}^n |a_{st}|^2.$$

The equality holds if and only if A is normal. The corresponding bound is established for the integral equation (5) with G_s a finite interval and K continuous on $G_s \times G_t$:

$$\sum_1^\infty |\lambda_i|^2 \leq \iint_{G_s \times G_t} |K(s, t)|^2 ds dt.$$

The sum on the left here is taken over all the reciprocal zeros (counting multiplicities) of the Fredholm determinant of K.

Schur's estimate and ours are best estimates, but with different ranges. There are, in fact, for each order n, matrices for which

$$n \max_{(s)} (R_s T_s) < \sum_{s,t=1}^n |a_{st}|^2,$$

the left member therefore affording a sharper bound for $\sum_1^n |\lambda_i|^2$. An example is the matrix

$$\begin{pmatrix} 0 & 2 & 1 \\ 10 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

For normal matrices, and for normal operators in Hilbert space, Schur's global relations can be augmented by more detailed statements concerning the location of individual characteristic values. Some results in this direction will be published by the author in the near future.

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NOTE ON THE SPECTRAL REPRESENTATION OF A BOUNDED NORMAL MATRIX

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Every bounded normal (infinite) matrix N which possesses a bounded reciprocal may be written as a product of its polar factors as $N=PU=UP$, where P is positive definite and U is unitary. Using the corresponding spectral representations for P and U, a spectral representation of N may be obtained by means of Stieltjes integrals. Also, the cartesian factorization $N=H_1+iH_2$, where H_1 and H_2 are Hermitian and commutable, together with the spectral representations of H_1 and H_2 may be employed to obtain a spectral representation of N (cf. [1, 6][1] for references). It seems more natural to proceed directly by using the result on the moment problem for distribution functions in more than one dimension. The spectral representation of N is a consequence of the theorem below, which is the two-dimensional analogue of one employed by M. H. Martin [2] in obtaining the spectral representation of a bounded Hermitian matrix. The theorem holds in general for any finite number of bounded Hermitian matrices which commute with each other, but we consider the case of two such matrices for simplicity.

Notation. Let x, y denote one-column vectors with an infinity of (complex) components x_i, y_i, $i=1, 2, \cdots$. An infinite matrix A is said to be bounded if the least upper bound (l.u.b.) of the set of numbers $|y^*Ax|$ is finite ($y^*=\bar{y}'$, the conjugate transpose of y), where x and y range independently over the unit Hilbert sphere, that is, $|x|=(\sum_{i=1}^{\infty}|x_i|^2)^{1/2}=1$, $|y|=(\sum_{i=1}^{\infty}|y_i|^2)^{1/2}=1$. The matrix H is said to be Hermitian if $H=H^*$. For Hermitian matrices x^*Hx is real and

$$\underset{|x|=|y|=1}{\text{l.u.b.}} |y^*Hx| = \underset{|x|=1}{\text{l.u.b.}} |x^*Hx|.$$

If H is Hermitian and the greatest lower bound (g.l.b.) of the set of numbers x^*Hx is non-negative, that is,

$$\underset{|x|=1}{\text{g.l.b.}}\ x^*Hx \geq 0,$$

then H is said to be non-negative definite. A positive definite matrix, P, is a Hermitian matrix satisfying

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[1] Numbers in brackets refer to references cited at the end of the paper.

$$\operatorname*{g.l.b.}_{|x|=1} x^*Px > 0.$$

We denote the unit matrix by E. A unitary matrix, U, is one for which $UU^* = U^*U = E$. N is said to be a normal matrix if $NN^* = N^*N$.

We first prove the following lemma.

LEMMA. *If* P_1 *and* P_2 *are two positive definite bounded matrices for which*

$$\text{(a)}\quad P_1P_2 = P_2P_1, \qquad \text{(b)}\quad \operatorname*{l.u.b.}_{|x|=1} x^*P_1x < 1, \quad \operatorname*{l.u.b.}_{|x|=1} x^*P_2x < 1$$

then there exist four bounded Hermitian matrices A_1, A_2, B_1, B_2 *such that*

(i) $A_1^2 = P_1$, $A_2^2 = P_2$, $B_1^2 = E - P_1$, $B_2^2 = E - P_2$,

(ii) A_1, A_2, B_1, B_2 *commute with each other.*

PROOF. Under the assumptions the existence of A_1, A_2, B_1, B_2 is guaranteed (cf. [3]). In fact, the "square roots" A_i and B_i may be represented as series

$$A_i = \sum_{n=0}^{\infty} c_n(E - P_i)^n, \qquad B_i = \sum_{n=0}^{\infty} c_nP_i^n, \qquad i = 1, 2,$$

where $c_0c_0 = 1$, $c_0c_1 + c_1c_0 = -1$,

$$\sum_{m=0}^{n} c_mc_{n-m} = 0, \qquad n = 2, 3, \cdots .$$

A_1 clearly commutes with B_1, and similarly for A_2 and B_2. By (a) A_1 and A_2 commute, as well as B_1 and B_2. Therefore A_1, A_2, B_1, B_2 all commute with each other.

THEOREM. *If* H_1 *and* H_2 *are bounded Hermitian matrices which commute and*

$$\mu_y(m, n) = y^*H_1^mH_2^ny, \qquad m, n = 0, 1, 2, \cdots, H_1^0 = H_2^0 = E,$$

where y *is a fixed vector for which* $|y| = 1$, *then there exists an additive monotone set function* $\phi = \phi_y = \phi_y(S)$ *of the set* S *in the* u, v*-plane* ($\phi(S) \equiv 0$ *if* S *outside a certain bounded rectangle*) *such that*

$$\mu_y(m, n) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^mv^n\phi_y(dS^{uv}), \qquad m, n = 0, 1, 2, \cdots .$$

PROOF. There exist real numbers α_1, β_1, α_2, β_2 for which

$$(*) \qquad P_1 = \alpha_1H_1 + \beta_1E, \qquad P_2 = \alpha_2H_2 + \beta_2E \qquad (\alpha_1, \alpha_2 > 0)$$

satisfy the conditions of the lemma. Also, if a function ϕ exists for P_1 and P_2, one will exist for H_1 and H_2 since the transformation (*) only results in a translation and stretching of the spectrum of ϕ. Therefore we may assume H_1 and H_2 have the same properties as P_1 and P_2 of the lemma.

For a given double sequence of numbers $\mu(m, n)$ a necessary and sufficient condition that an additive monotone set function $\phi(S)$ exist with moments

$$\mu(m, n) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^m v^n \phi(dS^{uv}), \qquad m, n = 0, 1, 2, \cdots,$$

is that

$$\Delta_1^k\Delta_2^h\mu(m, n) \geqq 0, \qquad k, h, m, n = 0, 1, 2, \cdots,$$

where

$$\Delta_1^k\Delta_2^h\mu(m, n) = \sum_{r=0}^{k}\sum_{s=0}^{h}\binom{k}{r}\binom{h}{s}(-1)^{r+s}\mu(m + r, n + s)$$

(cf. [4] and [5]). For $\mu(m, n) = \mu_y(m, n) = y^*H_1^mH_2^ny$,

$$\begin{aligned}\Delta_1^k\Delta_2^h\mu_y(m, n) &= \sum_{r=0}^{k}\sum_{s=0}^{h}\binom{k}{r}\binom{h}{s}(-1)^{r+s}(y^*H_1^{m+r}H_2^{n+s}y)\\ &= y^*\left(\sum_{r=0}^{k}\sum_{s=0}^{h}\binom{k}{r}\binom{h}{s}(-1)^{r+s}H_1^{m+r}H_2^{n+s}\right)y\\ &= y^*H_1^mH_2^n(E - H_1)^k(E - H_2)^hy\end{aligned}$$

since H_1 and H_2 commute. Hence we must show that the matrix polynomial $H_1^mH_2^n(E-H_1)^k(E-H_2)^h$ is non-negative definite for $k, h, m, n = 0, 1, 2, \cdots$.

From the lemma we have the existence of A_1, A_2, B_1, B_2 and it follows that

$$\begin{aligned}H_1^mH_2^n(E - H_1)^k(E - H_2)^h &= (A_1^2)^m(A_2^2)^n(B_1^2)^k(B_2^2)^h\\ &= (A_1^mA_2^nB_1^kB_2^h)(A_1^mA_2^nB_1^kB_2^h)\\ &= (A_1^mA_2^nB_1^kB_2^h)(A_1^mA_2^nB_1^kB_2^h)^*.\end{aligned}$$

Since any bounded matrix of the form CC^* is non-negative definite, the theorem is proved.

If N is a normal bounded matrix, $N = H_1 + iH_2$ where $H_1 = (N + N^*)/2$, $H_2 = (N - N^*)/2i$ and H_1 and H_2 are both bounded Hermitian matrices which *commute* with each other since N is normal, that is, H_1 and H_2 satisfy the conditions of the theorem. Hence, we may write

$$\sum_{k=0}^{n}\binom{n}{k}(i)^{n-k}(y^*H_1^kH_2^{n-k}\cdot y) = \sum_{k=0}^{n}\binom{n}{k}(i)^{n-k}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}u^kv^{n-k}\phi_y(dS^{uv}),$$

or

$$y^*(H_1+iH_2)^ny = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(u+iv)^n\phi_y(dS^{uv});$$

but this is the same as

$$y^*N^ny = \int w^n\phi_y(dS^w)$$

where $w=u+iv$ and the integration is over the entire w-plane.

With proper normalizations $\phi_y(S)$ may be made unique.

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GROUP DECOMPOSITION BY DOUBLE COSET MATRICES

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1. Introduction. We are concerned in this paper with a study of some important properties of the matrices V of the commuting ring of a finite intransitive group of permutation matrices R whose transitive constituents we denote by R^s, R^t, $\cdots$. In particular we shall find an explicit method for obtaining the degrees and characters of the common irreducible components of two permutation representations R^s and R^t of a group G.

Our interest in this subject was first aroused in connection with the study of molecular structure, in which theoretical chemists have made use of the theory of groups. The potential energy of the molecule is approximated by a quadratic form whose matrix V must be commutative with all the matrices of the symmetry group of the molecule. Its characteristic roots are closely related to the molecular spectrum. This suggests the general problem: "What matrices are permutable with a given intransitive symmetry group?"

Let G then be any finite group of order g, represented by an intransitive permutation group R,

$$R = R^1 + \cdots + R^s + R^t + \cdots, \tag{1.1}$$

which is the direct sum of transitive constituents R^s, R^t, $\cdots$. The rectangular matrices V^{ts} which intertwine R^t and R^s,

$$R^t V^{ts} = V^{ts} R^s, \tag{1.2}$$

are submatrices of the matrices V which commute with R. One basis for these matrices V^{ts} is supplied by the μ^{st} double coset matrices V^{ts}_α described in §2. An example is given in §3 to illustrate the theory at this point.

After a change of basis by suitable unitary transformations U^s, U^t, $\cdots$ obtained in §4, the permutation groups R^s, R^t are rewritten as completely reduced groups,

$$R^s_U = (U^s)^{-1} R^s U^s = \cdot\sum_j F_j \times I^s_j, \tag{1.3}$$

which are the direct sums of $(f_j \mu^s_j)$-dimensional square blocks placed consecutively along the diagonal. In each block R^s_U is the direct

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product of an irreducible representation F_j of degree f_j by a unit matrix I_j^s of degree μ_j^s. Under this same change of basis the double coset matrices V_α^{ts} of the set V^{ts} are transformed into rectangular basis matrices Φ_α^{ts} of the transformed linear set Φ^{ts},

$$(1.4) \qquad \Phi_\alpha^{ts} = (U^t)^{-1} V_\alpha^{ts} U^s = \sum_j I_j^0 \times \Phi_{\alpha j}^{ts},$$

where I_j^0 is the unit matrix of degree $f_j = \mu_j^0$ (corresponding to the identity subgroup H^0) and where $\Phi_{\alpha j}^{ts}$ are rectangular matrices whose coefficients $\phi_{\alpha j,\rho\sigma}^{ts}$ ($\rho = 1 \cdots \mu_j^t$; $\sigma = 1 \cdots \mu_j^s$) we shall call *double coset coefficients*. Theorem 1 in §3 gives an explicit formula for the double coset coefficients in terms of the entries of the irreducible components F_j. Later we try in reverse to obtain information about the F_j by first computing the double coset coefficients.

The double coset coefficients define a transformation from the basis Φ_α^{ts} of Φ^{ts} to a new basis E_η^{ts} described in §5. We use the single subscript η to replace the subscripts j, ρ, σ arranged in dictionary order.

$$(1.5) \qquad \Phi_\alpha^{ts} = \sum_\eta \rho_{\alpha\eta}^{ts} E_\eta^{ts}; \qquad \rho_{\alpha\eta}^{ts} = \phi_{\alpha j,\rho\sigma}^{ts}.$$

We shall prove in Theorem 2 of §5 that the double coset coefficients $\rho_{\alpha\eta}^{ts}$ satisfy a set of orthogonality relations, which include as a special case the well known orthogonality relations among the coefficients of the irreducible representations F_j of G.

Similarly in §6 the double coset traces $\tau_{\alpha j}^t$ and their associated factors $\psi_{\alpha j}^t$,

$$(1.6) \qquad \tau_{\alpha j}^t = \sum_\sigma \phi_{\alpha j,\sigma\sigma}^{tt},$$

$$(1.7) \qquad \psi_{\alpha j}^t = \tau_{\alpha j}^t / \tau_{\alpha 1}^t \tau_{1j}^t; \qquad \psi_{\alpha 1}^t = 1,$$

are shown to satisfy orthogonality relations including the well known relations among the group characters. The double coset traces are proved in Theorem 3 to be algebraic integers, and certain group characters are obtained as linear combinations of the $\psi_{\alpha j}^t$.

In §7 we assume that some properties of the permutation groups R^s and R^t and the related subgroups H^s and H^t are known, but not the irreducible representations F_j. The problem is to find the degrees and characters of those F_j which are common components of R^s and R^t. The answer is given in terms of the double coset traces and the numbers $k_{\alpha\lambda}^{ts}$ which define the number of elements of G common to a double coset H_α^{ts} and a class of conjugates C_λ. For subgroups of relatively small index in G, for which μ^{st} is small, the double coset coefficients can easily be determined by factoring a determinant of order μ^{st} using our principal Theorem 4. The characters of the F_j can then be obtained by a direct and unambiguous computation.

2. **Double coset matrices and the commuting ring.** Let $\gamma_1=1$, $\gamma_2, \cdots, \gamma_g$ be the elements of a finite group G, which is represented as the intransitive group of permutation matrices (1.1). Let H^s denote the subgroup consisting of the h^s elements of G which leave invariant the first symbol permuted by the matrices of R^s, and let θ^s_i $(i=1, 2, \cdots, n^s)$ be the right coset element in the group ring of G which is the sum of the elements of the right coset $H^s\gamma_i$. In particular, θ^s_1 represents the sum of the elements in the subgroup H^s, and θ^s_1/h^s is an idempotent of the group ring. If k^{st}_α denotes the order of the intersection of $\gamma_\alpha^{-1}H^s\gamma_\alpha$ and H^t, and n^{st}_α the index of this subgroup in G, then we define the double coset element θ^{st}_α of the group ring by[1]

$$\theta^{st}_\alpha = \theta^s_\alpha\theta^t_1/k^{st}_\alpha = n^{st}_\alpha\theta^s_\alpha\theta^t_1/g = \theta^s_1\gamma_\alpha\theta^t_1/k^{st}_\alpha . \tag{2.1}$$

This element θ^{st}_α is the sum of h^t/k^{st}_α distinct right coset elements θ^s_i. Each element of G occurs as a summand in exactly one of the θ^{st}_α, and occurs there with coefficient 1. We form a column vector θ^t of n^t rows having θ^t_i in the ith row, and a column vector θ of $n^1+n^2+\cdots$ rows which is the direct sum of $\theta^1, \theta^2, \cdots$. The transitive groups of matrices R^t and their intransitive direct sum R can then be defined:

$$\theta^t\gamma_i = R^t(\gamma_i)\theta^t, \tag{2.2}$$

$$\theta\gamma_i = \begin{bmatrix}\theta^1\\ \theta^2\\ \cdot\\ \cdot\end{bmatrix}\gamma_i = \begin{bmatrix}R^1(\gamma_i) & 0 & \cdots & 0\\ 0 & R^2(\gamma_i) & \cdots & 0\\ \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdots & \cdot\end{bmatrix}\begin{bmatrix}\theta^1\\ \theta^2\\ \cdot\\ \cdot\end{bmatrix} = R(\gamma_i)\theta. \tag{2.3}$$

By applying the associative law and (2.3) to the product $\theta\gamma_i\gamma_j$ we verify that R is homomorphic with G, since

$$R(\gamma_i)R(\gamma_j) = R(\gamma_i\gamma_j). \tag{2.4}$$

The double coset matrix V^{ts}_α is defined by the formula

$$\theta^{st}_\alpha\theta^t = h^tV^{ts}_\alpha\theta^s. \tag{2.5}$$

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[1] A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung*. The notion of double cosets is explained on p. 63.

The entries of V^{ts}_α consist of h^t/h^{st}_α 1's in each of its n^t rows, and h^s/h^{st}_α 1's in each of its n^s columns, all other entries being 0's. There are n^{st}_α 1's altogether in a particular V^{ts}_α. Since the sum over α of θ^{st}_α is the same as the sum of all elements of G, the sum over α of all V^{ts}_α is a rectangular matrix of 1's.

The fact that V^{ts}_α intertwines the matrices $R^t(\gamma_i)$ and $R^s(\gamma_i)$ is seen by combining (2.2) and (2.5). The scalar θ^{st}_α is permutable with the matrix $R^t(\gamma_i)$ and we have

$$\begin{aligned}R^t(\gamma_i)V^{ts}_\alpha h^t\theta^s &= R^t(\gamma_i)\theta^{st}_\alpha\theta^t = \theta^{st}_\alpha R^t(\gamma_i)\theta^t = \theta^{st}_\alpha\theta^t\gamma_i = h^t V^{ts}_\alpha\theta^s\gamma_i \\ &= V^{ts}_\alpha R^s(\gamma_i)h^t\theta^s.\end{aligned} \tag{2.6}$$

Comparing coefficients of $h^t\theta^s$, we have an equation similar to (1.2):

$$R^t(\gamma_i)V^{ts}_\alpha = V^{ts}_\alpha R^s(\gamma_i), \tag{2.7}$$

If V is any matrix in the commuting ring of R we have

$$RV = VR \quad \text{where} \quad V = \begin{pmatrix} V^{11} & V^{12}\ldots & V^{1s} & V^{1t}\ldots \\ V^{21} & V^{22}\ldots & V^{2s} & V^{2t}\ldots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ V^{s1} & V^{s2}\ldots & V^{ss} & V^{st}\ldots \\ V^{t1} & V^{t2}\ldots & V^{ts} & V^{tt}\ldots \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \end{pmatrix}. \tag{2.8}$$

It has been shown in an earlier paper[3] that the submatrix V^{ts} of (2.8) which satisfies (1.2) can be expressed as a linear combination of the μ^{st} double coset matrices V^{ts}_α,

$$V^{ts} = \sum_\alpha c^{ts}_\alpha V^{ts}_\alpha. \tag{2.9}$$

Furthermore the transpose of the matrix V^{ts}_α is the matrix $V^{st}_{\alpha'}$ which corresponds to the inverse double coset $H^{ts}_{\alpha'}$,

$$(V^{ts}_\alpha)' = V^{st}_{\alpha'}. \tag{2.10}$$

The identity element of G will occur in the product $H^{ts}_\beta H^{st}_\alpha$ only if $\beta = \alpha$. Hence the trace (tr) of the product $V^{ts}_\alpha V^{st}_{\beta'}$ is zero unless $\alpha = \beta$. It is then equal to n^{st}_α, the number of 1's in the matrix V^{ts}_α. Thus

$$\operatorname{tr}\,(V^{ts}_\alpha V^{st}_{\beta'}) = \delta_{\alpha\beta}n^{st}_\alpha. \tag{2.11}$$

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[3] J. S. Frame, *Double coset matrices and group characters*, Bull. Amer. Math. Soc. vol. 49 (1943) pp. 81–92.

3. **An example of double cosets.** In order to fix ideas we illustrate the theory with an example in which G is the symmetric group of degree 4 and order 24 whose elements γ_i we denote by writing them as permutations on 4 digits. Let $\theta_1^s = I + (13)$ and $\theta_1^t = I + (13) + (24) + (13)(24)$, so that H^s and H^t are subgroups of order 2 and 4 respectively. Then the right cosets and double cosets are as follows.

(3.1) Right cosets

$$\begin{array}{lll}
\theta_1^s = I + (13) & \theta_{12}^s = (13)(24) + (24) & \theta_1^t = \theta_1^s + \theta_{12}^s \\
\theta_2^s = (143) + (34) & \theta_{11}^s = (234) + (1423) & \theta_2^t = \theta_2^s + \theta_{11}^s \\
\theta_3^s = (134) + (14) & \theta_{10}^s = (142) + (1342) & \theta_3^t = \theta_3^s + \theta_{10}^s \\
\theta_4^s = (123) + (23) & \theta_9^s = (243) + (1243) & \theta_4^t = \theta_4^s + \theta_9^s \\
\theta_5^s = (132) + (12) & \theta_8^s = (124) + (1324) & \theta_5^t = \theta_5^s + \theta_8^s \\
\theta_6^s = (12)(34) + (1432) & \theta_7^s = (14)(23) + (1234) & \theta_6^t = \theta_6^s + \theta_7^s
\end{array}$$

(3.2) Double cosets

$$\begin{array}{lll}
\theta_1^{ss} = \theta_1^s & \theta_1^{st} = \theta_1^s + \theta_{12}^s & \theta_1^{ts} = \theta_1^t \\
\theta_2^{ss} = \theta_2^s + \theta_3^s & \theta_2^{st} = \theta_2^s + \theta_3^s + \theta_8^s + \theta_9^s & \theta_2^{ts} = \theta_2^t + \theta_3^t \\
\theta_3^{ss} = \theta_4^s + \theta_5^s & \theta_3^{st} = \theta_4^s + \theta_5^s + \theta_{10}^s + \theta_{11}^s & \theta_3^{ts} = \theta_4^t + \theta_5^t \\
\theta_4^{ss} = \theta_6^s + \theta_7^s & \theta_4^{st} = \theta_6^s + \theta_7^s & \theta_4^{ts} = \theta_6^t \\
\theta_5^{ss} = \theta_8^s + \theta_9^s & & \\
\theta_6^{ss} = \theta_{10}^s + \theta_{11}^s & & \theta_1^{tt} = \theta_1^t \\
\theta_7^{ss} = \theta_{12}^s & & \theta_2^{tt} = \theta_2^t + \theta_3^t + \theta_4^t + \theta_5^t \\
 & & \theta_3^{tt} = \theta_6^t
\end{array}$$

Typical permutation matrices $R^s(\gamma)$ and $R^t(\gamma)$ for $\gamma = (143)$ are

$$(3.3) \qquad R^s(\gamma) = \begin{pmatrix}
\begin{matrix}0&1&0\\0&0&1\\1&0&0\end{matrix} & 0 \quad\quad 0 & 0 \\
0 & \begin{matrix}0&0&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\end{matrix} & 0 \\
0 & \begin{matrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&1&0&0\end{matrix} & 0 \\
0 & 0 \quad\quad 0 & \begin{matrix}0&0&1\\1&0&0\\0&1&0\end{matrix}
\end{pmatrix}$$

$$R'(\gamma) = \begin{pmatrix} \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \end{pmatrix}.$$

In writing the general double coset matrix V it will be easier to print and to read if we replace the arbitrary parameters $c_1^{11} \cdots c_7^{11}$ by letters $A\ B\ C\ D\ E\ F\ G$, replace $c_1^{12} \cdots c_4^{12}$ by $X\ Y\ Z\ W$, replace $c_1^{21} \cdots c_4^{21}$ by $x\ y\ z\ w$, and replace c_1^{22}, c_2^{22}, c_3^{22}, by $a\ b\ c$ respectively. Then the matrix V becomes

$$(3.4)\quad V = \left\| \begin{array}{cccccc@{\quad}cccccc@{\quad}cccccc}
A & B & B & C & C & D & D & E & E & F & F & G & X & Y & Y & Z & Z & W \\
B & A & B & E & D & C & E & D & C & F & G & F & Y & X & Y & Z & W & Z \\
B & B & A & D & E & E & C & C & D & G & F & F & Y & Y & X & W & Z & Z \\
C & F & D & A & C & F & B & E & G & D & B & E & Z & Y & W & X & Z & Y \\
C & D & F & C & A & B & F & G & E & B & D & E & Z & W & Y & Z & X & Y \\
D & C & F & E & B & A & G & F & C & B & E & D & W & Z & Y & Z & Y & X \\[8pt]
D & F & C & B & E & G & A & C & F & E & B & D & W & Y & Z & Y & Z & X \\
F & D & C & F & G & E & C & A & B & E & D & B & Y & W & Z & Y & X & Z \\
F & C & D & G & F & C & E & B & A & D & E & B & Y & Z & W & X & Y & Z \\
E & E & G & D & B & B & F & F & D & A & C & C & Z & Z & X & W & Y & Y \\
E & G & E & B & D & F & B & D & F & C & A & C & Z & X & Z & Y & W & Y \\
G & E & E & F & F & D & D & B & B & C & C & A & X & Z & Z & Y & Y & W \\[8pt]
x & y & y & z & z & w & w & y & y & z & z & x & a & b & b & b & b & c \\
y & x & y & y & w & z & y & w & z & z & x & z & b & a & b & b & c & b \\
y & y & x & w & y & y & z & z & w & x & z & z & b & b & a & c & b & b \\
z & z & w & x & z & z & y & y & x & w & y & y & b & b & c & a & b & b \\
z & w & z & z & x & y & z & x & y & y & w & y & b & c & b & b & a & b \\
w & z & z & y & y & x & x & z & z & y & y & w & c & b & b & b & b & a
\end{array} \right\|.$$

A particular double coset matrix is obtained by setting one of the letters in (3.4) equal to 1 and all the rest 0.

4. **The transformation to irreducible components.** A finite group G with k classes of conjugates C_λ has exactly k non-equivalent sets of irreducible representations as a group of unitary transformations of

a vector space over a field K of characteristic 0 which includes the gth roots of unity. Let F_j of degree f_j be a particular unitary representation in the jth set, and denote by $f_{j,\rho\sigma}(\gamma_\iota)$ the entry in the ρth row and σth column of the matrix representing γ_ι in F_j. We order the $g=\sum f_i^2$ sets of three indices j, ρ, σ in dictionary order and assign to them the single index η. Given η we shall write η^* to denote the subscript j of the corresponding F_j. The complete set of coefficients in all the selected F_j for each γ_ι may then be written in a $g\times g$ matrix Z with entries

$$z_{\iota\eta} = f_{j,\rho\sigma}(\gamma_\iota). \tag{4.1}$$

From Z we derive a unitary matrix T with entries

$$t_{\iota\eta} = z_{\iota\eta}(f_{\eta^*}/g)^{1/2} \tag{4.2}$$

which completely reduces both the right and left regular representations of G.[8]

From T in turn we shall derive a decompossable unitary matrix

$$U_*^s = (X^s)^{-1}TQ^s = \begin{pmatrix} U^s & 0 \\ 0 & * \end{pmatrix} \tag{4.3}$$

which is adapted to the subgroup H^s and contains in its upper left corner an $n^s\times n^s$ matrix U^s which reduces R^s according to (1.3). This we do as follows.

Denoting by θ^0 a column vector whose entries are the elements $\gamma_1 \cdots \gamma_g$ of G, and by γ an arbitrary element of the group ring of G, the regular representations R^0 and L are given by

$$\theta^0\gamma = R^0(\gamma)\theta^0; \qquad \gamma\theta^0 = L(\gamma)\theta^0. \tag{4.4}$$

We define a new basis vector ϵ such that $\theta^0 = T\epsilon$. Then

$$\epsilon\gamma = T^{-1}R^0(\gamma)T\epsilon; \qquad \gamma\epsilon = T^{-1}L(\gamma)T\epsilon. \tag{4.5}$$

We further define I_j to be the idempotent matrix which vanishes except for f_j^2 1's as diagonal entries where $\eta^* = j$. Then

$$\begin{aligned} I_jT^{-1}R^0(\gamma)TI_j &= F_j(\gamma)\cdot\times I_j^0, \\ I_jT^{-1}L(\gamma)TI_j &= I_j^0\cdot\times F_j'(\gamma) \end{aligned} \tag{4.6}$$

where I_j^0 is a unit matrix of degree f_j and where $\cdot\times$ denotes the left direct product in which the submatrices consist of the first matrix multiplied by an entry from the second,[4] and the prime denotes the transposed matrix. Now $F_j(\theta_1^s/h^s)$ is an idempotent matrix of rank μ_j^s, so we may find a suitable unitary matrix Q_j^s which will bring it to the form $I_j^s = \mathrm{diag}\ (1, 1, \cdots, 1, 0, \cdots, 0)$. The matrices

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[8] J. S. Frame, *On the reduction of the conjugating representation of a finite group*, Bull. Amer. Math. Soc. vol. 53 (1947) p. 589.

$$F_j^s(\gamma) = (Q_j^s)^{-1} F_j(\gamma) Q_j^s \tag{4.7}$$

will be said to be *adapted* to the subgroup H^s. The direct sum of the products $I_j^0 \cdot \times (\bar{Q}_j^s)$ is now multiplied by a permutation matrix to give a product Q^s which assembles the $n^s = \mu_j^s f_j$ nonvanishing entries of the idempotents for H^s so that the matrix

$$L^s = (Q^s)^{-1} T^{-1} L(\theta_1^s/h^s)\, TQ^s = \begin{pmatrix} I^s & 0 \\ 0 & 0 \end{pmatrix} \tag{4.8}$$

is a diagonal idempotent having 1's in its first n^s diagonal entries, and 0's elsewhere.

Furthermore TQ^s transforms $R^0(\gamma)$ into a matrix having the submatrix R^s in the upper left corner. It is permutable with L^s and hence decomposable.

Having adapted the irreducible representations F_j to the subgroup H^s we next adapt the basis vector θ^0 to H^s. Let P^s be a permutation matrix such that in the vector $(P^s)^{-1}\theta^0$ the qth element of the νth right coset of H^s in G occupies the $i = (q-1)n^s + \nu$ row. Set

$$X^s = P^s Y^s, \qquad Y^s = I^s \cdot \times T^s \tag{4.9}$$

where Y^s is the left direct product of a unit matrix of degree n^s with the unitary matrix T^s of degree h^s which completely reduces the regular representation of H^s. In its first n^s columns the matrix $(h^s)^{1/2} Y^s$ contains I^s repeated h^s times. The conjugate transposed matrix is $(h^s)^{1/2}(Y^s)^{-1}$. This collects the coset elements of $(P^s)^{-1}\theta^0$ together, so that the new basis vector

$$\zeta^s = (h^s)^{1/2}(X^s)^{-1}\theta^0 \tag{4.10}$$

contains in its first n^s rows the subvector θ^s previously defined, and contains in its other rows ring elements which are annihilated on left multiplication by θ_1^s or any θ_α^{ts}. From (2.5) and (4.10) we obtain

$$\begin{aligned} \begin{pmatrix} V_\alpha^{ts} & 0 \\ 0 & 0 \end{pmatrix} \zeta^s &= (h^t)^{-1} \theta_\alpha^{st} \zeta^t = (h^t)^{-1/2} (X^t)^{-1} \theta_\alpha^{st} \theta^0 \\ &= (h^t)^{-1/2} (X^t)^{-1} L(\theta_\alpha^{st}) \theta^0. \end{aligned} \tag{4.11}$$

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[4] C. C. MacDuffee, *Theory of matrices*, Chelsea Press, 1947.

Expressing θ^0 in terms of ζ^s by (4.10) and comparing coefficients we have

$$\begin{pmatrix} V_\alpha^{ts} & 0 \\ 0 & 0 \end{pmatrix} = (X^t)^{-1} L(\theta_\alpha^{st}) X^s (h^s h^t)^{-1/2}. \tag{4.12}$$

Using the X^s of (4.9) we define the unitary matrix $U_*^s = (X^s)^{-1} T Q^s$ as in (4.3), and then multiply each member of (4.12) on the left by $(U_*^t)^{-1}$ and on the right by U_*^s. Thus we have

$$(U_*^t)^{-1} \begin{pmatrix} V_\alpha^{ts} & 0 \\ 0 & 0 \end{pmatrix} U_*^s = (Q^t)^{-1} T^{-1} L(\theta_\alpha^{st}) T Q^s (h^s h^t)^{-1/2}. \tag{4.13}$$

Since the right member of (4.13) belongs to the left ideal generated by the idempotent L^s of (4.8) and to the right ideal generated by L^t, it follows that the unitary matrices U_*^s and U_*^t are decomposable as indicated in (4.3). It can be shown by a straightforward computation that the submatrix U^s reduces R^s as in (1.3). Also by (4.13) we may express the transformed double coset matrices Φ_α^{ts} of (1.4) in terms of the coefficients in F_j by the equation

$$\begin{pmatrix} \Phi_\alpha^{ts} & 0 \\ 0 & 0 \end{pmatrix} = (Q^t)^{-1} T^{-1} L(\theta_\alpha^{st}) T Q^s (h^s h^t)^{-1/2}. \tag{4.14}$$

Since T is a unitary matrix which completely reduces L as shown in (4.6) it follows that both members of (4.14) actually are reduced, as was stated in the second part of (1.4). Consequently $\Phi_{\alpha j}^{ts}$ is a $\mu_j^t \times \mu_j^s$-dimensional submatrix of the matrix

$$(Q_j^{t\prime}) F_j'(\theta_\alpha^{st}/(h^s h^t)^{1/2}) \overline{Q}_j^s. \tag{4.15}$$

Taking the transpose of (4.15) and using (4.7) we have

$$\begin{pmatrix} \Phi_{\alpha j}^{ts} & 0 \\ 0 & 0 \end{pmatrix}' = \begin{pmatrix} I_j^s & 0 \\ 0 & 0 \end{pmatrix} (Q_j^s)^{-1} F_j(\gamma_\alpha \rho_{\alpha 1}^{st}) Q_j^t \begin{pmatrix} I_j^t & 0 \\ 0 & 0 \end{pmatrix} \tag{4.16}$$

where

$$\rho_{\alpha 1}^{st} = (h^s h^t)^{1/2} / h_\alpha^{st}. \tag{4.17}$$

THEOREM 1. *The transpose of the $\mu_j^t \times \mu_j^s$-dimensional submatrix $\Phi_{\alpha j}^{ts}$ of the double coset matrix Φ_α^{ts} may be expressed in terms of the coefficients of the irreducible representation F_j of G by equation* (4.16).

The factor $\rho_{\alpha 1}^{ts}$ in (4.16) is a mean proportion between the number of right cosets of H^s and the number of left cosets of H^t in the double coset H_α^{st}. The group element γ_α is any element in this double coset.

5. **Orthogonality relations.** Having transformed R and V by the unitary matrix U which is the direct sum of the matrices U^s defined in (4.3) we now investigate certain algebraic properties of the double coset matrices Φ_α^{ts}. We express the latter in terms of the new basis matrices E_η^{ts} defined by (1.5).

The matrix E_η^{ts} consists entirely of 0's except for f_j 1's forming a unit matrix I_j^0 in the rows corresponding to the ρth occurrence of F_j as a component of R_U^t, and columns correspond to the σth occurrence of F_j as a component of R_U^s. The number of distinct matrices E_η^{ts} for fixed ts is $\sum_j \mu_j^t \mu_j^s$. This is equal to the number μ^{st} of double cosets V_α^{ts}, which has been called the *transitive factor* of the two subgroups H^s and H^t.[5]

Solving for E_ζ^{ts} in terms of Φ_α^{ts} we write

$$E_\zeta^{ts} = \sum_\beta \sigma_{\zeta\beta}^{ts} \Phi_\beta^{ts}, \tag{5.1}$$

defining $(\sigma_{\zeta\beta}^{ts})$ to be the inverse matrix of $(\rho_{\alpha\eta}^{ts})$. Denoting by $E_{\zeta'}^{st}$ the transpose of the real matrix E_ζ^{ts}, and taking the transposed conjugates in (5.1) we have

$$E_{\zeta'}^{st} = \sum_\beta \bar{\sigma}_{\zeta\beta}^{ts} \Phi_{\beta'}^{st} \tag{5.2}$$

where $\Phi_{\beta'}^{st}$ and Φ_β^{ts} correspond to inverse double cosets.

The trace of the product $E_\eta^{ts} E_{\zeta'}^{st}$ is $f_\eta \delta_{\eta\zeta}$, whereas the trace of the product $\Phi_\alpha^{ts} E_{\zeta'}^{st}$ may be expressed in two ways, by using first (1.5) and then (5.2) and (2.11).

$$\begin{aligned} \operatorname{tr}\,(\Phi_\alpha^{ts} E_{\zeta'}^{st}) &= \sum_\eta \rho_{\alpha\eta}^{ts} \operatorname{tr}\,(E_\eta^{ts} E_{\zeta'}^{st}) = \rho_{\alpha\zeta}^{ts} f_{\zeta} \\ &= \sum_\beta \bar{\sigma}_{\zeta\beta}^{ts} \operatorname{tr}\,(\Phi_\alpha^{ts} \Phi_{\beta'}^{st}) \\ &= \sum_\beta \bar{\sigma}_{\zeta\beta}^{ts} \operatorname{tr}\,(V_\alpha^{ts} V_{\beta'}^{st}) = \bar{\sigma}_{\zeta\alpha}^{ts} n_\alpha^{ts}. \end{aligned} \tag{5.3}$$

Hence

$$\rho_{\alpha\zeta}^{ts} f_{\zeta} = \bar{\sigma}_{\zeta\alpha}^{ts} n_\alpha^{ts}. \tag{5.4}$$

Two sets of orthogonality relations follow from (5.4).

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[5] D. E. Littlewood, *The theory of group characters*, Oxford, 1940, p. 165. L. E. Dickson, *Algebras, and their arithmetics*, Chicago, 1923,

$$\sum_{\alpha} h^{st}_{\alpha}\rho^{ts}_{\alpha\eta}\bar{\rho}^{ts}_{\alpha\zeta} = (g/f_{\zeta s})\delta_{\eta\zeta}. \tag{5.5}$$

$$\sum_{\zeta} f_{\zeta s}\rho^{ts}_{\alpha\zeta}\bar{\rho}^{ts}_{\beta\zeta} = (g/h^{st}_{\alpha})\delta_{\alpha\beta} = n^{st}_{\alpha}\delta_{\alpha\beta}. \tag{5.6}$$

Equations (5.5) and (5.6) both reduce to the orthogonality relations among coefficients of F_j if $H^s = H^t = 1$. They may be expressed by the statement:

THEOREM 2. *The μ^{st}-dimensional square matrix, whose entries are the modified double coset coefficients $\rho^{ts}_{\alpha\zeta}(f_{\zeta s}/n^{st}_{\alpha})^{1/2}$, is a unitary matrix.*

6. **The double coset traces.** Although the double coset coefficients may depend on the particular choice of matrices Q^s_j which are used to adapt the representation F_j to the subgroup H^s, the traces $\tau^t_{\alpha j}$ of the double coset submatrices $\Phi^{tt}_{\alpha j}$ of (1.4) and (4.16) are invariants of the group and are uniquely determined. They were defined in (1.6).

THEOREM 3. *The double coset traces are algebraic integers.*

PROOF. The determinant $\left|\sum p^t_\alpha V^{tt}_\alpha\right|$ is a form of degree n^t in the parameters p^t_α with integral coefficients. After transforming the basis by U^t, the new matrix Φ^{tt}_α breaks up into the direct sum of submatrices $\Phi^{tt}_{\alpha j}$ of degree μ^t_j and multiplicity f_j. Hence

$$\left|\sum_\alpha p^t_\alpha V^{tt}_\alpha\right| = \left|\sum_\alpha p^t_\alpha \Phi^{tt}_\alpha\right| = \prod_j \left|\sum_\alpha p^t_\alpha \Phi^{tt}_{\alpha j}\right|^{f_j}. \tag{6.1}$$

The absolutely irreducible factors $\left|\sum p^t_\alpha \Phi^{tt}_{\alpha j}\right|$ of the rational integral form $\left|\sum p^t_\alpha V^{tt}_\alpha\right|$ must have coefficients which are algebraic integers in the extension field K in which the latter form factors. Expanding a given irreducible factor in powers of p^t_1, and noting that Φ^{tt}_{1j} is a unit matrix, we have

$$\left|\sum p^t_\alpha \Phi^{tt}_{\alpha j}\right| = (p^t_1)^{\mu_j} + (p^t_1)^{\mu_j - 1}\sum_{\alpha\neq 1} p^t_\alpha \tau^t_{\alpha j} + \cdots . \tag{6.2}$$

Hence $\tau^t_{\alpha j}$ is an algebraic integer for $\alpha \neq 1$. The trace of the μ^t_j-dimensional unit matrix is the integer $\tau^t_{1j} = \mu^t_j$. Q. E. D.

Orthogonality relations for the double coset traces can be derived from (5.5) by setting $s = t$ and summing over values of η and ζ corresponding to diagonal elements in $\Phi^{tt}_{\alpha j}$. We obtain

$$\sum_\alpha h^{tt}_\alpha \tau^t_{\alpha j}\bar{\tau}^t_{\alpha k} = (g\mu^t_j/f_j)\delta_{jk}. \tag{6.3}$$

These become the usual relations for group characters if H^t is the identity subgroup. Since

$$\tau^t_{1j} = \mu^t_j, \quad \text{and} \quad \tau^t_{\alpha 1} = h^t/h^{tt}_\alpha = n^{tt}_\alpha/n^t, \tag{6.4}$$

it is convenient whenever $\mu^t_j \neq 0$ to introduce the associated trace factors $\psi^t_{\alpha j}$,

$$\psi^t_{\alpha j} = \tau^t_{\alpha j}/\tau^t_{\alpha 1}\tau^t_{1j}, \qquad \text{where } \psi^t_{\alpha 1} = 1. \tag{6.5}$$

We may then rewrite (6.3) in the form

$$\sum_\alpha \tau^t_{\alpha j}\bar{\psi}^t_{\alpha k} = (n^t/f_j)\delta_{jk}, \qquad \text{if } \mu^t_j \neq 0. \tag{6.6}$$

This equation will be useful later in finding the degrees f_j.

We define the following quantities

χ^j_λ = trace of $F_j(\gamma)$ for γ in the class C_λ.

(6.7) g_λ = number of elements of G in C_λ.

$g^t_{\alpha\lambda}$ = number of elements of C_λ in the right coset $H^t\gamma_\alpha$.

$k^t_{\alpha\lambda} = g^t_{\alpha\lambda}\tau^t_{\alpha 1}$, number of elements of C_λ in the double coset H^{tt}_α.

Using the expression (4.15) for $s = t$ and thus computing the trace $\tau^t_{\alpha j}$ of the μ^t_j-dimensional matrix $\Phi^{tt}_{\alpha j}$ we find a relation between the double coset traces and group characters.

$$\tau^t_{\alpha j} = \sum_\lambda (k^t_{\alpha\lambda}/h^t)\chi^j_\lambda = \tau^t_{\alpha 1}\sum_\lambda (g^t_{\alpha\lambda}/h^t)\chi^j_\lambda. \tag{6.8}$$

Solving for $\bar{\psi}_{\alpha k}$ from (6.8) and (6.5), we substitute in (6.6) and multiply by h^t.

$$\sum_{\alpha,\mu} g^t_{\alpha\mu}\tau^t_{\alpha j}\bar{\chi}^k_\mu = \tau^t_{1k}(g/f_j)\delta_{jk}. \tag{6.9}$$

Multiplying by χ^k_λ in (6.9) and employing the orthogonality of group characters, we have after summing over k and solving for χ^j_λ,

$$\chi^j_\lambda = (f_j/g_\lambda)\sum_\alpha k^t_{\alpha\lambda}\psi^t_{\alpha j}. \tag{6.10}$$

This equation expresses the group characters in terms of the associated trace factors.

7. Determination of the degrees and characters of the irreducible representations. In a paper entitled *Double coset matrices and group characters* (abbreviated DCM)[2] the author defined the non-negative integral multiplication constants $c^{rst}_{\alpha\beta\gamma}$ for the double cosets by

$$H^{rs}_{\alpha}H^{st}_{\beta'}/h^{s} = \sum_{\gamma} c^{rst}_{\alpha\beta\gamma}H^{rt}_{\gamma} \tag{7.1}$$

and denoted by K^{st} the matrix whose elements in row α and column β are the linear functions $\sum c^{tst}_{\alpha\beta\gamma}b^{t}_{\gamma}$ of the parameters b^{t}_{γ}:

$$K^{st} = \Big(\sum_{\gamma} c^{tst}_{\alpha\beta\gamma}b^{t}_{\gamma}\Big). \tag{7.2}$$

It was stated erroneously on p. 82 of DCM that the determinant of $n^{t}K^{st}$ factors into irreducible factors which are linear in the parameters b^{t}_{γ}. The irreducible factors are indeed linear in the new parameters a_{λ} when we replace b^{t}_{γ} by

$$b^{t}_{\gamma} = \sum_{\lambda} k^{t}_{\gamma\lambda}a_{\lambda} \tag{7.3}$$

and this was the statement made on p. 90 of DCM in the principal theorem of the paper. But for multiplicities $\mu^{t}_{j}>1$ the corresponding irreducible factors are of degree μ^{t}_{j} and reduce to linear factors only when the b^{t}_{γ} are suitably restricted as in (7.3).

The correct general statement of the factoring of the determinant is given by the following theorem.

THEOREM 4. *The determinant of the matrix of linear combinations* $\sum_{\alpha}c^{tst}_{\rho\sigma\alpha}n^{tt}_{\alpha}p^{t}_{\alpha}$ *of double coset structure constants, in which* p^{t}_{α} *are arbitrary parameters and* ρ, σ *represent row and column indices, can be factored into the irreducible double coset factors* $\left|\sum_{\alpha}p^{t}_{\alpha}\Phi^{st}_{\alpha j}\right|$ *of degree* μ^{t}_{j} *occurring with multiplicities* μ^{s}_{j}. *The numerical coefficient of the* μ^{st} *power of* p^{t}_{1} *is the product of the* n^{tt}_{α}, *and this is equal to the product of* f_j *to the power* $\mu^{s}_{j}\mu^{t}_{j}$ *multiplied by an integer which is divisible by* $n^{s}n^{t}$. *This integer is the squared absolute value of the determinant* $\left|\rho^{st}_{\alpha\gamma}\right|$.

Furthermore, after making the substitution

$$p^{t}_{\alpha} = \sum_{\lambda} g^{t}_{\alpha\lambda}a_{\lambda} \tag{7.4}$$

the factors of degree μ^{t}_{j} each reduce to a power of a factor which is linear in the group characters; thus

$$\left|\sum p^{t}_{\alpha}\Phi^{tt}_{\alpha j}\right| = \Big(\sum_{\lambda} a_{\lambda}g_{\lambda}\chi^{j}_{\lambda}\Big)^{\mu^{t}_{j}}. \tag{7.5}$$

Stated in formulas we have

$$\left|\sum_{\alpha} c^{tst}_{\rho\sigma\alpha} n^{tt}_{\alpha} p^{t}_{\alpha}\right| = n^{s} n^{t} A^{st} \prod_{j} f_{j}^{\mu^{s}_{j}\mu^{t}_{j}} \left|\sum_{\alpha} p^{t}_{\alpha} \Phi^{tt}_{\alpha j}\right|^{\mu^{s}_{j}}, \tag{7.6}$$

$$\left|\sum_{\alpha,\lambda} c^{tst}_{\rho\sigma\alpha} n^{tt}_{\alpha} g_{\alpha\lambda} a_{\lambda}\right| = n^{s} n^{t} A^{st} \prod_{j} \left(f_{j} \sum_{\lambda} a_{\lambda} g_{\lambda} \chi^{j}_{\lambda}\right)^{\mu^{s}_{j}\mu^{t}_{j}} \tag{7.7}$$

where A^{st} is the integer

$$A^{st} = \left|\rho^{st}_{\alpha\eta}\right| \left|\bar{\rho}^{st}_{\alpha\eta}\right| / n^{s} n^{t}. \tag{7.8}$$

Proof of Theorem 4. Following the line of argument of pp. 87 and 89 of DCM we evaluate the trace (tr) of a matrix in two ways. Taking p^{t}_{α} as arbitrary parameters we have

$$\begin{aligned} \mathrm{X}^{st}_{\rho\sigma}(p) = \mathrm{tr}\left(V^{ts}_{\rho'} V^{st}_{\sigma} \sum_{\alpha} p^{t}_{\alpha} V^{tt}_{\alpha}\right) &= \mathrm{tr}\left(\sum_{\beta,\alpha} c^{tst}_{\rho\sigma\beta} V^{tt}_{\beta'} V^{tt}_{\alpha} p^{t}_{\alpha}\right) \\ &= \sum_{\alpha} c^{tst}_{\rho\sigma\alpha} n^{tt}_{\alpha} p^{t}_{\alpha}. \end{aligned} \tag{7.9}$$

Transforming by U and applying (1.5) we have

$$\begin{aligned} \mathrm{X}^{st}_{\rho\sigma}(p) &= \mathrm{tr}\left(\Phi^{ts}_{\rho'} \Phi^{st}_{\sigma} \sum_{\alpha} p^{t}_{\alpha} \Phi^{tt}_{\alpha}\right) \\ &= \sum_{\zeta,\eta} \bar{\rho}^{st}_{\rho\zeta} \rho^{st}_{\sigma\eta} \, \mathrm{tr}\left(E^{ts}_{\zeta'} E^{st}_{\eta} \sum_{\alpha} p^{t}_{\alpha} \Phi^{tt}_{\alpha}\right) \end{aligned} \tag{7.10}$$

where as usual the subscripts η and ζ each stand for three indices $j, \lambda\mu$, and so on. Now the trace in (7.10) vanishes for all pairs of values ζ, η except those corresponding to the same j and λ. If ζ is $j, \lambda\nu$ and η is $j, \lambda\mu$ we obtain a nonvanishing trace only from the component of $\Phi^{tt}_{\alpha j}$ in the μth set of f_j rows and the νth set of f_j columns. The trace is then equal to $f_j \sum_{\alpha} p^{t}_{\alpha} \phi^{tt}_{\alpha j,\mu\nu}$. For each j there are μ^{s}_{j} values of λ. Hence the matrix $\mathrm{X}^{st}_{\rho\sigma}(p)$ may be written as the product of three matrices in which the first and last factors are $(\rho^{st}_{\sigma\eta})$ and its transposed conjugate, and the second matrix is the direct sum of the matrices $f_j \sum p^{t}_{\alpha} \Phi^{tt}_{\alpha j}$ each occurring as a summand with multiplicity μ^{s}_{j}. Comparing determinants of (7.9) and (7.10) and using the formula (7.8) which was proved in DCM, we obtain the first stated result.

If the p^{t}_{α} are expressed as linear combinations of a_{λ} by (7.4), however, then the matrices $\sum p^{t}_{\alpha} \Phi^{tt}_{\alpha j}$ become diagonal matrices and their determinants factor into linear factors (7.5) as proved in DCM. The comparison of numerical coefficients is immediate,[6] since p^{t}_{1} occurs only along the principal diagonals of $X^{\alpha}_{\mu\mu}(p)$ and of $\sum p^t_\alpha \Phi^{\alpha}_{\alpha j}$. Q.E.D.

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[6] J. S. Frame, *The double cosets of a finite group*, Bull. Amer. Math. Soc. vol. 47 (1941) p. 459, Theorem B.

To apply Theorem 4 to the determination of group characters let us suppose that H^s and H^t are two large subgroups of G such that the number of common irreducible components F_j of R^s and R^t is small, and μ^{st} is small, possibly not more than 5 or 6, preferably smaller yet. Then if the structure of the group G is known it is not difficult to obtain the constants $c^{tst}_{\rho\sigma\alpha}$ from the multiplication of double cosets. Factoring the determinant as in (7.6) we obtain the double coset traces $\tau^t_{\alpha j}$ as coefficients of $(p^t_1)^{m-1}p^t_\alpha$ in an irreducible factor of degree $m=\mu^t_j$ in which $(p^t_1)^m$ has coefficient 1. The associated factors $\psi^t_{\alpha j}$ are found from (6.5), the degrees f_j from (6.6), and finally the characters themselves from (6.10).

In simple cases where a subgroup H^t has only three or four double cosets we may even compute the double coset traces directly without using the multiplication constants $c^{tst}_{\rho\sigma\alpha}$. We must then have $\mu^t_j=\tau^t_{1j}=1$, $\sum_j \tau^t_{\alpha j}=0$ for $\alpha\neq 1$, and we know the $\tau^t_{\alpha 1}$ from (6.4). We substitute from (6.5) into (6.6) and try to find algebraic integers $\tau^t_{\alpha j}$ for which $\tau^t_{\alpha j}$ and $\bar{\psi}^t_{\alpha k}$ are orthogonal $(j\neq k)$ and for which

$$(7.11) \qquad f_j = \frac{n^t}{\sum\limits_{\alpha} \tau^t_{\alpha j}\bar{\tau}_{\alpha j}/\tau^t_{\alpha 1}}$$

is an integer. If no two irreducible components F_j occurring in R^t_U are algebraically conjugate, the $\tau^t_{\alpha j}$ will be rational integers such that $\sum_\alpha \tau^t_{\alpha j}=0$ for $j\neq 1$.

In the example on pp. 190–192 in DCM we had $\mu^{st}=2$, $n^t=45$. Expressed in the terms of parameters p_α the two linear factors of the determinants (7.6) are then equal to $(p_1+12p_2+32p_3)$ and $(p_1+3p_2-4p_3)$. Hence the matrices $(\tau^t_{\alpha j})$ and $(\psi^t_{\alpha j})$ are

$$(7.12) \qquad (\tau^t_{\alpha j}) = \begin{Vmatrix} 1 & 1 \\ 12 & 3 \\ 32 & -4 \end{Vmatrix}, \qquad (\psi^t_{\alpha j}) = \begin{Vmatrix} 1 & 1 \\ 1 & 1/4 \\ 1 & -1/8 \end{Vmatrix}$$

and we find immediately $f_1=1$, $f_2=45/(1+9/12+16/32)=20$, $\chi^{(2)}_\lambda=20(g_{1\lambda}+3g_{2\lambda}-4g_{3\lambda})$.

Similarly, considering the permutation group of the 27 lines on the cubic surface, we have just three double cosets, with $\tau^s_{\alpha 1}=1, 10, 16$. Hence $\mu^{ss}=3$. Either τ_{22} or τ_{33} is non-negative. Suppose it is τ_{33}. Setting $\tau_{33}=x$ and $\tau_{22}=y$ we must find non-negative integral solutions (x, y) of the equation

$$(7.13) \quad 1 - y(x+1)/10 - x(y+1)/16 = 0, \text{ or } 13xy + 5x + 8y \doteq 80.$$

Hence $xy<6$, and $y(x+1)\equiv 0 \pmod 5$. The solutions for (x, y) are $(0, 10)$, $(16, 0)$ and $(4, 1)$. Only the last choice gives integral values for f_j and we then have by (6.5) and (7.11),

$$(7.14)\quad (\tau^*_{\alpha j}) = \begin{Vmatrix} 1 & 1 & 1 \\ 10 & -5 & 1 \\ 16 & 4 & -2 \end{Vmatrix}, \qquad (\psi^*_{\alpha j}) = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & -1/2 & 1/10 \\ 1 & 1/4 & -1/8 \end{Vmatrix},$$

$$f_2 = \frac{27}{4.5} = 6,$$

$$f_3 = \frac{27}{1.35} = 20.$$

The irreducible components have degrees 1, 6, 20, and the characters may be found by applying (6.10).

MICHIGAN STATE COLLEGE

EQUAL SUMS OF LIKE POWERS

E. M. WRIGHT

Let $s\geqq 2$ and let $P(k, s)$ be the least value of j such that the equations

$$(1)\qquad \sum_{i=1}^{j} a_{i1}^{h} = \sum_{i=1}^{j} a_{i2}^{h} = \cdots = \sum_{i=1}^{j} a_{is}^{h} \qquad (1 \leqq h \leqq k)$$

have a nontrivial solution in integers, that is, a solution in which no set $\{a_{iu}\}$ is a permutation of another set $\{a_{iv}\}$. It was remarked by Bastien [1][1] that $P(k, 2)\geqq k+1$ and this is true *a fortiori* for general s. The only upper bound for $P(k, s)$ for general k and s which I have found in the literature is due to Prouhet [5] who (in 1851) gave solutions of (1) with $j=s^k$, so that $P(k, s)\leqq s^k$. He allocates each of the numbers $0, 1, \cdots, s^{k+1}-1$ to the set $\{a_{iu}\}$ if the sum of its digits in the scale of s is congruent to $u \pmod s$. Recently Lehmer [4] took $m_1, \cdots, m_{k+1}$ any $k+1$ integers, let each of $b_1, \cdots, b_{k+1}$ run through

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

the values 0, 1, $\cdots$, $s-1$ and allocated the number

$$b_1m_1 + \cdots + b_{k+1}m_{k+1} \tag{2}$$

to the set $\{a_{iu}\}$ if $\sum b_l \equiv u \pmod{s}$. Lehmer's method provides a solution which may be trivial, though any set of m_l which makes the numbers (2) all different will certainly give a nontrivial solution. Prouhet's case, in which $m_l = s^{l-1}$ $(1 \leq l \leq k+1)$, clearly does this.

The problem of determining $P(k, 2)$ has received much attention. The inequality $P(k, 2) \leq 2^k$, a particular case of Prouhet's result, was rediscovered in 1912 by Tarry [6] and by Escott [8]. This has since been improved [7] to

$$P(k, 2) \leq (k^2 + 4)/2. \tag{3}$$

In this note I find upper bounds for $P(k, s)$ for general k independent of s and comparable with (3). Unlike Prouhet I do not find a particular solution of (1), but my method gives bounds for the a. I cannot prove that $P(k, s)$ is independent of s, though I conjecture (somewhat more tentatively than for $P(k, 2)$ in [7]) that $P(k, s) = k+1$.

Various authors [2, 3] have shown that $P(k, 2) = k+1$ for $1 \leq k \leq 9$ and Gloden [3] proved that $P(k, s) = k+1$ for $k = 2$, 3, and 5 and for all s.

THEOREM 1. *$P(k, s) \leq (k^2+k+2)/2$.*

Let $j = (k^2+k+2)/2$, $n = (s-1)j!j^k$, $1 \leq a_r \leq n$ $(1 \leq r \leq j)$, and

$$l_h = a_1^h + \cdots + a_j^h.$$

Then $j \leq l_h \leq jn^h$ and so there are at most

$$\prod_{h=1}^{k} (jn^h - j + 1) < j^k n^{k(k+1)/2}$$

different sets $l_1, \cdots, l_k$. But there are n^j different sets $a_1, \cdots, a_j$ and so more than $j^{-k}n^{j-k(k+1)/2} = (s-1)j!$ sets $a_1, \cdots, a_j$ associated with some one set $l_1, \cdots, l_k$. Since the number of permutations of j objects among themselves is $j!$, there are at least s sets $a_1, \cdots, a_j$ which have the same $l_1, \cdots, l_k$ and none of which is a permutation of any other. These provide a nontrivial solution of (1) with $1 \leq a_{iu} \leq (s-1)j!j^k$.

THEOREM 2. *If k is odd, $P(k, s) \leq (k^2+3)/2$.*

For $k = 1$ the theorem is trivial. Let k be odd, $k \geq 3$, $m = (k-1)/2$,

$t=m(m+1)+1$, $n=(s-1)t!t^m$, $1\leq a_r\leq n$ $(1\leq r\leq t)$, and

$$L_h = a_1^{2h} + \cdots + a_t^{2h}.$$

Since $t\leq L_h\leq tn^{2h}$, the number of different sets $L_1, L_2, \cdots, L_m$ is at most

$$\prod_{h=1}^{m} (tn^{2h} - t + 1) < t^m \prod_{h=1}^{m} n^{2h} = t^m n^{t-1}.$$

But there are n^t different sets $a_1, \cdots, a_t$ and so more than $t^{-m}n(t!)^{-1} = s-1$ sets $a_1, \cdots, a_t$ which have the same $L_1, \cdots, L_m$ and none of which is a permutation of any other. We take s of these sets, denote the numbers in them by $a_1^{(u)}, \cdots, a_t^{(u)}$ $(1\leq u\leq s)$ and put

$$\begin{aligned} j &= 2t = (k^2+3)/2, & \\ a_{iu} &= n + 1 + a_i^{(u)} & (1 \leq i \leq t), \\ a_{iu} &= n + 1 - a_{i-t}^{(u)} & (t + 1 \leq i \leq j) \end{aligned}$$

in (1). Since

$$\sum_{i=1}^{j} a_{iu}^{h} = j(n+1)^h + 2\binom{h}{2}(n+1)^{h-2}L_1 + 2\binom{h}{4}(n+1)^{h-4}L_2 + \cdots$$

and this is the same for all u when $1\leq h\leq k$, we have a nontrivial solution of (1).

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THE UNIVERSITY OF ABERDEEN

ONE-TO-ONE MAPPINGS OF RINGS AND LATTICES

C. E. RICKART

1. **Introduction.** Let $\mathfrak{R}$ and $\mathfrak{S}$ be two arbitrary rings and let $\phi(r)$ denote a one-to-one, multiplicative mapping of $\mathfrak{R}$ onto $\mathfrak{S}$. The major portion of this paper is devoted to the proofs of two theorems which state that, under certain conditions on $\mathfrak{R}$, the mapping $\phi(r)$ is automatically additive. In Theorem I (§2), $\mathfrak{R}$ is an arbitrary Boolean ring and in Theorem II (§3), $\mathfrak{R}$ is any ring which contains a family $\mathcal{J}$ of minimal right ideals satisfying the following two conditions: (i) $Rr=(0)$, for every $R\in\mathcal{J}$, implies $r=0$ and (ii) each $R\in\mathcal{J}$ is of dimension greater than one over the division ring of all endomorphisms of R which commute with each endomorphism induced in R through right multiplication by elements of $\mathfrak{R}$. In §4 it is proved (Theorem III) that a one-to-one, meet preserving mapping of a distributive lattice onto a distributive lattice is necessarily join preserving.

The class of rings considered in Theorem II contains as a special case the ring of all bounded linear operators on a Banach space of dimension greater than one. Theorem II, with the additional hypothesis that the mapping be continuous, was proved in this special case by Eidelheit[1] [2].[2] The finite-dimensional case of the Eidelheit theorem was obtained by Nagumo [5].

This paper has been greatly influenced by conversations which we have had with B. J. Pettis. In particular, our interest in the questions considered here was stimulated by Pettis' conjecture that the Eidelheit theorem mentioned above could be obtained without the continuity hypothesis.

2. **Boolean rings.** By a Boolean ring [6] we mean a ring each of whose elements satisfies the conditions $r^2=r$. It is not difficult to show that a Boolean ring is necessarily commutative and that each of its elements also satisfies the condition $r+r=0$.

THEOREM I. *Any one-to-one, multiplicative mapping of a Boolean ring* $\mathfrak{B}$ *onto an arbitrary ring* $\mathfrak{S}$ *is necessarily additive.*

Let $\phi(r)$ denote the one-to-one, multiplicative mapping of $\mathfrak{B}$ onto

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[1] Eidelheit, in fact, assumed $\mathfrak{S}$, as well as $\mathfrak{R}$, to consist of all bounded linear operators on a Banach space. He also assumed the Banach spaces to be real.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

$\mathfrak{S}$. It is evident that the multiplicative condition on $\phi(r)$ implies that $\mathfrak{S}$ must also be a Boolean ring. Moreover $\phi(r)=0$ if, and only if, $r=0$. In fact, since the range of $\phi(r)$ covers $\mathfrak{S}$, there exists $u\in\mathfrak{B}$ such that $\phi(u)=0$. Therefore

$$\phi(0) = \phi(0u) = \phi(0)\cdot\phi(u) = 0.$$

That $u=0$ follows from the fact that $\phi(r)$ is one-to-one.

Now let r, s be arbitrary elements of $\mathfrak{B}$ and choose $t\in\mathfrak{B}$ so that

$$\phi(t) = \phi(r) + \phi(s).$$

Observe that, since the mapping is multiplicative,

$$\phi(rt + st) = \phi(r + rs) + \phi(s + rs), \tag{1}$$

$$\phi(rt) = \phi(r) + \phi(rs), \tag{2}$$

$$\phi(st) = \phi(rs) + \phi(s). \tag{3}$$

Consider first the case in which $rs=0$. Then (1), (2) and (3) become

$$\phi(rt + st) = \phi(r) + \phi(s), \tag{1$'$}$$

$$\phi(rt) = \phi(r), \tag{2$'$}$$

$$\phi(st) = \phi(s). \tag{3$'$}$$

Since the mapping is one-to-one, equations (1)$'$, (2)$'$ and (3)$'$ imply respectively $rt+st=t$, $rt=r$ and $st=s$. It follows that $t=r+s$ and therefore

$$\phi(r + s) = \phi(r) + \phi(s). \tag{4}$$

Now consider the case in which $rs=s$. Then (1), (2) and (3) become

$$\phi(rt + st) = \phi(r + s), \tag{1$''$}$$

$$\phi(rt) = \phi(r) + \phi(s), \tag{2$''$}$$

$$\phi(st) = 0. \tag{3$''$}$$

Again, since the mapping is one-to-one, equations (1)$''$, (2)$''$ and (3)$''$ imply respectively $rt+st=r+s$, $rt=t$ and $st=0$. Therefore, $t=r+s$ and it follows that equation (4) holds in this case as well.

In the general case, observe that

$$r + s = (r + rs) + (s + rs).$$

Since $(r+rs)(s+rs)=0$, the first case considered above gives

$$\phi(r + s) = \phi(r + rs) + \phi(s + rs). \tag{5}$$

Moreover, since $r(rs)=rs$ and $s(rs)=rs$, the second case considered above gives

$$\phi(r+rs)=\phi(r)+\phi(rs),\ \phi(s+rs)=\phi(s)+\phi(rs).$$

Substituting these values in (5), we again obtain (4) which completes the proof.

COROLLARY. *In a Boolean ring, the operation of addition is uniquely determined by the operation of multiplication.*

3. **Rings with minimal ideals.** Let $\mathfrak{R}$ be an arbitrary ring which contains a minimal right ideal R. For each $r\in\mathfrak{R}$, let T_r denote the endomorphism of the additive group of R defined by $xT_r=xr$, $x\in R$. Since R is minimal, the T_r constitute an irreducible representation of $\mathfrak{R}$ in the ring of endomorphisms of R. By Schur's lemma, the family Φ of all endomorphisms of R which commute with each T_r is a division ring. Therefore, R can be considered as a linear vector space over Φ. The linear space associated in this way with R will be denoted by $\overline{R}$. By a result of Jacobson [3, Theorem 6], the T_r constitute a dense ring of linear operators on $\overline{R}$. In other words, for arbitrary positive integer k and arbitrary elements $x_1, \cdots, x_k, y_1, \cdots, y_k$ in R, where the $x_1, \cdots, x_k$ are linearly independent in $\overline{R}$, there exists $r\in\mathfrak{R}$ such that $x_ir=y_i$ for $i=1, \cdots, k$.

LEMMA. *Let $\mathfrak{R}$ be a ring which contains a minimal right ideal R whose associated linear vector space $\overline{R}$ is of dimension greater than one. Then any one-to-one, multiplicative mapping of $\mathfrak{R}$ onto an arbitrary ring $\mathfrak{S}$ is additive on R.*

Let $\phi(r)$ denote the one-to-one, multiplicative mapping of $\mathfrak{R}$ onto $\mathfrak{S}$ and observe (as in the proof of Theorem I) that $\phi(r)=0$ if, and only if, $r=0$.

Consider arbitrary elements $x, y\in R$ and choose $z\in\mathfrak{R}$ such that

$$\phi(z)=\phi(x)+\phi(y).$$

Assume first that x, y are linearly independent in $\overline{R}$. Then, by the density property, there exist $r, s\in\mathfrak{R}$ such that

$$xr=x,\qquad yr=0,\qquad xs=0,\qquad ys=y.$$

Observe that

$$\begin{aligned}\phi(zr)&=\phi(x)\cdot\phi(r)+\phi(y)\cdot\phi(r)\\&=\phi(xr)+\phi(yr)\\&=\phi(x).\end{aligned}$$

Therefore, since the mapping is one-to-one, $zr=x$. Similarly we obtain $zs=y$. It follows that

$$x+y=z(r+s).$$

Note also that

$$x(r+s)=x, \qquad y(r+s)=y.$$

Therefore

$$\begin{aligned}\phi(x+y)&=\phi(z(r+s))\\&=\phi(x)\cdot\phi(r+s)+\phi(y)\cdot\phi(r+s)\\&=\phi(x(r+s))+\phi(y(r+s))\\&=\phi(x)+\phi(y).\end{aligned}$$

This proves that $\phi(r)$ is additive on linearly independent elements of $\overline{R}$.

Next assume that x and y are linearly dependent in $\overline{R}$. Since $\overline{R}$ is of dimension greater than one, there exists $u\in\overline{R}$ linearly independent of x and y. Since $\phi(r)$ is additive on linearly independent elements of $\overline{R}$, we have

$$\begin{aligned}\phi(x+y)+\phi(u)&=\phi(x+y+u)\\&=\phi(x)+\phi(y+u)\\&=\phi(x)+\phi(y)+\phi(u).\end{aligned}$$

Therefore

$$\phi(x+y)=\phi(x)+\phi(y)$$

and the proof is complete.

COROLLARY. *Under the hypothesis of the lemma,* $\phi(R)$ *is a minimal right ideal in* $\mathfrak{S}$.

THEOREM II. *Let* $\mathfrak{R}$ *contain a family* $\mathcal{J}$ *of minimal right ideals which satisfy the following conditions*: (i) $Rr=(0)$, *for each* $R\in\mathcal{J}$, *implies* $r=0$; (ii) *The linear space* $\overline{R}$ *associated with each* $R\in\mathcal{J}$ *is of dimension greater than one. Then any one-to-one, multiplicative mapping of* $\mathfrak{R}$ *onto an arbitrary ring* $\mathfrak{S}$ *is necessarily additive.*

Denote by $\phi(r)$ the one-to-one, multiplicative mapping of $\mathfrak{R}$ onto $\mathfrak{S}$. Let r, s be arbitrary elements of $\mathfrak{R}$ and choose $t\in\mathfrak{R}$ such that

$$\phi(t)=\phi(r)+\phi(s).$$

The proof consists in showing that $t=r+s$. Let R be any element of $\mathcal{J}$ and observe that, if $x \in R$, then

$$\phi(xt) = \phi(xr) + \phi(xs).$$

Moreover, by the above lemma,

$$\phi(xr + xs) = \phi(xr) + \phi(xs).$$

Therefore

$$xr + xs = xt.$$

It follows that

$$R(r + s - t) = (0).$$

Since R was an arbitrary element of $\mathcal{J}$, we conclude that $t = r+s$.

COROLLARY. *In any ring which satisfies the conditions of Theorem* II, *the operation of addition is uniquely determined by the operation of multiplication.*

COROLLARY. *Under the hypothesis of Theorem* II, *any one-to-one, anti-multiplicative*[3] *mapping of $\mathfrak{R}$ onto an arbitrary ring is necessarily additive.*

The following example shows that the dimension restriction in Theorem II cannot be dropped. It also shows, incidentally, that the Boolean ring in Theorem I cannot be replaced by a generalized Boolean ring[4] in the sense of McCoy and Montgomery [4]. Let $\mathfrak{R}$ and $\mathfrak{S}$ both equal the finite field F_p of integers modulo the prime p. In this case, $\mathfrak{R}$ is a generalized Boolean ring (of index p) which contains only one minimal ideal, namely, $\mathfrak{R}$ itself. Condition (i) of Theorem II is clearly satisfied, although the dimension condition (ii) does not hold. Now consider the mapping $\phi(x)$ of $\mathfrak{R}$ into $\mathfrak{S}$ defined by $\phi(x) = x^3$. It is obvious that $\phi(x)$ is multiplicative. Moreover, in the special case $p = 5$, $\phi(x)$ defines a one-to-one mapping of $\mathfrak{R}$ onto $\mathfrak{S}$. In fact, $\phi(0) = 0$, $\phi(1) = 1$, $\phi(2) = 3$, $\phi(3) = 2$ and $\phi(4) = 4$. However, since $\phi(1+2) \neq \phi(1)+\phi(2)$, the mapping is not additive. The above example, with $\mathfrak{R}$ and $\mathfrak{S}$ equal to the field of real numbers instead of F_5, was given by Eidelheit [2, p. 103 nb].

It is perhaps desirable to remark briefly on the relationship between Theorems I and II. Note, in the first place, that in Theorem I the Boolean ring $\mathfrak{B}$ need not contain any minimal ideals. Moreover, even if $\mathcal{B}$ does contain a minimal ideal R, it is evident that R contains only one nonzero element. It follows that $\bar{R}$ cannot have dimension greater than one. Therefore, the hypotheses of Theorem II cannot be satisfied for a Boolean ring. On the other hand, it is trivial to verify that the lemma for Theorem II holds for Boolean rings without the dimension requirement. Hence, the proof of Theorem II could still be carried through for Boolean rings if condition (i) were satisfied. We also note at this point that Theorem I could be obtained from Theorem III of the next section.

---

[3] That is, $\phi(rs) = \phi(s)\phi(r)$.

[4] A generalized Boolean ring (of index p) is a commutative ring for which there exists a prime p such that, for every element r of the ring, $r^p = r$ and $pr = 0$.

4. **Distributive lattices.** We obtain now a theorem for distributive lattices[5] analogous to the ring theorems obtained above.

THEOREM III. *Any one-to-one, meet-preserving mapping of a distributive lattice onto a distributive lattice is also join-preserving.*

Let $\mathcal{L}$ and $\mathfrak{M}$ be arbitrary distributive lattices and let $\phi(x)$ denote a one-to-one, meet-preserving mapping of $\mathcal{L}$ onto $\mathfrak{M}$. Consider two arbitrary elements $x, y \in \mathcal{L}$ and choose $z \in \mathcal{L}$ such that

$$\phi(z) = \phi(x) \cup \phi(y).$$

The proof consists in showing that $z = x \cup y$. Observe first that

$$\begin{aligned} \phi(z \cap (x \cup y)) &= \phi(z) \cap \phi(x \cup y) \\ &= [\phi(x) \cap \phi(x \cup y)] \cup [\phi(y) \cap \phi(x \cup y)] \\ &= \phi(x \cap (x \cup y)) \cup \phi(y \cap (x \cup y)) \\ &= \phi(x) \cup \phi(y) \\ &= \phi(z). \end{aligned}$$

Since the mapping is one-to-one, it follows that

$$z = z \cap (x \cup y) = (z \cap x) \cup (z \cap y).$$

Also

$$\phi(z \cap x) = [\phi(x) \cap \phi(x)] \cup [\phi(y) \cap \phi(x)] = \phi(x) \cup \phi(y \cap x).$$

But

$$\phi(x) \cap \phi(y \cap x) = \phi(x \cap y \cap x) = \phi(y \cap x).$$

Therefore

$$\phi(z \cap x) = \phi(x) \cup \phi(y \cap x) = \phi(x).$$

Similarly

---

[5] The terminology used here conforms with [1].

$$\phi(z \cap y) = \phi(y).$$

Again, since the mapping is one-to-one, we have $z \cap x = x$ and $z \cap y = y$. It follows that $z = x \cup y$ and the proof is complete.

Since the dual of a distributive lattice is also distributive, we can state the following corollaries:

COROLLARY. *Any one-to-one, join-preserving mapping of a distributive lattice onto a distributive lattice is also meet-preserving.*

COROLLARY. *Any one-to-one mapping of a distributive lattice onto a distributive lattice which takes meets into joins also takes joins into meets.*

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YALE UNIVERSITY

ZEROS OF THE HERMITE POLYNOMIALS AND WEIGHTS FOR GAUSS' MECHANICAL QUADRATURE FORMULA

ROBERT E. GREENWOOD AND J. J. MILLER

In the numerical integration of a function $f(x)$ it is very desirable to choose the set of values $\{x_i\}$ at which the function $f(x)$ is to be observed, for it is generally possible to obtain the same accuracy with fewer points when these points are especially selected. Gauss[1] gave such a proof for the case of the finite range $(-1, +1)$ and established that the "best" accuracy with n ordinates is obtained when the corresponding abscissae are the n roots of the Legendre polynomials, $P_n(x)=0$. For this case there obtains

$$\int_{-1}^{1} f(x)dx \simeq \sum_{i=1}^{n} \lambda_{i,n} f(x_{i,n}) \tag{1}$$

where the numbers $\{x_{i,n}\}$ are the zeros of $P_n(x)$ and where the numbers $\{\lambda_{i,n}\}$ are the Christoffel or Cotes numbers. Formula (1) is exact whenever $f(x)$ is a polynomial of degree $(2n-1)$ or less. Values of the zeros $\{x_{i,n}\}$ and the corresponding Christoffel numbers $\{\lambda_{i,n}\}$ for the Legendre polynomials for $n=1$ to $n=16$ have been tabulated by the Mathematical Tables Project.[2] The range of integration can be chosen to be any finite range (p, q) with suitable modification[2] of the zeros $\{x_{i,n}\}$ and the constants $\{\lambda_{i,n}\}$.

It is understood that while selection of the abscissae $\{x_{i,n}\}$ is very desirable for theoretical reasons, it may not always be practicable to measure the ordinates of $f(x)$ at these values.

For the infinite range $(-\infty, +\infty)$ a similar situation holds for the Hermite polynomials. These may be defined by the relation

$$\begin{aligned} H_n(x) &= (-1)^n e^{x^2} \frac{d^n(e^{-x^2})}{dx^n} \\ &= (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} \pm \cdots . \end{aligned} \tag{2}$$

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[1] C. F. Gauss, *Methodus nova integralium valores per approximationem inveniendi*, Werke, vol. 3, pp. 163–196.

[2] A. N. Lowan, Norman Davids and Arthur Levenson, *Table of the zeros of the Legendre polynomials of order* 1–16 *and the weight coefficients for Gauss' mechanical quadrature formula*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 739–743.

For n even, the last term is

$$(-1)^{n/2}\frac{n!}{(n/2)!}$$

and for n odd, the last term is

$$(-1)^{(n-1)/2}\frac{n!}{((n-1)/2)!}(2x).$$

These polynomials obey the recursion relations

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0, \tag{3}$$

$$H_n'(x) = 2nH_{n-1}(x). \tag{4}$$

[Some writers, including many statisticians, prefer to use

$$h_n(x) = e^{x^2/2}\frac{d^n(e^{-x^2/2})}{dx^n} \tag{5}$$

as the defining relation for Hermite polynomials. The relation between these two sets of polynomials is given by

$$H_n(x) = (-2^{1/2})^n h_n(2^{1/2}x).] \tag{6}$$

The approximate numerical integration formula for functions $f(x)$ on the infinite range $(-\infty, +\infty)$ with the weight function $\exp(-x^2)$ is

$$\int_{-\infty}^{\infty} e^{-x^2}f(x)dx \simeq \sum_{i=1}^{n}\lambda_{i,n}f(x_{i,n}) \tag{7}$$

where the set $\{x_{i,n}\}$ is the set of roots defined by

$$H_n(x) = 0 \tag{8}$$

and where the set $\{\lambda_{i,n}\}$ is given by[3]

$$\lambda_{i,n} = \frac{\pi^{1/2}2^{n+1}n!}{[H_n'(x_{i,n})]^2}. \tag{9}$$

If $f(x)$ is a polynomial of degree $(2n-1)$ or less, integration formula (7) is exact.[4]

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[3] Gabor Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939, p. 344.

[4] Szegö, op. cit. Chapter XV.

C. Winston, *On mechanical quadrature formula involving classical orthogonal polynomials*, Ann. of Math. (2) vol. 35 (1934) pp. 658–677.

The zeros $\{X_{i,n}\}$ for the polynomials $h_n(x)$ for $n=1$ to $n=27$ have been tabulated by Smith[3] to six decimal places. The corresponding zeros $\{x_{i,n}\}$ for the Hermite polynomial $H_n(x)$ are given by

$$x_{i,n} = \frac{1}{2^{1/2}} X_{i,n}. \tag{10}$$

Newton's tangent rule was used to improve Smith's values, and the zeros $\{x_{i,n}\}$ for $n=1$ to $n=5$ are tabulated herewith to twelve decimal places and for $n=6$ to $n=10$ to nine decimal places. The set of Christoffel numbers $\{\lambda_{i,n}\}$ were calculated from relation (9).

The results were subject to the following tests:

$$\prod \text{positive roots} = \left(\frac{|\text{coefficient of either } x^0 \text{ or } x \text{ term}|}{\text{coefficient of } x^n \text{ term}}\right)^{1/2}, \tag{11}$$

$$\sum x_{i,n}^2 = \frac{n(n-1)}{2}, \tag{12}$$

$$\sum \lambda_{i,n} = \pi^{1/2}, \tag{13}$$

$$\sum \lambda_{i,n} x_{i,n}^2 = \frac{\pi^{1/2}}{2}, \tag{14}$$

where (11) and (12) are obtained from the algebraic expressions for the products of the roots and the sum of the squares of the roots and where (13) and (14) are obtained from (7) by taking $f(x)=1$ and $f(x)=x^2$ respectively. For the most part, calculations were carried to two more places than were retained.

Because of symmetry, it is convenient to tabulate the zeros of $H_3(x)$ as x_{-1}, x_0, x_1 rather than as x_1, x_2, x_3, and the zeros of $H_4(x)$ as x_{-2}, x_{-1}, x_1, x_2 rather than as x_1, x_2, x_3, x_4, with similar notation for other values of n. Since there is but slight danger of confusion, the second subscript n on the zeros and Christoffel numbers has been omitted. Because of the relations $x_{-i}=-x_i$ and $\lambda_{-i}=\lambda_i$ only values with non-negative subscripts have been tabulated.

$n=1$ $H_1(x)=2x$

$x_0=0.000\ 000\ 000\ 000$ $\lambda_0=1.772\ 453\ 850\ 906$

$n=2$ $H_2(x)=4x^2-2$

$x_1=0.707\ 106\ 781\ 187$ $\lambda_1=0.886\ 226\ 925\ 453$

$n=3$ $H_3(x)=8x^3-12x$

$x_0=0.000\ 000\ 000\ 000$ $\lambda_0=1.181\ 635\ 900\ 604$

$x_1=1.224\ 744\ 871\ 392$ $\lambda_1=0.295\ 408\ 975\ 151$

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[3] E. R. Smith, *Zeroes of the Hermitean polynomials*, Amer. Math. Monthly vol. 43 (1936) pp. 354–358.

$n=4$	$H^4(x)=16x^4-48x^2+12$	
	$x_1=0.524\ 647\ 623\ 275$	$\lambda_1=0.804\ 914\ 090\ 006$
	$x_2=1.650\ 680\ 123\ 886$	$\lambda_2=0.081\ 312\ 835\ 447\ 3$
$n=5$	$H_5(x)=32x^5-160x^3+120x$	
	$x_0=0.000\ 000\ 000\ 000$	$\lambda_0=0.945\ 308\ 720\ 483$
	$x_1=0.958\ 572\ 464\ 614$	$\lambda_1=0.393\ 619\ 323\ 152$
	$x_2=2.020\ 182\ 870\ 456$	$\lambda_2=0.019\ 953\ 242\ 059\ 0$
$n=6$	$H_6(x)=64x^6-480x^4+720x^2-120$	
	$x_1=0.436\ 077\ 412$	$\lambda_1=0.724\ 629\ 595$
	$x_2=1.335\ 849\ 074$	$\lambda_2=0.157\ 067\ 320$
	$x_3=2.350\ 604\ 974$	$\lambda_3=0.004\ 530\ 009\ 90$
$n=7$	$H_7(x)=128x^7-1344x^5+3360x^3-1680x$	
	$x_0=0.000\ 000\ 000$	$\lambda_0=0.810\ 264\ 618$
	$x_1=0.816\ 287\ 883$	$\lambda_1=0.425\ 607\ 253$
	$x_2=1.673\ 551\ 629$	$\lambda_2=0.054\ 515\ 582\ 8$
	$x_3=2.651\ 961\ 357$	$\lambda_3=0.000\ 971\ 781\ 258$
$n=8$	$H_8(x)=256x^8-3584x^6+13{,}440x^4-13{,}440x^2+1680$	
	$x_1=0.381\ 186\ 990$	$\lambda_1=0.661\ 147\ 013$
	$x_2=1.157\ 193\ 712$	$\lambda_2=0.207\ 802\ 326$
	$x_3=1.981\ 656\ 757$	$\lambda_3=0.017\ 077\ 983\ 0$
	$x_4=2.930\ 637\ 420$	$\lambda_4=0.000\ 199\ 604\ 071$
$n=9$	$H_9(x)=512x^9-9216x^7+48{,}384x^5-80{,}640x^3+30{,}240x$	
	$x_0=0.000\ 000\ 000$	$\lambda_0=0.720\ 235\ 216$
	$x_1=0.723\ 551\ 019$	$\lambda_1=0.432\ 651\ 559$
	$x_2=1.468\ 553\ 289$	$\lambda_2=0.088\ 474\ 527\ 4$
	$x_3=2.266\ 580\ 585$	$\lambda_3=0.004\ 943\ 624\ 28$
	$x_4=3.190\ 993\ 202$	$\lambda_4=0.000\ 039\ 606\ 977\ 4$
$n=10$	$H_{10}(x)=1024x^{10}-23{,}040x^8+161{,}280x^6-403{,}200x^4$ $+302{,}400x^2-30{,}240$	
	$x_1=0.342\ 901\ 327$	$\lambda_1=0.610\ 862\ 634$
	$x_2=1.036\ 610\ 830$	$\lambda_2=0.240\ 138\ 611$
	$x_3=1.756\ 683\ 649$	$\lambda_3=0.033\ 874\ 394\ 5$
	$x_4=2.532\ 731\ 674$	$\lambda_4=0.001\ 343\ 645\ 77$
	$x_5=3.436\ 159\ 119$	$\lambda_5=0.000\ 007\ 640\ 432\ 86$

As an example, consider the evaluation of

$$I=\int_{-\infty}^{\infty}e^{-x^2}\cos x\,dx.$$

The known value of this integral is

$$I = \pi^{1/2}e^{-1/4} \simeq 1.380\ 388\ 447.$$

Using formula (7) to evaluate I approximately, and taking $n=8$ and making use of symmetry, there results

$$\begin{aligned} I &\simeq \sum \lambda_{i,8} \cos (x_{i,8}) \\ &= 2[(0.661\ 147\ 013)(0.928\ 223\ 702) \\ &\quad + (0.207\ 802\ 326)(0.401\ 910\ 767) \\ &\quad + (0.017\ 077\ 983\ 0)(-0.399\ 398\ 300) \\ &\quad + (0.000\ 199\ 604\ 071)(-0.977\ 831\ 340)] \\ &= 1.380\ 388\ 447 \end{aligned}$$

which agrees to nine decimal places with the actual value.

If the function $f(x)$ is expansible in a Taylor's series about the origin, estimates of the error in using (7) may be made as follows: The error (not counting rounding off error) arises from terms in the Taylor's series with powers of x greater than $(2n-1)$ when n ordinates are used. In the example above, the largest error term is that due to the x^{16} term in the expansion of $\cos x$. Since

$$\int_{-\infty}^{\infty} \frac{e^{-x^2}x^{16}}{16!}\, dx = \frac{\pi^{1/2}}{2^{16}\cdot 8!} < 2 \times 10^{-10}$$

and since the neglected terms in the cosine expansion when integrated out as above form an alternating series of decreasing absolute values, the error can be estimated as being less than 2×10^{-10}. This heuristic estimate is not the actual error in the use of integration correspondence (7), but is a sort of an order of magnitude error.

THE UNIVERSITY OF TEXAS

ON THE CONVEXITY OF MEAN VALUE FUNCTIONS

HAROLD SHNIAD

1. **Introduction.** Let (a) denote a set $a_1, a_2, \cdots, a_n$ of n distinct positive numbers, $n \geqq 2$, with the subscripts ν labeled so that $a_\nu < a_{\nu+1}$ for $\nu = 1, \cdots, n-1$. Let (ξ) denote a set of positive numbers $\xi_1, \xi_2, \cdots, \xi_n$ with $\sum_{\nu=1}^{n} \xi_\nu = 1$. The mean value function $M_t(a, \xi) = (\sum_{\nu=1}^{n} \xi_\nu a_\nu^t)^{1/t}$, $t \neq 0$, $\pm\infty$; $M_0(a, \xi) = \prod_{\nu=1}^{n} a_\nu^{\xi_\nu}$; $M_{-\infty}(a, \xi) = \min_{\nu=1,2,\cdots,n} a_\nu$ and $M_{+\infty}(a, \xi) = \max_{\nu=1,2,\cdots,n} a_\nu$; is a continuous and strictly increasing function of t for $-\infty \leqq t \leqq +\infty$.[1] For given fixed sets (a) and (ξ), let $M(t)$ denote $M_t(a, \xi)$ and $\Lambda(t)$ denote $\log M_t(a, \xi)$. Each of the functions $M(t)$ and $\Lambda(t)$ has horizontal asymptotes and consequently at least one point of inflection. We shall show that these functions may have more than one inflection point, but shall show that $\Lambda(t)$ is a convex function of t in a neighborhood of $-\infty$, and a concave function of t in a neighborhood of $+\infty$. A sufficient condition will be obtained for $\Lambda(t)$ to be convex for all negative t, and one for $\Lambda(t)$ to be concave for all positive t. Finally, the applicability of the methods used to more general weighted sums will be considered briefly.

2. **Notations and fundamental formulae.** Let

$$f(t) = \log\left(\sum_{\nu=1}^{n} \xi_\nu a_\nu^t \right), \tag{1}$$

$$\eta_\nu(t) = \frac{\xi_\nu a_\nu^t}{\sum_{\nu=1}^{n} \xi_\nu a_\nu^t}, \quad \lambda_\nu = \log a_\nu \qquad (\nu = 1, 2, \cdots, n), \tag{2}$$

$$S_k = \sum_{\nu=1}^{n} \eta_\nu (\lambda_\nu)^k. \tag{3}$$

Then

$$\frac{d\eta_\nu}{dt} = \eta_\nu\lambda_\nu - \eta_\nu S_1, \quad \frac{dS_k}{dt} = S_{k+1} - S_1 S_k,$$

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[1] See Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge University Press, 1934, chap. 2, for the basic properties of the mean value function.

$$
\begin{aligned}
f' &= S_1,\\
f'' &= S_2 - S_1^2,\\
f''' &= S_3 - 3S_1S_2 + 2S_1^3, \qquad (4)\\
f^{\mathrm{IV}} &= S_4 - 4S_1S_3 - 3S_2^2 + 12S_1^2S_2 - 6S_1^4,
\end{aligned}
$$

$$
f^{\mathrm{IV}} = (S_4 - 3S_2^2 + 2S_1^4) - 4S_1f'''. \qquad (5)
$$

Since $\Lambda(t) = \log M(t) = f/t$, we have

$$
\Lambda^{(k)} = (1/t)(f^{(k)} - k\Lambda^{(k-1)}); \qquad (6)
$$

letting $k = 3$ in (6) and integrating, we obtain the result:

$$
(t_2^3)\Lambda''(t_2) = (t_1^3)\Lambda''(t_1) + \int_{t_1}^{t_2} t^2 f''' dt, \qquad (7)
$$

or

$$
t^3\Lambda''(t) = \int_0^t t^2 f''' dt \quad \text{and} \quad \Lambda''(0) = (1/3)f'''(0). \qquad (8)
$$

3. **The function** $f'''(t)$. The idea behind our analysis of the convexity or concavity of $\Lambda(t)$ in certain ranges of t is to deduce results from corresponding properties of the relatively simpler function $f'''(t)$. By comparing formula (4) with the following expression:

$$
\Lambda''(t) = (1/t)(S_2 - S_1^2) - (2/t^2)(S_1) + (2/t^3)f,
$$

it may be seen that a direct study of $\Lambda''(t)$ is considerably more complicated than that of $f'''(t)$. Formulae (7) and (8) will be used to pass from conclusions regarding the sign of $f'''(t)$ in an interval to similar conclusions regarding the sign of $\Lambda''(t)$.

Our analysis of $f'''(t)$ will be that of a function:

$$
F(t) = \sum_{\nu=1}^{n} \eta_\nu(\lambda_\nu)^3 - 3\left(\sum_{\nu=1}^{n} \eta_\nu\lambda_\nu\right)\left[\sum_{\nu=1}^{n} \eta_\nu(\lambda_\nu)^2\right] + 2\left(\sum_{\nu=1}^{n} \eta_\nu\lambda_\nu\right)^3,
$$

where the expressions η_ν, λ_ν ($\nu = 1, 2, \cdots, n$; $n \geqq 2$) are subject to the conditions:

(a) λ_ν is any fixed finite valued real number with $\lambda_\nu < \lambda_{\nu+1}$;

(b) $\eta_\nu = \eta_\nu(t)$ with $\eta_\nu > 0$, $\sum_{\nu=1}^{n} \eta_\nu(t) = 1$, and $\eta_\nu(t)$ is a continuous function of t defined for all real t;

(c) η_1 is a strictly decreasing function of t and $\lim_{t\to-\infty} \eta_1 = +1$, $\lim_{t\to+\infty} \eta_1 = 0$;

(d) η_n is a strictly increasing function of t and $\lim_{t\to-\infty} \eta_n = 0$, $\lim_{t\to+\infty} \eta_n = +1$.

That $f'''(t)$ is a function $F(t)$ follows easily from formula (4) and the definitions (2) and (3).

LEMMA. *The function $F(t)$ has the property that if each λ_ν is replaced by $\lambda_\nu+c$ (c independent of ν), then $F(t)$ remains invariant.*

PROOF. Let $S_k(t, c) = \sum_{\nu=1}^{n} \eta_\nu(\lambda_\nu+c)^k$. Then

$$\frac{\partial S_k(t, c)}{\partial c} = kS_{k-1}(t, c), \qquad \frac{\partial S_1(t, c)}{\partial c} = 1,$$

and

$$\frac{\partial}{\partial c}\left\{S_3(t, c) - 3S_1(t, c)S_2(t, c) + 2[S_1(t, c)]^3\right\} = 0.$$

THEOREM 1. (a) *If $\eta_1(t_1)=1/2$, then $t<t_1$ implies that $F(t)>0$.*

(b) *If $\eta_n(t_2)=1/2$, then $t>t_2$ implies that $F(t)<0$.*

(c) *We have $F(t_1)=0$ if and only if $n=2$; that is, if and only if $t_1=t_2$.*

PROOF. *Case* 1, $n=2$. We replace λ_ν by $\lambda_\nu-\lambda_2$ and obtain $F(t) = (\eta_1-3\eta_1^2+2\eta_1^3)(\lambda_1-\lambda_2)^3 = (1-2\eta_1)N(t)$, where $N(t)<0$. Hence $F(t)>0$ for $t<t_1$, since $\eta_1(t)$ is a strictly decreasing function of t.

Case 2, $n>2$. Assume the theorem valid for all n satisfying $2\leqq n\leqq k$ and consider $n=k+1$. Suppose that $\eta_1(t)>1/2$, and let $d_\nu=\lambda_\nu-\lambda_2$. Then

$$\begin{aligned} F(t) &= \left(\eta_1 d_1^3 + \sum_{\nu=3}^{k+1} \eta_\nu d_\nu^3\right) - 3\left(\eta_1 d_1 + \sum_{\nu=3}^{k+1} \eta_\nu d_\nu\right)\left(\eta_1 d_1^2 + \sum_{\nu=3}^{k+1} \eta_\nu d_\nu^2\right) \\ &\quad + 2\left(\eta_1 d_1 + \sum_{\nu=3}^{k+1} \eta_\nu d_\nu\right)^3 \\ &= P_1 + P_2 + P_3 + P_4, \end{aligned}$$

where

$$P_1 = (\eta_1 - 3\eta_1^2 + 2\eta_1^3)d_1^3,$$

$$P_2 = \sum_{\nu=3}^{k+1} \eta_\nu d_\nu^3 - 3\left(\sum_{\nu=3}^{k+1} \eta_\nu d_\nu\right)\left(\sum_{\nu=3}^{k+1} \eta_\nu d_\nu^2\right) + 2\left(\sum_{\nu=3}^{k+1} \eta_\nu d_\nu\right)^3,$$

$$P_3 = 3\eta_1 d_1^2(2\eta_1 - 1)\sum_{\nu=3}^{k+1} \eta_\nu d_\nu,$$

$$P_4 = 3\eta_1 d_1\left[2\left(\sum_{\nu=3}^{k-1} \eta_\nu d_\nu\right)^2 - \sum_{\nu=3}^{k+1} \eta_\nu d_\nu^2\right].$$

We have $P_1>0$ for $\eta_1>1/2$, since $d_1<0$.

If we regard the numbers $0, d_3, \cdots, d_{k+1}$ as λ's and the numbers $(\eta_1+\eta_2), \eta_3, \cdots, \eta_{k+1}$ as weights, then the expression P_2 is a function $F(t)$ with $n=k$. Since $(\eta_1+\eta_2)>1/2$ whenever $\eta_1>1/2$, the induction hypothesis implies that $P_2>0$.

We have $P_3>0$ since $d_\nu>0$ for $\nu=3, \cdots, k+1$.

Cauchy's inequality gives:

$$\left(\sum_{\nu=3}^{k+1}\eta_\nu d_\nu\right)^2 \leqq \left(\sum_{\nu=3}^{k+1}\eta_\nu d_\nu^2\right)\left(\sum_{\nu=3}^{k+1}\eta_\nu\right) < \frac{1}{2}\left(\sum_{\nu=3}^{k+1}\eta_\nu d_\nu^2\right).$$

This inequality together with $d_1<0$ gives $P_4>0$. This concludes the proof of part (a).

If $\eta_1=1/2$ and $n>2$, then $P_1=P_3=0$, $P_2>0$ and $P_4>0$. Hence for $n>2$, we have $F(t_1)>0$. The rest of the theorem may be verified by using a similar procedure (for example, replace λ_ν by $\lambda_\nu-\lambda_{k-1}$, and so on).

4. **The behavior of $\Lambda''(t)$.** We shall establish the following result:

THEOREM 2. *There exists a t_1 such that $t<t_1$ implies that $\Lambda''(t)>0$.*

PROOF. By Theorem 1 and the fact that η_1 decreases continuously from plus 1 to 0, there is a value p such that $f'''(p)=0$ and such that $t<p$ implies $f'''(t)>0$.

(a) If $p>0$, then $\Lambda''(t)>0$ for all t on the range $-\infty<t\leqq p$. Since $\Lambda''(0)=(1/3)f'''(0)$, we have $\Lambda''(0)>0$. That $\Lambda''(t)>0$ for all nonzero t on the given range follows from formula (8).

(b) If $p=0$, then $\Lambda''(0)>0$ for all negative t, and $\Lambda''(0)=0$.

(c) If $p<0$ and $\Lambda''(\bar{t})\leqq 0$ for some $\bar{t}<p$, then from formula (7) we have $\Lambda''(t)<0$ for all t satisfying $\bar{t}<t\leqq p$. But $\Lambda(t)$ is an increasing and bounded function of t. Hence for some $t_1\leqq p$, $\Lambda''(t_1)>0$ and for such a t_1, $t<t_1$ will imply that $\Lambda''(t)>0$.

COROLLARY 1. *There exists a t_1 such that $M(t)$ is convex for $t\leqq t_1$.*

COROLLARY 2. *If $\xi_1\geqq 1/2$, then both $\Lambda(t)$ and $M(t)$ are convex for all negative t.*

PROOF. Since $\eta_1(0)=\xi_1$, the inequality $\xi_1\geqq 1/2$ implies that $f'''(t)>0$ for all negative t. The conclusion for $\Lambda(t)$ follows from parts (a) and (b) of the proof of Theorem 2. Since the convexity of the logarithm of a function always implies the convexity of the function, the result for $M(t)$ follows.

The same methods may be used to prove:

THEOREM 2'. *There exists a t_2 such that $t>t_2$ implies that $\Lambda''(t)<0$.*

COROLLARY 2'. *If $\xi_n \geqq 1/2$, then $\Lambda(t)$ is concave for all positive t.*

THEOREM 3. *If $f'''(t)$ has exactly one zero, then so does $\Lambda''(t)$.*

PROOF. Let $f'''(\bar{t})=0$. Then we have $f'''(t)>0$ for $t<\bar{t}$ and $f'''(t)<0$ for $t>\bar{t}$. If $\bar{t}>0$, the proofs of Theorems 2 and 2' show that $\Lambda''(t)>0$ on the range $-\infty<t\leqq\bar{t}$ and that $\Lambda''(t)$ has one zero on the range $\bar{t}<t<+\infty$. The other cases $\bar{t}=0$ and $\bar{t}<0$ are similar.

5. **Convex-concave functions.** A function $g(t)$ defined on the range $-\infty<t<+\infty$ is convex-concave providing there exists a point $\bar{t}$ such that $g(t)$ is convex for $t\leqq\bar{t}$ and concave for $\bar{t}\leqq t$.

The question may be raised whether either of the functions $\Lambda(t)$ or $M(t)$ is necessarily convex-concave. By Theorems 1 and 3 the answer is affirmative for $\Lambda(t)$ in the case $n=2$. We now develop an example to show that this conclusion regarding either $\Lambda(t)$ or $M(t)$ is not true for $n>2$.

Consider the case $a_\nu = e^\nu$ $(\nu=1, 2, 3)$, $\xi_1=\xi_3<1/6$, $\lambda_\nu=\nu$. If λ_ν is replaced by $\lambda_\nu+c$, $f'(t)$ is increased by c and every higher order derivative of $f(t)$ remains invariant. Therefore, to calculate both $f'''(0)$ and $f^{\mathrm{IV}}(0)$, the set $(1, 2, 3)$ may be replaced by the set $(-1, 0, +1)$. We have that $f'''(0)$ is equal to $-\xi_1+\xi_3=0$ and, by formula (5), that $f^{\mathrm{IV}}(0)$ is equal to $(\xi_1+\xi_3)-3(\xi_1+\xi_3)^2>0$. Therefore in a neighborhood of zero, $f'''(t)$ has the same sign as t. From formula (8), it follows that $\Lambda''(t)$ agrees with $f'''(t)$ in sign throughout this neighborhood and consequently that $\Lambda''(t)$ has at least three zeros. In particular for the mean value function $M(t)=(.1e^t+.8e^{2t}+.1e^{3t})^{1/t}$, the following table shows that there are values t_1 and t_2 such that $t_1<t_2$ and $M''(t_1)<0<M''(t_2)$ or that $M(t)$ is not convex-concave.

Table of M and its derivatives for $M(t)=(.1e^t+.8e^{2t}+.1e^{3t})^{1/t}$

t	-1.2	-0.8	-0.4	0.0
$M(t)$	6.519	6.810	7.098	7.389
$M'(t)$	0.7311	0.7215	0.7209	0.7389
$M''(t)$	-0.026	-0.017	$+0.018$	$+0.074$

6. **Generalized weighted sums.** Let (a) again denote a set of distinct positive numbers $a_1, a_2, \cdots, a_n$ and (ξ) a set of arbitrary positive numbers $\xi_1, \xi_2, \cdots, \xi_n$, $n\geqq 2$. Consider the function $S_t(a, \xi)$

$= [\sum_{\nu=1}^{n} \xi_\nu a_\nu^t]^{1/t}$ for $t \neq 0, \pm\infty$;

$$S_0(a, \xi) = \prod_{\nu=1}^{n} a_\nu^{\xi_\nu} \quad \text{if} \quad \sum_{\nu=1}^{n} \xi_\nu = 1;$$

$$S_0(a, \xi) = 0 \quad \text{if} \sum_{\nu=1}^{n} \xi_\nu \neq 1;$$

$$S_{-\infty}(a, \xi) = \min_{\nu=1,2,\ldots,n} a_\nu;$$

and

$$S_{+\infty}(a, \xi) = \max_{\nu=1,2,\ldots,n} a_\nu.$$

For fixed sets (a) and (ξ), the function $S_t(a, \xi)$ is continuous for $-\infty \leqq t < 0$ and for $0 < t \leqq +\infty$. If $\sum_{\nu=1}^{n} \xi_\nu \leqq 1$, then $S_t(a, \xi)$ is continuous from the right at $t=0$; and if $\sum_{\nu=1}^{n} \xi_\nu \geqq 1$, then $S_t(a, \xi)$ is continuous from the left at $t=0$. However, if we define f, η_ν, and S_k as in formulae (1), (2), and (3), then all of the results of §3 regarding $f'''(t)$ are valid without modification. Formula (7) will hold as long as t_1 and t_2 are both positive or both negative.

THEOREM 4. (a) *There is a point* p_1 *such that* log $S_t(a, \xi)$ *is either a concave function of* t *for all* $t < p_1$ *or a convex function for all* $t < p_1$.

(b) *There is a point* p_2 *such that* log $S_t(a, \xi)$ *is convex for all* $t > p_2$, *or concave for all* $t > p_2$.

PROOF OF THEOREM 4a. Assume that for all p, log $S_t(a, \xi)$ is not convex in the range $-\infty < t < p$. Let $p_1 < 0$ be such that $t < p_1$ implies that $f'''(t) > 0$. Consider any number $t_0 < p_1$. Then, by assumption, $d^2 \log S_t(a, \xi)/dt^2 < 0$ has a solution for some $t' < t_0$. Formula (7) implies that this second derivative is negative for all t satisfying $t' < t \leqq p_1$, or that log $S_t(a, \xi)$ is concave for all $t \leqq p_1$.

The proof of Theorem 4b is similar.

To show the applicability of Theorem 4 consider a function $S_t(a, \xi)$ where $\sum_{\nu=1}^{n} \xi_\nu < 1$. For all such sets (ξ) and for every set (a), the function $S_t(a, \xi)$ has the following properties:

(a) $t_1 < t_2$ and $t_1 t_2 > 0$ imply that $S_{t_1}(a, \xi) < S_{t_2}(a, \xi)$;

(b) $\min_{\nu=1,2,\ldots,n} a_\nu < S_t(a, \xi)$ for $t < 0$;

(c) $S_t(a, \xi) < \max_{\nu=1,2,\ldots,n} a_\nu$ for $t > 0$.

Hence there is no point p such that $t < p$ implies that log $S_t(a, \xi)$ is concave nor a point p such that $t > p$ implies that log $S_t(a, \xi)$ is convex. We conclude from Theorem 4 that if $\sum_{\nu=1}^{n} \xi_\nu < 1$, log $S_t(a, \xi)$ is convex in a neighborhood of $-\infty$ and concave in a neighborhood of $+\infty$.

If $\xi_\nu \geqq 1$ for each ν, then by reasoning analogous to that of the preceding example, it may be shown, for any set (a), that there is no point p such that $t<p$ implies that $\log S_t(a, \xi)$ is convex nor a point p such that $t>p$ implies that $\log S_t(a, \xi)$ is concave. Hence Theorem 4 applies to all such functions $\log S_t(a, \xi)$. However, for this case the conclusion of the general theorem is weaker than the known result that $\log S_t(a, \xi)$ is convex for all positive t and concave for all negative t.[3]

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[3] See Beckenbach, *An inequality of Jensen*, Amer. Math. Monthly vol. 53 (1946) pp. 501–505.

HOMOMORPHISMS ON BANACH SPACES

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1. **Introduction.** Let E be a Banach space and E^* its conjugate space. Let G be a closed linear subspace of E, and let $\Gamma = \{f \mid f \in E^*, f(x) = 0 \text{ for } x \in G\}$. Krein and Šmulian have shown [4, Theorem 12′][1] that $G^* = E^*/\Gamma$ in the sense that the two spaces are algebraically isomorphic and that the usual definitions of norm in the two are equivalent. Noting the algebraic isomorphism, let us look at the topological aspects of this equivalence in a slightly different light. G^* being a factor space of E^*, there is defined a natural homomorphism [5, p. 64] $T(E^*) = G^*$. Since they are using the induced topology [5, p. 58] in E^*/Γ, Krein and Šmulian prove what is equivalent to the theorem that the transformation T is continuous and open (see [5, Theorem 12]). Stated in this way, incidentally, the result follows immediately from the Hahn-Banach theorem by means of the usual neighborhood argument for continuity and openness.

However, the homomorphism $T(E^*) = G^*$ suggests other topological questions the answers to which are not quite so obvious. Specifically, what are the topological properties of T when E^* and G^* are given topologies other than their norm topologies?

The conjugate to a Banach space may be topologized in any one of several well known ways. The most common such topologies are the norm, weak, weak*, bounded weak and bounded weak*. We shall

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

use these terms in connection with properties of T to designate the property mentioned when both E^* and G^* have the topology mentioned. For example, the statement, "T is weak* continuous," means that T is continuous when E^* is given its weak* topology and G^* is given its weak* topology.

2. **Notes on weak topologies.** Before proceeding, let us review the definitions and some of the properties of the various weak topologies of a conjugate space E^*.

DEFINITION 2.1. A directed set $\{f_\alpha\}\subset E^*$ has $f\in E^*$ as its weak* limit provided $\lim_\alpha f_\alpha(x)=f(x)$ for every $x\in E$. This topology is described by the neighborhoods of the origin $U(\sigma, \epsilon)=\{f\mid |f(x)|<\epsilon \text{ for } x\in\sigma\}$ where σ is a finite set in E and $\epsilon>0$.

DEFINITION 2.2. A directed set $\{f_\alpha\}\subset E^*$ is weakly convergent to $f\in E^*$ provided $\lim_\alpha X(f_\alpha)=X(f)$ for every $X\in E^{**}$. Neighborhoods of the origin for this topology are $U(\Sigma, \epsilon)=\{f\mid |X(f)|<\epsilon \text{ for } X\in\Sigma\}$ where Σ is a finite set in E^{**} and $\epsilon>0$.

Bounded weak* and bounded weak topologies are obtained from definitions 2.1 and 2.2 respectively, if the provision is added that $\{f_\alpha\}$ converges to f only if there is a number K such that $\|f_\alpha\|\leqq K$ for every α.

For sequences, the notions of weak* and bounded weak* convergence are equivalent, as are those of weak and bounded weak convergence, because convergent sequences are necessarily bounded [2, pp. 80, 123]. However, this is not true for directed sets in general. Every conjugate space is boundedly weak* complete; but Alaoglu has shown [1, p. 254] that if E^* is infinite-dimensional, it is not weak* complete. An example similar to Alaoglu's shows that if E^* is infinite-dimensional, it contains a directed set weak* convergent to the null functional for which no bounded subset has the null functional as a limit point.

Let $\{x_1, x_2, x_3, \cdots\}$ be a sequence of linearly independent elements of E with $\|x_i\|=1$ for each i. Let the finite-dimensional subspaces of E be partially ordered by means of the inclusion relationship. These subspaces $\{G_\alpha\}$ then form a directed set. We form a corresponding directed set $\{f_\alpha\}\subset E^*$ as follows: if G_α is n-dimensional, at least one of the points $x_1, x_2, \cdots, x_{n+1}$ lies outside it; hence [2, p. 57] there is a functional f_α which is zero on G_α and equal to n at this point. Thus $\|f_\alpha\|\geqq n$. Clearly $\lim_\alpha f_\alpha(x)=0$ for every $x\in E$, but the neighborhood $U(\sigma_{n+1}, 1)$—$\sigma_{n+1}=\{x_1, x_2, \cdots, x_{n+1}\}$—contains no member of $\{f_\alpha\}$ with norm less than n.

An analogous construction will demonstrate the non-equivalence of weak and bounded weak convergence in infinite-dimensional spaces.

With regard to closure of linear subspaces, however, the picture is slightly different: and it is here that the weak* topology comes into its own. For linear subspaces of E^* the notions of weak* and bounded weak* closure are equivalent. Furthermore, they are equivalent [1, Theorem 1.4] to regular closure, defined by Banach as follows:

DEFINITION 2.3. A linear subspace Γ of E^* is regularly closed provided to every $f \in E^* - \Gamma$ there corresponds an $x \in E$ such that $f(x) = 1$ and $g(x) = 0$ for every $g \in \Gamma$.

Equivalence of weak* and bounded weak* closure for linear subspaces seems to be known [3, p. 52]. The proof is contained in the work of Alaoglu [1, p. 256], but the result is not noted there. Alaoglu's proof that regular closure implies bounded weak* closure makes no use of boundedness. Thus weak* closure may be inserted between regular and bounded weak* closure in his chain of implications. It might be remarked in passing that in his discussion of these relationships for the case of separable E, Banach occasionally speaks of sets with no qualification when he means linear sets [2, Theorem 5, p. 121 and summary of results, p. 126].

Finally, we recall one of the most important properties of the weak* topology:

THEOREM 2.4 (*Alaoglu* [1, *Theorem* 1.3]). *The unit sphere in* E^* *is bicompact in the weak* topology.*

3. **Topological properties of the homomorphism** T. Returning now to the natural homomorphism of E^* onto G^*, we first investigate the questions of continuity and openness. Continuity is obvious in each case: hence we state without proof:

THEOREM 3.1. *T is norm continuous, weakly continuous, boundedly weakly continuous, weak* continuous and boundedly weak* continuous.*

As noted in the introduction, due to the work of Krein and Šmulian, we may add:

THEOREM 3.2. *T is norm open.*

For use in subsequent proofs, we introduce the operation $\oplus$ to denote the direct sum of sets in E (or E^*, as the case may be). $A \oplus B = \{x \mid x = a + b,\ a \in A,\ b \in B\}$.

THEOREM 3.3. *T is weak* open.*

Let $U(\sigma, \epsilon)$ be any weak* neighborhood in E^*. To prove that T is weak* open, we must show [5, p. 59] that $T[U(\sigma, \epsilon)]$ contains a weak* neighborhood in G^*. Specifically, we shall construct a set $\tau \subset G$ such that $T[U(\sigma, \epsilon)] \supset U(\tau, \epsilon)$. If, as in the introduction, we denote the denominator subspace by Γ, this means that having constructed τ, we must show that to every $f \in U(\tau, \epsilon)$ there corresponds a $g \in \Gamma$ such that $f+g \in U(\sigma, \epsilon)$.

The set τ is constructed as follows: σ being a finite set, it is contained in a finite-dimensional subspace S of E. S being finite-dimensional, the factor space $(S \oplus G)/G$ is finite-dimensional and therefore has a finite basis. Let $\{x_1, x_2, \cdots, x_k\}$ be a set of unit elements of S, each chosen from a different member of the basis for $(S \oplus G)/G$. The following are then easily verified:

(1) If $x \in \sigma$, $x = y_x + \sum_{j=1}^{k} a_j x_j$, where $y_x \in G$ and the a_j are real coefficients (depending, of course, on x).

(2) For each i ($i = 1, 2, \cdots, k$), x_i lies outside the closed linear subspace $G \oplus S_i$, where S_i is the subspace generated by $x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k$.

We now define τ to be the set of all y_x (see (1)) for $x \in \sigma$. Now given any $f \in E^*$, it follows from (2) and a familiar property of closed linear subspaces [2, p. 57] that there exists $g_i \in E^*$ such that $g_i(y) = 0$ for $y \in G \oplus S_i$ and $g_i(x_i) = -f(x_i)$. We let $g = \sum_{i=1}^{k} g_i$; then $g \in \Gamma$, and the expansion in (1) immediately yields the result that if $x \in \sigma$, $f(x) + g(x) = f(y_x)$. Hence $f \in U(\tau, \epsilon)$ implies $f+g \in U(\sigma, \epsilon)$. Q.E.D.

THEOREM 3.4. *T is weakly open.*

Krein and Šmulian have shown [4, Theorem 13] that G^{**} is a subspace of E^{**}. As a matter of fact, it is a regularly closed subspace, as is seen immediately from the construction they give for it and a remark of Banach's [2, p. 117]. Armed with this information, we can reverse the roles of elements and functionals and proceed as in the proof of Theorem 3.3. The only difference is that we use regular closure instead of Banach's lemma to guarantee the existence of the g_i. This calls for regular closure of subspaces of the form $G^{**} \oplus S_i$ where S_i is finite-dimensional. This follows immediately from regular closure of G^{**} if we replace regular closure by weak* closure and consider limits of directed sets.

There remain the questions of bounded weak and bounded weak* openness for T. The construction given in the proof of Theorem 3.3 does not answer these. The norm of each g_i may be as large as the norm of f; hence the bound on norms might conceivably have to be multiplied by the number of elements in σ. We do not have an example handy to show that this unfavorable situation can actually occur; but because of the equivalence of the bounded and unbounded closures for linear subspaces, this point can frequently be by-passed (see Theorem 3.5 below).

The transformation T is not closed with respect to any of the topologies considered. This is shown by a very simple example. Let E^* be Euclidian 2-space, and let Γ be the y-axis. Then G^* is the x-axis and T is the orthogonal projection transformation. Consider the closed set $A=\{(x, y)|x>0, y>0, xy\geqq 1\}$. Then $T(A)=\{x|x>0\}$ which is not closed. The five topologies are equivalent here; so this example answers all five questions. Note that the set A above is convex; so we cannot say even that T is closed for convex sets.

The theorems on openness of T may be used to prove closure by taking complements, provided we first take complete inverse images. Now $T^{-1}[T(\Delta)]=\Delta\oplus\Gamma$; thus we can show that $T(\Delta)$ is closed by showing that

(A) $\Delta\oplus\Gamma$ is closed, and

(B) the topology used is one of those discussed in Theorems 3.2, 3.3 and 3.4 or is equivalent to one of these for sets of the type $\Delta\oplus\Gamma$.

Using this technique, we arrive very easily at the following results:

THEOREM 3.5. 1. *For linear subspaces, T is* (a) *weak* closed,* (b) *boundedly weak* closed.*

2. *For compact sets, T is closed with respect to each of the five topologies.*

3. *For bounded sets, T is* (a) *weak* closed,* (b) *boundedly weak* closed.*

Statements 1(a) and 1(b) are equivalent. For 1(b), the condition (A) is satisfied because unit spheres are bicompact (Theorem 2.4), and the direct sum of two bicompact sets is bicompact, hence closed. Turning now to 1(a), we see that Theorem 3.3 applies to this topology. Thus condition (B) is also satisfied.

To prove 2, we note that the direct sum of a compact set and a closed set is closed. The openness theorems then dispose of three cases immediately. To cover the cases of the bounded topologies we note that for sets $\Delta\oplus\Gamma$ where Δ is compact, every limit point is the limit of a bounded subset. Every point is of the form $f+g$ where $f\in\Delta$ and $g\in\Gamma$. Since Δ is compact (henced bounded), the f's are automatically bounded; and since Γ is linear, the g's may be replaced by a bounded subset.

Finally, we note that 3 reduces to 2 by means of Theorem 2.4. In connection with 3, we might point out that weak* closure of T for bounded sets is quite a different thing from bounded weak* closure of T. The natural homomorphism has the first of these properties, but not the second.

The most obvious unanswered questions are those of norm and weak closure of T for linear subspaces. By means of Theorems 3.2 and 3.4, these are reduced to questions of closure of direct sums; but the compactness arguments of Theorem 3.5 must be replaced by others.

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A NOTE ON THE OPERATORS OF BLASCHKE AND PRIVALOFF

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Let $f(P)$ be a function of a point $P \equiv P(x, y)$ in Euclidean 2-space. Let $L(f; P; r)$, $A(f; P; r)$ be the mean values of $f(P)$ on the perimeter and on the interior, respectively, of a circle of center P and radius r, that is,

$$L(f; P; r) = \frac{1}{2\pi r}\int_{C(P;r)} f(Q)ds_Q,$$

$$A(f; P; r) = \frac{1}{\pi r^2}\int\int_{D(P;r)} f(Q)dQ$$

where $C(P; r)$, $D(P; r)$ are the perimeter and interior, respectively, of the circle with center P and radius r. The operators

$$\nabla_B f(P) = \lim_{r\to 0}\frac{4}{r^2}\left[L(f; P; r) - f(P)\right],$$

$$\nabla_a f(P) = \lim_{r\to 0}\frac{8}{r^2}\left[A(f; P; r) - f(P)\right]$$

have been defined by Blaschke and Privaloff, respectively. The following are a few of the results which have been obtained by these and other investigators.

THEOREM A [1, 2].[1] *If $f(P)$ has continuous second partial derivatives, then $\nabla_B f(P)$, $\nabla_a f(P)$ exist, and*

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)_P \equiv \nabla^2 f(P) = \nabla_B f(P) = \nabla_a f(P).$$

THEOREM B [1]. *If* (i) *$f(P)$ is continuous on a circle $\bar{D}(Q; r)$,* (ii) *$\nabla_B f(P)$ exists on the interior, $D(Q; r)$, then*

$$\frac{4}{r^2}\left[L(f; Q; r) - f(Q)\right]$$

lies between the upper and lower bounds of $\nabla_B f(P)$ on $D(Q; r)$.

THEOREM C [3, 4]. *If $u(P)$ is a logarithmic potential function*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$u(P) = \int_W \log \frac{1}{PQ}\, d\mu(Q)$$

where μ *is a mass distribution, and if the density exists at* R, *that is,*

$$\lim_{r\to 0} \frac{1}{\pi r^2} \int_{D(R;r)} d\mu(Q) = D_s\mu(R)$$

exists, then $\nabla_p u(R)$, $\nabla_s u(R)$ *exist and* $\nabla_p u(R) = \nabla_s u(R) = -2\pi D_s\mu(R)$. ($W$ *indicates integration over the whole space.*)

The purpose of this note is to give extensions of Theorems B and C. Theorem B is readily extended to the operator ∇_s by the following:

THEOREM 1. *If* (i) $f(P)$ *is continuous on a circle* $\bar{D}(Q; r)$, (ii) $\nabla_s f(P)$ *exists on the interior*, $D(Q; r)$, *then*

$$\frac{8}{r^2}\,[A(f; Q; r) - f(Q)]$$

lies between the upper and lower bounds of $\nabla_s f(P)$ *on* $D(Q; r)$.

PROOF. Consider the function

$$\lambda(P) = f(P) - h(P) + L(f; Q; \rho) - f(Q) - \frac{1}{\rho^2}\,[L(f; Q; \rho) - f(Q)]\overline{PQ}^2$$

where $\rho \leqq r$, and $h(P)$ is the function harmonic on $D(Q; \rho)$ and such that $h(P) = f(P)$ on $C(Q; \rho)$. Clearly $\lambda(P) = 0$ on $C(Q; \rho)$. Further $\lambda(Q) = L(f; Q; \rho) - h(Q)$. But

$$h(Q) = \frac{1}{2\pi r} \int_{C(Q;\rho)} h(P)ds_P = \frac{1}{2\pi r} \int_{C(Q;\rho)} f(P)ds_P = L(f; Q; \rho).$$

Therefore $\lambda(Q) = 0$. Thus the continuous function $\lambda(P)$ has both a maximum and a minimum value on $D(Q; \rho)$. Now if R is a maximum point of $\lambda(P)$ then $\nabla_s\lambda(R) \leqq 0$, for

$$\nabla_s\lambda(R) = \lim_{\rho\to 0} \frac{1}{\pi\rho^4} \iint_{D(P;\rho)} [\lambda(P) - \lambda(R)]dP \leqq 0.$$

But

$$\nabla_s\lambda(P) = \nabla_s f(P) - \nabla_s h(P) - \frac{1}{\rho^2}\,[L(f; Q; \rho) - f(Q)]\nabla_s \overline{PQ}^2.$$

By Theorem A, $\nabla_s h(P) = 0$, $\nabla_s \overline{PQ}^2 = 4$. Therefore

$$\nabla_s\lambda(P) = \nabla_s f(P) - \frac{4}{\rho^2}\left[L(f; Q; \rho) - f(Q)\right].$$

But $\nabla_s\lambda(R) \leqq 0$, hence

$$\nabla_s f(R) \leqq \frac{4}{\rho^2}\left[L(f; Q; \rho) - f(Q)\right].$$

Similarly if S is a minimum point of $\lambda(P)$ on $D(Q; \rho)$ we have

$$\nabla_s f(S) \geqq \frac{4}{\rho^2}\left[L(f; Q; \rho) - f(Q)\right].$$

Thus, if M, m are the upper and lower bounds, respectively, of $\nabla_s f(P)$ on $D(Q; r)$, then for all $\rho \leqq r$

$$m \leqq \frac{4}{\rho^2}\left[L(f; Q; \rho) - f(Q)\right] \leqq M,$$

and so

$$\frac{2}{r^2}\int_0^r m\rho^3 d\rho \leqq \frac{8}{r^2}\int_0^r L(f; Q; \rho)\rho d\rho - \frac{8}{r^2}\int_0^r f(Q)\rho d\rho \leqq \frac{2}{r^2}\int_0^r M\rho^3 d\rho.$$

But

$$A(f; Q; r) = \frac{1}{\pi r^2}\iint_{D(Q;r)} f(P)dP = \frac{2}{r^2}\int_0^r \rho d\rho \, \frac{1}{2\pi\rho}\int_{C(Q;\rho)} f(P)ds_P$$
$$= \frac{2}{r^2}\int_0^r L(f; Q; \rho)\rho d\rho.$$

Thus $mr^2/2 \leqq 4[A(f; Q; r) - f(Q)] \leqq Mr^2/2$ and

$$m \leqq \frac{8}{r^2}\left[A(f; Q; r) - f(Q)\right] \leqq M.$$

For the operator ∇_s a somewhat stronger form of Theorem C is obtainable.

THEOREM 2. *If $u(P)$ is a logarithmic potential function*

$$u(P) = \int_W \log \frac{1}{PQ}\, d\mu(Q)$$

where μ is a mass distribution, and if at R

$$\lim_{r\to 0}\operatorname{ap}\frac{1}{\pi r^2}\int_{D(R;r)} d\mu(P) = D_s\mu(R)$$

exists, then $\nabla_s u(R)$ *exists, and* $\nabla_s u(R) = -2\pi D_s\mu(R)$.

Proof. Consider

$$L(u;R;\rho) = \frac{1}{2\pi\rho}\int_{C(R;\rho)} u(P)ds_P = \frac{1}{2\pi\rho}\int_{C(R;\rho)} ds_P\cdot\int_W \log\frac{1}{PQ}\,d\mu(Q)$$
$$= \int_W d\mu(Q)\cdot\frac{1}{2\pi\rho}\int_{C(R;\rho)}\log\frac{1}{PQ}\,ds_P.$$

Now

$$\frac{1}{2\pi\rho}\int_{C(R;\rho)}\log\frac{1}{PQ}\,ds_P = \log\frac{1}{QR} \qquad (QR>\rho)$$
$$= \log\frac{1}{\rho} \qquad (QR\leqq\rho).$$

Hence

$$L(u;R;\rho) = \int_{D(R;\rho)} d\mu(Q)\cdot\log\frac{1}{\rho} + \int_{W-D} d\mu(Q)\cdot\log\frac{1}{QR}$$
$$= \int_W d\mu(Q)\cdot\log\frac{1}{QR} + \int_{D(R;\rho)}\left[\log\frac{1}{\rho}-\log\frac{1}{QR}\right]d\mu(Q)$$
$$= u(R) + \int_{D(R;\rho)}\left[\log\frac{1}{\rho}-\log\frac{1}{QR}\right]d\mu(Q).$$

Thus

$$A(u;R;r) = \frac{2}{r^2}\int_0^r L(u;R;\rho)\rho d\rho$$
$$= u(R) + \frac{2}{r^2}\int_0^r \rho d\rho\cdot\int_{D(R;\rho)}\left[\log\frac{1}{\rho}-\log\frac{1}{QR}\right]d\mu(Q),$$

and so

$$\frac{8}{r^2}\left[A(u;R;r)-u(R)\right]$$
$$= \frac{16}{r^4}\int_0^r \rho d\rho\cdot\int_{D(R;\rho)}\left[\log\frac{1}{\rho}-\log\frac{1}{QR}\right]d\mu(Q).$$

The integrand depends only on $|QR|$, so we can write

$$\frac{8}{r^2}\left[A(u;R;r)-u(R)\right]=\frac{16}{r^4}\int_0^r \rho d\rho\cdot\int_0^\rho\left[\log\frac{1}{\rho}-\log\frac{1}{t}\right]d\bar{\mu}(t)$$

where

$$\bar{\mu}(t)=\int_{D(R;t)}d\mu(Q).$$

Integrating by parts we have

$$\begin{aligned}\frac{8}{r^2}&\left[A(u;R;r)-u(R)\right]\\&=\frac{16}{r^4}\int_0^r\rho d\rho\cdot\left\{\left[\left(\log\frac{1}{\rho}-\log\frac{1}{t}\right)\bar{\mu}(t)\right]_0^\rho-\int_0^\rho\bar{\mu}(t)\frac{dt}{t}\right\}.\end{aligned}$$

But

$$\bar{\mu}(t)=\int_{D(R;t)}d\mu(Q)=\pi t^2D_s\mu(R)+o(t^2)$$

for almost all small t. Hence

$$\begin{aligned}\frac{8}{r^2}&\left[A(u;R;r)-u(R)\right]\\&=\frac{16}{r^4}\int_0^r\rho d\rho\cdot\left\{-\int_0^\rho\pi t^2D_s\mu(R)\frac{dt}{t}-\int_0^\rho o(t^2)\frac{dt}{t}\right\}\\&=\frac{16}{r^4}\int_0^r\rho d\rho\cdot\left[-\frac{\pi}{2}\rho^2D_s\mu(R)+o(\rho^2)\right]\\&=-2\pi D_s\mu(R)+\frac{16}{r^4}\int_0^r o(\rho^3)d\rho.\end{aligned}$$

Thus

$$\lim_{r\to0}\frac{8}{r^2}\left[A(u;R;r)-u(R)\right]=\nabla_s u(R)=-2\pi D_s\mu(R).$$

Many results which have been proven for one operator can be extended to the other operator by use of the following theorem.

THEOREM 3. *If* $\nabla_R f(P)$ *exists, then so does* $\nabla_s f(P)$, *and* $\nabla_s f(P)=\nabla_R f(P)$.

PROOF. $L(f; P; r)$ exists for small r, and further

$$L(f; P; r) = f(P) + \frac{r^2}{4}\nabla_p f(P) + o(r^2).$$

Also

$$A(f; P; r) = \frac{2}{r^2}\int_0^r L(f; P; \rho)\rho d\rho.$$

And hence

$$\begin{aligned}\frac{8}{r^2}&[A(f; P; r) - f(P)]\\ &= \frac{8}{r^4}\left[2\int_0^r L(f; P; \rho)\rho d\rho - r^2 f(P)\right]\\ &= \frac{8}{r^4}\left\{2\int_0^r\left[f(P) + \frac{\rho^2}{4}\nabla_p f(P) + o(\rho^2)\right]\rho d\rho - r^2 f(P)\right\}\\ &= \nabla_p f(P) + \frac{16}{r^4}\int_0^r o(\rho^3)d\rho.\end{aligned}$$

The last term is easily seen to approach zero as $r \to 0$. Thus

$$\lim_{r\to 0}\frac{8}{r^2}[A(f; P; r) - f(P)] = \nabla_a f(P) = \nabla_p f(P).$$

These results hold true for spaces of higher dimensions, and the above proofs follow through with obvious modifications of the coefficients and the form of the potential function.

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THE REMAINDER IN APPROXIMATIONS BY MOVING AVERAGES

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1. Introduction. Many of the processes of interpolation or smoothing are of the following sort. A function $L(s)$, defined for all real s, characterizes the process. Given a function $x(s)$, the function

$$y(t) = \sum_{j=-\infty}^{\infty} x(j)L(t-j) \tag{1}$$

is constructed, when possible; $y(t)$ is thought of as an approximation of $x(t)$. The remainder in the approximation is

$$R[x] = x(t) - y(t). \tag{2}$$

In the conventional processes of smoothing or interpolation, $L(s)$ is a function which vanishes for all $|s|$ sufficiently large. I. J. Schoenberg[2] has recently introduced a class of formulas (1), (2) in which $L(s)$ is an analytic function and the series (1) does not consist of a finite number of terms.

Schoenberg gives an elegant criterion for recognizing cases in which the approximating process is exact for polynomials of degree $n-1$; that is, cases in which $R[x]=0$, for all t, whenever $x(s)$ is a polynomial of degree $n-1$.[3] In the present paper we obtain an integral representation of such operations $R[x]$ in terms of the nth derivative $x^{(n)}(s)$. The representation is precisely of the sort that holds when $R[x]$ is a linear functional on certain spaces of functions $x(s)$ defined on a *finite* s-interval.

2. The integral representation. We shall consider an operation which is more general than (1), (2). *Let $g(s, t)$ be a function which, for each number t in a given set $\mathfrak{T}$, is of bounded variation in s on each finite s-interval.* Given any function $x(s)$, put

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[2] *Contributions to the problem of approximation of equidistant data by analytic functions*, Quarterly of Applied Mathematics vol. 4 (1946) pp. 45–99 and 112–141.

[3] Loc. cit. Theorem 2B, p. 64. Schoenberg's criterion is valid whether $L(s)$ is a symmetric function or not.

Throughout the present paper "polynomial of degree k" is to be understood as "polynomial of proper degree k or less."

$$y(t) = \int_{-\infty}^{\infty} x(s)d_s g(s, t), \tag{3}$$

and

$$R[x] = x(t) - y(t), \qquad t \in \mathfrak{T}.$$

Unless the contrary is stated, *integrals on infinite ranges are to be understood either as Lebesgue-Stieltjes integrals or as improper Lebesgue-Stieltjes integrals*, that is, limits of integrals over finite intervals as the intervals become infinite. Either convention may be adopted, providing that it is consistently held. We shall say that $R[x]$ exists if $y(t)$ and $x(t)$ exist and are finite for each $t \in \mathfrak{T}$.

The integral (3) reduces to the sum (1) if $g(s, t)$ is, for each $t \in \mathfrak{T}$, constant on each interval $j < s < j+1$ and if $g(j+0, t) - g(j-0, t) = L(t-j), j = \cdots, -2, -1, 0, 1, \cdots$. The name "moving average" is most appropriate to (3) when $d_s g(s-m, t) = d_s g(s, t+m)$ for all numbers m or for all integers m; we do not require that g satisfy this condition.

Assume that $R[x]$ *exists and vanishes, for all* $t \in \mathfrak{T}$, *whenever* $x(s)$ *is a polynomial of degree* $n-1$ $(n \geqq 1)$. Put

$$p(s, s') = (s - s')^{n-1}/(n - 1)!;$$

$$\psi_{s'} = \psi_{s'}(s) = \begin{cases} 0 & \text{if } s \leqq s', \\ p(s, s') & \text{if } s > s'. \end{cases} \tag{4}$$

For each fixed s', $R[\psi_{s'}]$ exists, since $\psi_{s'}$ is a truncated polynomial of degree $n-1$. Hence the function $k(s', t) = R[\psi_{s'}]$ is defined for all s' and all $t \in \mathfrak{T}$. An alternative formula for $k(s', t)$ is the following:

$$k(s', t) = \begin{cases} \displaystyle\int_{-\infty}^{s'} p(s, s')dg(s) & \text{if } s' < t, \\ \displaystyle -\int_{s'}^{\infty} p(s, s')dg(s) & \text{if } s' \geqq t, \end{cases} \qquad t \in \mathfrak{T}; \tag{5}$$

here[4] and elsewhere $dg(s)$ *is to be understood as an abbreviation for* $d_s g(s, t)$. To establish (5), observe that, if $s' \geqq t$, $\psi_{s'}(t) = 0$, and

$$R[\psi_{s'}] = \psi_{s'}(t) - \int_{-\infty}^{\infty} \psi_{s'}(s)dg(s) = -\int_{s'}^{\infty} p(s, s')dg(s),$$

by (4). The other part of (5) is derived similarly, with the use of the additional fact that

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[4] Whether the value s' is included or excluded in the range of integration of these integrals is immaterial, since $p(s', s') = 0$.

$$R[p(s, s')] = 0 = p(t, s') - \int_{-\infty}^{\infty} p(s, s')dg(s), \qquad t \in \mathfrak{T}. \tag{6}$$

This relation is true because $p(s, s')$ is a polynomial of degree $n-1$, for each s'.

Suppose that $x(s)$ is a function whose derivative of order $n-1$ exists and is absolutely continuous on every finite s-interval. Put

$$\left.\begin{aligned} I &= \int_{-\infty}^{\infty} dg(s) \int_0^s p(s, s')x^{(n)}(s')ds', \\ R^*[x] &= \int_{-\infty}^{\infty} x^{(n)}(s')k(s', t)ds', \end{aligned}\right\} \qquad t \in \mathfrak{T}.$$

THEOREM. *A necessary and sufficient condition that $R[x]$ and $R^*[x]$ exist and be equal is that I exist and that the order of integration in I be invertible, for all $t \in \mathfrak{T}$.*

PROOF. For brevity put

$$z = p(s, s')x^{(n)}(s').$$

Sufficiency: Since the order of integration in I is invertible.

$$\begin{aligned} I = \int_{-\infty}^{\infty} dg(s) \int_0^s z ds' = -\int_{-\infty}^{0} ds' \int_{-\infty}^{s'} z dg(s) \\ + \int_0^{\infty} ds' \int_{s'}^{\infty} z dg(s). \end{aligned} \tag{7}$$

As $x^{(n-1)}(s)$ is absolutely continuous,

$$x(s) = x(0) + sx'(0) + \cdots + \frac{s^{n-1}x^{(n-1)}(0)}{(n-1)!} + \int_0^s z ds'. \tag{8}$$

Since R vanishes for polynomials of degree $n-1$ and the integral I exists, $R[x]$ exists and

$$\begin{aligned} R[x] &= R\left[\int_0^s z ds'\right] \\ &= \int_0^t p(t, s')x^{(n)}(s')ds' - \int_{-\infty}^{\infty} dg(s) \int_0^s z ds'. \end{aligned} \tag{9}$$

Furthermore,

$$(10)\qquad \int_0^t p(t, s')x^{(n)}(s')ds' = \int_0^t ds' \int_{-\infty}^{s'} s\,dg(s) + \int_0^t ds' \int_{s'}^{\infty} s\,dg(s).$$

This may be proved as follows. By (6),

$$p(t, s') = \int_{-\infty}^{\infty} p(s, s')dg(s) = \int_{-\infty}^{s'} p(s, s')dg(s) + \int_{s'}^{\infty} p(s, s')dg(s).$$

For fixed $t \in \mathfrak{T}$, each of the last two integrals is a measurable, essentially bounded function of s' for s' between 0 and t; hence (10) follows.

By (9), (10) and (7),

$$(11)\qquad R[x] = \int_{-\infty}^{t} ds' \int_{-\infty}^{s'} s\,dg(s) - \int_{t}^{\infty} ds' \int_{s'}^{\infty} s\,dg(s) = R^*[x].$$

The last equality follows from (5). Thus $R^*[x]$ and $R[x]$ exist and are equal.

Necessity: Since $R[x]$ and $R^*[x]$ exist and are equal, (11) and (9) hold, and I exists. Furthermore, (11), (9) and (10) imply (7).

This completes the proof of the theorem.

3. **Sufficient conditions.** Put

$$M(s', t) = \begin{cases} \displaystyle\int_{-\infty}^{s'} |p(s, s')|\,|dg(s)| & \text{if } s' \leqq 0, \\ \displaystyle\int_{s'}^{\infty} |p(s, s')|\,|dg(s)| & \text{if } s' > 0, \end{cases} \qquad t \in \mathfrak{T}.$$

If the integral

$$J = \int_{-\infty}^{\infty} |x^{(n)}(s')|\, M(s', t)ds'$$

is finite for all $t \in \mathfrak{T}$, *then* $R[x]$ *and* $R^*[x]$ *exist and are equal, and* $R^*[x]$ *exists as a Lebesgue-Stieltjes integral.*

Proof. The double integral corresponding to I will exist and (7) will hold, by Fubini's theorem, since the right side of (7) is majorized by J. Hence, by the previous theorem, $R[x]$ and $R^*[x]$ exist and are equal.

That $R^*[x]$ exists as a Lebesgue-Stieltjes integral may be seen as follows. Suppose that $t \geqq 0$. ($t < 0$ is treated similarly.) The integrals

$$(12)\qquad \int_{-\infty}^{0} ds' \int_{-\infty}^{s'} s\,dg(s), \qquad -\int_{t}^{\infty} ds' \int_{s'}^{\infty} s\,dg(s)$$

are majorized by J. Furthermore, by (6),

$$\int_0^t ds' \int_{-\infty}^{s'} z\,dg(s) = \int_0^t x^{(n)}(s')p(t, s')ds' - \int_0^t ds' \int_{s'}^{\infty} z\,dg(s). \tag{13}$$

Now the last integral in (13) is majorized by J, and the middle integral is on a finite interval. Hence the integrals (12), (13) exist as Lebesgue-Stieltjes integrals. The sum of (13) and the two integrals (12) is precisely $R^*[x]$, by (11).

Note that, by (8), the integral (3) will exist as a Lebesgue-Stieltjes integral, in the present case, if it is true that (3) with $x(s)$ a polynomial of degree $n-1$ exists as a Lebesgue-Stieltjes integral.

Anyone of the following conditions is sufficient to imply the finiteness of J.

(i) For each $t\in\mathfrak{T}$, $M(s', t)$ is absolutely integrable and $x^{(n)}(s')$ is essentially bounded, on $-\infty < s' < \infty$.

(ii) For each $t\in\mathfrak{T}$, $M(s', t)$ is essentially bounded and $x^{(n)}(s')$ is absolutely integrable, on $-\infty < s' < \infty$.

(iii) For each $t\in\mathfrak{T}$, $g(s, t)$ is constant for sufficiently large s and constant for sufficiently small s.

In the particular case in which $R[x]$ is of the form (1), (2),

$$M(s', t) = \begin{cases} \displaystyle\sum_{-\infty < j \leqq s'} p(s', j)\,|\,L(t-j)\,| & \text{if} \quad s' \leqq 0, \\ \displaystyle\sum_{s' \leqq j \leqq \infty} p(j, s')\,|\,L(t-j)\,| & \text{if} \quad s' > 0. \end{cases}$$

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ON APPROXIMATE DERIVATIVES[1]

CHOW, SHU-ER

1. Introduction and definitions. In the present paper $f(x)$ will signify a real, one-valued function defined in the interval I: (a, b). All sets we shall have occasion to speak of will be subsets of I.

The theorem of Denjoy on Dini derivatives, as extended by S. Saks [1, (1)],[2] H. E. Hanson [2], and H. Blumberg [3] to arbitrary functions $f(x)$, is briefly as follows: At almost every point x we have either (α) the derivative of $f(x)$ exists and is finite—we shall then say that the *directional angle* of the curve $y=f(x)$ is 0° at x; or (β) two opposite derivatives (the lower Dini derivative on one side and the upper Dini derivative on the other side) are finite and equal, and the other two are $\pm\infty$ respectively—we then say that the directional angle is 180° at x; or (γ) the upper Dini derivatives are $+\infty$ and the lower $-\infty$ —we then say that the directional angle is 360° at x.

Another line of generalization is concerned with the use of *approximate derivatives*, which we proceed to define.

DEFINITION 1. *The upper right approximate limit of* $f(x)$ *at* ξ— which we denote by $u^+(f, \xi)$, or simply by $u^+(\xi)$ when there is no ambiguity—is the g.l.b. of the (real) numbers k for which the set

$$E_x[f(x) > k, x > \xi]$$

has zero exterior (metric) density at ξ. Similarly, *the lower right approximate limit of* $f(x)$ *at* ξ—denoted by $l^+(f, \xi)$, or $l^+(\xi)$—is the l.u.b. of the numbers k for which the set

$$E_x[f(x) < k, x > \xi]$$

has zero exterior density at ξ. The two left-hand limits, $u^-(\xi)$ and $l^-(\xi)$, are defined similarly.

Without reference to the (left or right) direction of approach, we define *the upper* and *lower approximate limits* of $f(x)$ at ξ respectively as

$$u(\xi) = \max[u^+(\xi), u^-(\xi)], \text{ and } l(\xi) = \min[l^+(\xi), l^-(\xi)].^3$$

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[1] The author wishes to express his gratitude to Professor H. Blumberg, who suggested the application of "the measurable boundaries," and assisted in the editing.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

[3] The functions $u(x)$ and $l(x)$ are called the "*upper*" and "*lower metric boundaries*" of $f(x)$ by Prof. Blumberg [3].

DEFINITION 2. *The upper right approximate derivative of $f(x)$ at ξ*—which we shall denote by $A^+(f, \xi)$, or simply by $A^+(\xi)$—is the upper right approximate limit of the quotient function

$$Q(f, \xi, x) = \frac{f(x) - f(\xi)}{x - \xi}.$$

We define similarly the other three approximate derivatives, namely, the lower right $A_+(\xi)$, the upper left $A^-(\xi)$, and the lower left $A_-(\xi)$.

If all these four approximate derivatives are equal, their common value is called the *approximate derivative* of $f(x)$ at ξ, and denoted by $A(f, \xi)$, or $A(\xi)$. If, furthermore, $A(\xi)$ is finite, $f(x)$ is said to possess an approximate derivative, or be approximately derivable at ξ.

The approximate derivative, as defined, has been the basis of generalizations of Denjoy's theorem by J. C. Burkill and U. S. Haslam-Jones (among others). They proved the following result [4, (1)]—in which it is apparent what meaning to attach to the phrase, "the directional angle with respect to approximate derivation":

If $f(x)$ is measurable, the directional angle, in respect to approximate derivation, of the curve $y=f(x)$, is almost everywhere either 0° *or* 360°. *Consequently, if one of the approximate derivatives of $f(x)$ is finite at every point of a set E, then $f(x)$ is approximately derivable at almost every point of E.*

For approximate derivatives of *non*-measurable functions, there is an isolated result by A. J. Ward [5], and another by Burkill and Haslam-Jones [4, (2)]. In his *Theory of the Integral*, Saks [1, (2)] gives a proof of the foregoing theorem of Burkill and Haslam-Jones, and remarks that, by a "slight modification" of the proof, the theorem may be extended "in a certain way" to unconditioned functions. However, he gives no explicit proof.[4] *But we prove in the present paper that the Denjoy analogue for approximate derivatives for the case of non-measurable functions is different from the literature result for measurable functions.*

The present paper communicates *the proper analogue of Denjoy's theorem for approximate derivatives of unconditioned functions*, and the proof is simple and straightforward. It turns out for this extension—*in contradistinction to the corresponding result for measurable functions*—that *each of the cases* (α), (β) and (γ) *can occur* on a set of positive exterior measure. The reasoning also shows just why case (β) drops out when $f(x)$ is measurable.

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[4] Curiously, he makes no reference to Jeffery's paper on the same question [6] which appeared prior to the publication of the book of Saks and was reviewed by him in Zentralblatt für Mathematik [7].

In a paper by R. L. Jeffery [6], on the approximate derived numbers of arbitrary functions, the results summarized in his Theorem VI are similar to our main results. Jeffery uses an unusual definition of approximate derived number, and his work is based on the unusual idea of metric separability. His proofs are difficult to read. He does not develop the proofs of the preceding results, and omits the discussion of the existence, in his sense, of the pertinent approximate derivative numbers in case the hypotheses of his Theorems IV and V are not satisfied.

The principal means employed for obtaining our stated results is the theorem of Professor Blumberg on the "*measurable boundaries of an arbitrary function.*" This theorem [3, p. 272] is as follows:

THEOREM (BLUMBERG). *With every real function $f(x)$, one- or many-valued, defined on an interval I, there are uniquely associated two functions $u(x)$ and $l(x)$, defined in I, having the following properties:*

(i) *$u(x)$ and $l(x)$ are measurable.*

(ii) *The set of points x for which $f(x)>u(x)$ or $f(x)<l(x)$ is of measure zero.*

(iii) *The points $(x, u(x))$ and $(x, l(x))$ are positively approached*[5] *by the curves $y=u(x)$, $y=l(x)$, respectively, for every x; these points are fully approached*[6] *by the curve $y=f(x)$ for almost every x, and positively approached for every x.*

This theorem gives a structural representation of an unconditioned function $f(x)$ which shows that $f(x)$ is necessarily built—in the words of Professor Blumberg—"on the scaffolding of two *measurable* functions," the measurable boundaries of $f(x)$. It is this structure which permits the *transfer* of various theorems on *measurable* functions to *arbitrary* functions. The memoir of Professor Blumberg cites diverse interesting cases where this transfer can be effected. The present paper thus adds another application of this theorem.

2. Lemmas.

In this section, we prove two lemmas which are utilized in the proof of the theorems of §3.

LEMMA 1. *For every real function $f(x)$, the set $E=E_x[u^+(x)\neq u^-(x)]$ is of measure zero. A like result holds for $l^+(x)$ and $l^-(x)$.*

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[5] The point (ξ, η) is *positively approached* by the curve or function $y=f(x)$ if for every pair k, l of real numbers, with $k<\eta<l$, the set $E_{kl}=E_x[k<f(x)<l]$ has positive *upper, exterior* density at ξ.

[6] The point (ξ, η) is *fully approached* by $y=f(x)$, if for every pair k, l, with $k<\eta<l$, E_{kl} is of exterior density 1 at ξ.

PROOF. Let $E_1 = E_x[u^+(x) > u^-(x)]$. For a fixed rational number r, let $E_r = E_x[u^+(x) > r > u^-(x)]$. Then

$$E_1 = \sum_r E_r,$$

where r ranges over the set of rational numbers. On account of the definition of $u^\pm(x)$, and the condition for the points of E_r, the set

$$E_x[f(x) > r,\ x > \xi]$$

has positive upper exterior density at every point ξ of E_r; and the set

$$E_x[f(x) > r,\ x < \xi]$$

has zero exterior density at ξ. Hence the set

$$E_x[f(x) > r]$$

has positive upper exterior density at the right and zero exterior density at the left of every point of E_r. Now an arbitrary set has its exterior density either 0 or 1 almost everywhere. Hence E_r, and therefore E_1, is of measure zero. Similarly, the set of points x for which $u^+(x) < u^-(x)$ is of measure zero. We conclude that E is of measure zero.

It follows similarly that the set $E_x[l^+(x) \neq l^-(x)]$ is of measure zero.

LEMMA 2. *For every real function* $f(x)$, *if* $u(\xi) = f(\xi)$, *then* $A^+(u, \xi) = A^+(f, \xi)$ *and* $A_-(u, \xi) = A_-(f, \xi)$. *Similarly, if* $l(\xi) = f(\xi)$, *then* $A^-(l, \xi) = A^-(f, \xi)$ *and* $A_+(l, \xi) = A_+(f, \xi)$.

PROOF. Suppose $u(\xi) = f(\xi)$. Since $u(x) \geqq f(x)$ for almost every x, we have

$$Q(u, \xi, x) \geqq Q(f, \xi, x) \qquad \text{for a.e. } x > \xi.$$

Hence

$$A^+(u, \xi) \geqq A^+(f, \xi). \tag{2.1}$$

On the other hand, suppose $A^+(f, \xi) = k$. We may assume that $k < +\infty$; for if $k = +\infty$, we have $A^+(u, \xi) \geqq A^+(f, \xi) = +\infty$, whence $A^+(u, \xi) = A^+(f, \xi)$. For a given $\epsilon > 0$, let λ be the straight line through the point $(\xi, f(\xi)) = (\xi, u(\xi))$ with slope $k + \epsilon$. Then, by the definition of $A^+(f, \xi)$, the set E of numbers $x > \xi$ for which the point $(x, f(x))$ lies above λ has exterior density zero at ξ. Hence for every $\eta > 0$, there exists a positive number h such that

$$m_e(IE) < \eta l$$

for every interval I with ξ as left end point and length $l<h$, where $m_e(IE)$ denotes the exterior Lebesgue measure of IE.

If $\overline{E}=I-IE$, we have

$$m_i(\overline{E}) > (1-\eta)l$$

where $m_i(\overline{E})$ denotes the interior Lebesgue measure of $\overline{E}$. Let T be a measurable subset of $\overline{E}$ of measure greater than $(1-\eta)l$ and such that the density of T is 1 at each of its points. For every x of $\overline{E}$, the point $(x, f(x))$ lies on or below the line λ. Since, by the theorem of Blumberg, the point $(x, u(x))$ is positively approached by $y=f(x)$ for every x, it follows that for every x of T, the point $(x, u(x))$ also lies on or below λ; that is,

$$Q(u, \xi, x) \leqq k+\epsilon$$

for every x in T. Hence, if we denote the set

$$E_x[Q(u, \xi, x) > k+\epsilon, x > \xi]$$

by E_1, then $m_e(E_1I)<\eta l$. Since this holds for every $\eta>0$ however small, and for every I of length $l<h$, we conclude that the exterior density of E_1 is zero at the point ξ. This shows that $A^+(u, \xi)\leqq k+\epsilon$, and since ϵ is arbitrary, it follows that

$$A^+(u, \xi) \leqq k = A^+(f, \xi).$$

Therefore, by (2.1),

$$A^+(u, \xi) = A^+(f, \xi).$$

Other parts of the lemma follow similarly.

3. **Theorems.** Theorem I states some properties of approximate derivatives which hold for both measurable and non-measurable functions. Some of these results are in the literature, but our proof is different and simpler. Theorem II gives sufficient conditions for the approximate derivability of arbitrary functions. Theorem III proves the principal result of this paper, namely the extension of Denjoy's theorem to approximate derivatives of arbitrary functions.

THEOREM I. *If $f(x)$ is a finite function, then*:

(i) *The set of points for which the upper approximate derivative on one side is less than the lower approximate derivative on the other side is of measure zero.*

(ii) *If one of the approximate derivatives is finite at every point of a* *set E, it is equal to its opposite derivative at almost every point of E.*

(iii) *The set of points where the upper approximate derivative is $-\infty$, or the lower approximate derivative is $+\infty$, is of measure zero.*

PROOF. (i) It is sufficient to reason about the set

$$E = E_s[A^+(x) < A_-(x)]$$

since the corresponding set with the left and right sides interchanged may be treated similarly. If x is a point of E, $A^+(x)$ cannot be $+\infty$. It follows that the set

$$E_s[u(x) > f(x),\ x \in E]$$

is of measure zero. For by the theorem of Blumberg, the point $(x, u(x))$ is fully approached by the curve $y=f(x)$ at almost every x; hence the inequality $u(x)>f(x)$ would imply, for almost every x of E, that $A^+(x)=+\infty$. Accordingly, $u(x)\leqq f(x)$ for almost every point x of E. But the set $E_s[u(x)<f(x)]$ is of measure zero. Consequently,

$$u(x) = f(x) \qquad \text{a.e. in } E.$$

Now we apply Lemma 2, obtaining the equalities

$$(3.1)\qquad \begin{aligned} A^+(x)[= A^+(f, x)] &= A^+(u, x), \\ A_-(x)[= A_-(f, x)] &= A_-(u, x) \end{aligned}$$

for almost every x in E. Since $u(x)$ is measurable, the set $E_s[A^+(u, x) < A_-(u, x)]$ is of measure zero—according to the results cited in the introduction for the approximate derivatives of *measurable* functions. Hence $E=E_s[A^+(x)<A_-(x)]$ is of measure zero.

(ii) Suppose $A^+(x)$ is finite at every point of a set E. As before, we must have

$$u(x) = f(x) \qquad \text{a.e. in } E$$

and (3.1) applies. By the results for measurable functions just referred to, the measurable function $u(x)$ has a finite approximate derivative $A(u, x)$ a.e. on E. Hence

$$A^+(u, x) = A_-(u, x) \qquad \text{a.e. in } E.$$

Consequently, by (3.1)

$$A^+(x) = A_-(x) \qquad \text{a.e. in } E.$$

The reasoning for the cases where the other approximate derivatives are finite is similar.

(iii) It is sufficient to reason about the set

$$E = E_x[A^+(x) = -\infty],$$

since the other cases are similar. By reasoning like that for (i), we have

$$u(x) \leqq f(x) \qquad \text{a.e. in } E.$$

Since the set of the points for which $u(x) < f(x)$ is of measure zero, it follows that

$$u(x) = f(x) \qquad \text{a.e. in } E.$$

We may consequently apply (3.1) and conclude that

$$A^+(u, x) = A^+(x) = -\infty \qquad \text{a.e. in } E.$$

But for the measurable function $u(x)$, the set of points for which $A^+(u, x) = -\infty$ is of measure zero. Hence $E = E_x[A^+(x) = -\infty]$ is of measure zero.

THEOREM II. *If $f(x)$ is a given finite function, let E be the set of abscissas at which either the two approximate derivatives on the same side or the two upper (or lower) approximate derivatives are finite. Then $f(x)$ is approximately derivable at almost every point of E.*

PROOF. Let E_1 be the subset of E at which $A^+(x)$ and $A_+(x)$ are finite. By Theorem I, case (i), we have

$$A^+(x) = A_-(x), \qquad A_+(x) = A^-(x) \qquad \text{a.e. in } E_1.$$

But since

$$A^+(x) \geqq A_+(x), \qquad A^-(x) \geqq A_-(x)$$

for every x in E, it follows that

$$A^+(x) = A_-(x) \leqq A^-(x) = A_+(x) \qquad \text{a.e. in } E_1,$$

whence $A^+(x) = A_+(x)$. The four approximate derivatives are consequently finite and equal, hence $f(x)$ is approximately derivable at almost every point of E_1. The other three possibilities may be treated similarly.

THEOREM III. *For any finite function $f(x)$, the directional angle, in respect to approximate derivation, of the curve $y = f(x)$ is, at almost every x, either* 0° *or* 180°, or 360°.

PROOF. The set of points x for which any one of the relations

$$f(x) > u(x), \qquad f(x) < l(x),$$

$$u^+(x) \neq u^-(x), \qquad l^+(x) \neq l^-(x)$$

holds is of measure zero (by the theorem of Blumberg and Lemma 1). Therefore we need only consider those points x at which

$$u^+(x) = u^-(x) = u(x), \qquad l^+(x) = l^-(x) = l(x)$$

and

$$u(x) \geqq f(x) \geqq l(x).$$

Case (a). $u(x) > f(x) > l(x)$. From $u(x) > f(x)$, it follows that

$$A^+(x) = +\infty, \qquad A_-(x) = -\infty \qquad \text{a.e.}$$

(cf. proof of Theorem I). Likewise, it follows from $f(x) > l(x)$ that

$$A^-(x) = +\infty, \qquad A_+(x) = -\infty \qquad \text{a.e.}$$

Hence in this case the directional angle is 360° almost everywhere on the set $E_x[u(x) > f(x) > l(x)]$.

Case (b). $u(x) = f(x) > l(x)$. (The relation $u(x) > f(x) = l(x)$ may be treated similarly.) From $f(x) > l(x)$, it follows, as before, that

$$A^-(x) = +\infty, \qquad A_+(x) = -\infty \qquad \text{a.e.}$$

From $u(x) = f(x)$, it follows, by Lemma 2, that

$$A^+(x) = A^+(u, x), \qquad A_-(x) = A_-(u, x).$$

If $A^+(u, x)$ is finite for almost every x in a set E, then, on account of the measurability of $u(x)$, $A^+(u, x) = A_-(u, x)$ almost everywhere in E. Hence $A^+(x)$ and $A_-(x)$ are finite and equal almost everywhere in E. The directional angle is, therefore, 180° almost everywhere in E. If $A^+(u, x) = +\infty$ for almost every x in E, then $A_-(u, x) = -\infty$ at almost every point of E (again on account of the results for measurable functions). Hence $A^+(x) = +\infty$ and $A_-(x) = -\infty$ almost everywhere in E. Therefore the directional angle is 360° almost everywhere in E.

Case (c). $u(x) = f(x) = l(x)$. By Lemma 2, we have at every point of $E = E_x[u(x) = f(x) = l(x)]$

$$\begin{aligned} A^+(x) &= A^+(u, x), & A_-(x) &= A_-(u, x), \\ A^-(x) &= A^-(l, x), & A_+(x) &= A_+(l, x). \end{aligned} \tag{3.2}$$

Now since $u(x) \geqq l(x)$ for every x, it follows that

$$\begin{aligned} A^+(u, x) &\geqq A^+(l, x), & A_+(u, x) &\geqq A_+(l, x), \\ A^-(u, x) &\leqq A^-(l, x), & A_-(u, x) &\leqq A_-(l, x). \end{aligned} \tag{3.3}$$

On the other hand, $u(x)$ and $l(x)$ are both measurable. The directional angle is, therefore, either 0° or 360° at the points of $y=u(x)$ and $y=l(x)$, for almost every x. Consequently,

$$\begin{aligned} A^+(u, x) &= A^-(u, x) = \sup A\,(u, x), \\ A^+(l, x) &= A^-(l, x) = \sup A\,(l, x), \\ A_+(u, x) &= A_-(u, x) = \inf A\,(u, x), \\ A_+(l, x) &= A_-(l, x) = \inf A(l, x), \end{aligned} \tag{3.4}$$

say, for almost every x. From (3.3) and (3.4) we obtain

$$\sup A\,(u, x) = \sup A\,(l, x), \quad \inf A\,(u, x) = \inf A\,(l, x) \quad \text{a.e. in } E. \tag{3.5}$$

Combining (3.2) and (3.5), we obtain, for almost every point x of E.

$$\begin{aligned} A^+(x) &= \sup A\,(u, x) = \sup A\,(l, x) = A^-(x), \\ A_+(x) &= \inf A\,(u, x) = \inf A\,(l, x) = A_-(x). \end{aligned}$$

Hence the directional angle of $y=f(x)$ is, almost everywhere in E, the same as that of $y=u(x)$ or $y=l(x)$; that is, either 0° or 360°. This completes the proof.

REMARK. A measurable function $f(x)$ is aproximately continuous almost everywhere, and therefore

$$u(x) = f(x) = l(x) \qquad \text{a.e.}$$

Thus case (c) of Theorem III occurs almost everywhere, and the directional angle of $y=f(x)$ is, therefore, either 0° or 360° almost everywhere.

The following simple example shows that case (β) (directional angle $=180°$) may be realized for a *non*-measurable function on a set of *positive* measure. Let $I=E_1+E_2$ be a decomposition of the interval I: (0, 1) into two disjoint, non-measurable sets, each of exterior measure 1. Define $f(x)=1$ in E_1, $f(x)=0$ in E_2. It follows that case (β) is valid at every point of I.

The results of this paper on approximate derivatives, in particular Theorems I, II and III, are readily extensible to the case where x ranges over an arbitrary set instead of an interval. They are also easily extended to many-valued functions.

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GENERALIZATION OF MENGER'S RESULT ON THE STRUCTURE OF LOGICAL FORMULAS

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Menger's paper[2] gives necessary and sufficient conditions that an expression containing sentential variables and unary and binary sentential connectives be a formula in the Łukasiewicz notation. This paper extends his result to expressions containing n-ary symbols for all $n \geqq 0$. Sentential variables and constants are treated as the case $n=0$.

An expression is a sequence $s_1 \cdots s_k$ such that s_i for $i=1, \cdots, k$ is an n-ary symbol for some n. An initial segment of such an expression is an expression $s_1 \cdots s_i$ where $i<k$; a terminal segment is an expression $s_t \cdots s_k$ where $t>1$. A formula is a sequence $sz_1 \cdots z_n$ where s is an n-ary symbol, and $z_1, \cdots, z_n$ are formulas. The measure $[s]$ of an n-ary symbol s is $n-1$. The measure $[s_1 \cdots s_k]$ of an expression $s_1 \cdots s_k$ is $[s_1]+ \cdots +[s_k]$.

THEOREM. *Necessary and sufficient conditions that an expression* $x=s_1 \cdots s_k$ *be a formula are*:

$$[y] \geqq 0 \textit{ for each initial segment } y \textit{ of } x, \tag{1}$$

and

$$[x] = -1. \tag{2}$$

PROOF. Suppose s is an n-ary symbol, $z_1, \cdots, z_k$, $k \geqq 0$ are formulas, z is an initial segment of a formula z_{k+1}, and $z_1, \cdots, z_{k+1}$ satisfy (1) and (2). Then

$$\begin{aligned} [sz_1 \cdots z_k] &= [s] + [z_1] + \cdots + [z_k] \\ &= (n-1) - 1 - \cdots - 1 = n - 1 - k \end{aligned} \tag{3}$$

and

$$[sz_1 \cdots z_k z] = n - 1 - k + [z] \geqq n - 1 - k. \tag{4}$$

PROOF OF NECESSITY. Let $x = sz_1 \cdots z_n$ be a formula where by the induction hypothesis s is an n-ary symbol and $z_1, \cdots, z_n$ are formulas

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[1] Paper suggested by Dr. J. C. C. McKinsey and Miss Helen Dayton of Oklahoma Agricultural and Mechanical College.

[2] Menger, Karl, *Eine elementare Bemerkung über die Struktur logischer Formeln*, Ergebnisse eines mathematischen Kolloquiums, vol. 3, 1930–1931, pp. 22–23.

satisfying (1) and (2). An initial segment y of x has one of the forms (3) or (4), with $h<n$ so that $[y]\geqq 0$, and (3) applies to x with $h=n$, giving $[x]=-1$.

Proof of sufficiency. An expression of length one is a formula by (2) and the definition. Suppose all expressions of length less than k satisfying (1) and (2) are formulas, and let $x=s_1\cdots s_k$ be an expression of length $k>1$ satisfying (1) and (2).

Lemma. *Starting at any symbol* s_i *of* x, $1\leqq i\leqq k$, *there is a unique segment satisfying* (1) *and* (2).

Proof. For any terminal segment $s_i\cdots s_k$ of x, we have $[s_1\cdots s_{i-1}]+[s_i\cdots s_k]=[x]$ or $[s_i\cdots s_k]\leqq -1$ by (1) and (2). Thus, for any symbol s_i, $1\leqq i\leqq k$, there is a symbol s_j, $i\leqq j\leqq k$, such that $[s_i\cdots s_j]=-1$. For each integer i, only the smallest such integer j provides a segment $s_i\cdots s_j$ satisfying (1) as well as (2).

By (1), s_1 is a connective s for some $n>0$. The lemma may be applied, starting at s_2, to exhaust the symbols of x by constructing consecutive segments $z_1, \cdots, z_h$, each a formula by the induction hypothesis. Now for $x=sz_1\cdots z_h$, by (2) and (3), $[x]=-1=n-1-h$ or $h=n$; hence x is a formula.

This completes the proof of the theorem.

Corollary. *In any formula, starting at a given symbol, there is a unique consecutive part which is a formula.*

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THE EXTENSION OF A HOMEOMORPHISM DEFINED ON THE BOUNDARY OF A 2-MANIFOLD

J. W. T. YOUNGS[1]

1. Introduction. Suppose that M and $\mathfrak{M}$ are homeomorphic 2-manifolds with boundaries B and $\mathfrak{B}$, respectively. Then B ($\mathfrak{B}$) is the union of a collection $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$), $n>0$, of Jordan curves which are disjoint in pairs. Suppose h is a homeomorphism from B onto $\mathfrak{B}$. (It may be assumed that $h(J_i)=\mathfrak{J}_i$, $i=1, \cdots, n$.) It is the purpose of this paper to investigate the possibility of *extending* the homeomorphism h so as to obtain a homeomorphism from M onto $\mathfrak{M}$.

It will be shown that, if M (and therefore $\mathfrak{M}$) is orientable, then h *cannot always* be extended unless $n=1$. (A necessary and sufficient condition for the extendability is given in Theorem 1.) If M (and therefore $\mathfrak{M}$) is non-orientable, then the extension is *always* possible—a result which, at first glance, may appear rather implausible.

These results are generalizations of the Schoenflies theorem [2, p. 324][2] and, astonishingly enough, do not appear to have been mentioned elsewhere. It is possible that they may serve as instruments in generalizing an extension theorem of Adkisson and MacLane [1] from a statement involving 2-spheres to one concerned with 2-manifolds. In any event, the theorems will be employed in the representation problem for Fréchet surfaces in a manner comparable to that by which a similar theorem was used to obtain a partial solution (Youngs [4]).

2. The theorems. Using the notation of the introduction, suppose that M is orientable. A *concordant* orientation of $(J_1, \cdots, J_n)$ consists of an orientation on each Jordan curve, $J_1, \cdots, J_n$, such that the orientation induced on M by the orientation on J_i is independent of $i=1, \cdots, n$; or, in other words, there is an orientation on M such that $J_1+\cdots+J_n$ (J_i regarded as a cycle, $i=1, \cdots, n$) is the algebraic boundary of M. Hence each concordant orientation of $(J_1, \cdots, J_n)$ determines an orientation on M; namely, the orientation induced by J_i for any $i=1, \cdots, n$. Conversely each orientation on M determines a concordant orientation of $(J_1, \cdots, J_n)$; the orientation on J_i being induced by the orientation on M, $i=1, \cdots, n$.

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[2] Numbers in brackets refer to the bibliography.

Thus there are two concordant orientations of $(J_1, \cdots, J_n)$; given one, the other is obtained by reversing the orientation on J_i, $i=1, \cdots, n$.

Now consider the homeomorphism h and select a concordant orientation of $(J_1, \cdots, J_n)$. Then J_i is oriented and $h|J_i$ (that is, h restricted to J_i) determines an orientation on $\mathfrak{J}_i$, $i=1, \cdots, n$. This selection of orientations may or may not be a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$. If it is, then h is said to carry a concordant orientation of $(J_1, \cdots, J_n)$ into a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$. It is obvious that if h carries one of the two concordant orientations of $(J_1, \cdots, J_n)$ into a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$, then it carries the other concordant orientation of $(J_1, \cdots, J_n)$ into the second concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$.

Now suppose that the homeomorphism h can be extended so as to obtain a homeomorphism $h^*: M \twoheadrightarrow \mathfrak{M}$. (The heavy arrow indicates that the mapping is from M onto $\mathfrak{M}$). Select an orientation on M and consider the concordant orientation of $(J_1, \cdots, J_n)$ determined by the orientation on M. The homeomorphism $h|J_i$ induces an orientation on $\mathfrak{J}_i$, $i=1, \cdots, n$, while the homeomorphism h^* induces an orientation on $\mathfrak{M}$. It follows that this orientation on $\mathfrak{M}$ induces an orientation on $\mathfrak{J}_i$ which is precisely that induced by $h|J_i$, $i=1, \cdots, n$. Consequently the orientation induced on $\mathfrak{J}_i$ by $h|J_i$, $i=1, \cdots, n$, yields a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$. In other words, h takes a concordant orientation of $(J_1, \cdots, J_n)$ into a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$. Thus half of the first theorem listed below has been proved.

THEOREM 1. *If M and $\mathfrak{M}$ are homeomorphic orientable 2-manifolds with boundaries $B = J_1 \cup \cdots \cup J_n$ and $\mathfrak{B} = \mathfrak{J}_1 \cup \cdots \cup \mathfrak{J}_n$ respectively $(n>0)$, then a homeomorphism $h: B \twoheadrightarrow \mathfrak{B}$ can be extended to a homeomorphism $h^*: M \twoheadrightarrow \mathfrak{M}$ if, and only if, h carries a concordant orientation of $(J_1, \cdots, J_n)$ into a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$.*

THEOREM 2. *If M and $\mathfrak{M}$ are homeomorphic non-orientable 2-manifolds with boundaries $B = J_1 \cup \cdots \cup J_n$ and $\mathfrak{B} = \mathfrak{J}_1 \cup \cdots \cup \mathfrak{J}_n$ respectively $(n>0)$, then a homeomorphism $h: B \twoheadrightarrow \mathfrak{B}$ can always be extended to a homeomorphism $h^*: M \twoheadrightarrow \mathfrak{M}$.*

PROOF OF THEOREM 1. The sufficiency of the condition needs to be established. Assuming that $h(J_i) = \mathfrak{J}_i$, $i=1, \cdots, n$, let M^* and $\mathfrak{M}^*$ be the closed orientable 2-manifolds obtained by adjoining 2-cells to the bounding curves $J_1, \cdots, J_n$ and $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$. These manifolds are obviously homeomorphic; suppose that their 1-dimensional Betti number is $2j$, $j\geqq 0$. By a suitable cutting of M^* ($\mathfrak{M}^*$) one obtains the fundamental polygon $P^*:AA^{-1}$ ($\mathfrak{P}^*:\mathfrak{A}\mathfrak{A}^{-1}$), if $j=0$, or $P^*:A_1B_1A_1^{-1}B_1^{-1}\cdots A_jB_jA_j^{-1}B_j^{-1}$ ($\mathfrak{P}^*:\mathfrak{A}_1\mathfrak{B}_1\mathfrak{A}_1^{-1}\mathfrak{B}_1^{-1}\cdots\mathfrak{A}_j\mathfrak{B}_j\mathfrak{A}_j^{-1}\mathfrak{B}_j^{-1}$), if $j>0$, and the Jordan curves $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$) are interior to P^* ($\mathfrak{P}^*$). (See Seifert-Trelfall [3, chap. VI].) Let J_{n+1} ($\mathfrak{J}_{n+1}$) be the Jordan curve boundary of P^* ($\mathfrak{P}^*$) and P' ($\mathfrak{P}'$) be the 2-manifold obtained from P^* ($\mathfrak{P}^*$) by omitting the interiors of the 2-cells bounded by $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$).

Select an orientation on P' and consider the induced orientations on $J_1, \cdots, J_{n+1}$. The mapping $h|J_i$ induces an orientation on $\mathfrak{J}_i$, $i=1, \cdots, n$. It follows from the hypothesis that the orientation on $\mathfrak{P}'$ induced by $\mathfrak{J}_i$ is independent of $i=1, \cdots, n$. Consider $\mathfrak{J}_{n+1}$ to be given the orientation induced by the above orientation on $\mathfrak{P}'$. It may be assumed that the order $\mathfrak{A}\mathfrak{A}^{-1}$, if $j=0$, or $\mathfrak{A}_1\mathfrak{B}_1\mathfrak{A}_1^{-1}\mathfrak{B}_1^{-1}\cdots\mathfrak{A}_j\mathfrak{B}_j\mathfrak{A}_j^{-1}\mathfrak{B}_j^{-1}$, if $j>0$, agrees with the orientation on $\mathfrak{J}_{n+1}$, and that the order AA^{-1}, if $j=0$, or $A_1B_1A_1^{-1}B_1^{-1}\cdots A_jB_jA_j^{-1}B_j^{-1}$, if $j>0$, agrees with the orientation on J_{n+1}.

If $j=0$ select the vertex v ($\mathfrak{v}$) which is the first point of A ($\mathfrak{A}$). If $j>0$ select the vertex v ($\mathfrak{v}$) which is the first point of A_1 ($\mathfrak{A}_1$). Let $x_i\in J_i$ and $\mathfrak{x}_i=h(x_i)\in\mathfrak{J}_i$, $i=1, \cdots, n$. It follows that there are arcs $Q_1, \cdots, Q_n$ ($\mathfrak{Q}_1, \cdots, \mathfrak{Q}_n$) from v to $x_1, \cdots, x_n$ ($\mathfrak{v}$ to $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$) respectively, such that: 1°. $Q_i\cap Q_k=v$ ($\mathfrak{Q}_i\cap\mathfrak{Q}_k=\mathfrak{v}$), $i\neq k$; $i, k=1, \cdots, n$. 2°. If P' ($\mathfrak{P}'$) is cut along these arcs then one obtains the polygon $P:Q_1J_1Q_1^{-1}\cdots Q_nJ_nQ_n^{-1}AA^{-1}$ ($\mathfrak{P}:\mathfrak{Q}_1\mathfrak{J}_1\mathfrak{Q}_1^{-1}\cdots\mathfrak{Q}_n\mathfrak{J}_n\mathfrak{Q}_n^{-1}\mathfrak{A}\mathfrak{A}^{-1}$), if $j=0$, or $P:Q_1J_1Q_1^{-1}\cdots Q_nJ_nQ_n^{-1}A_1B_1A_1^{-1}B_1^{-1}\cdots A_jB_jA_j^{-1}B_j^{-1}$ ($\mathfrak{P}:\mathfrak{Q}_1\mathfrak{J}_1\mathfrak{Q}_1^{-1}\cdots\mathfrak{Q}_n\mathfrak{J}_n\mathfrak{Q}_n^{-1}A_1\mathfrak{B}_1\mathfrak{A}_1^{-1}\mathfrak{B}_1^{-1}\cdots\mathfrak{A}_j\mathfrak{B}_j\mathfrak{A}_j^{-1}\mathfrak{B}_j^{-1}$), if $j>0$. This is the fundamental polygon for M ($\mathfrak{M}$) and it is to be noted that the oriented boundary arcs $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$) are found in the order of increasing indices in the above array. It follows that the homeomorphism h carrying J_i onto $\mathfrak{J}_i$, $i=1, \cdots, n$, can be extended to a homeomorphism h from the boundary of P onto the boundary of $\mathfrak{P}$ in such a manner that if x and y are to be identified by the identification mapping which obtains M from P, then $h(x)$ and $h(y)$ are identified by the identification mapping which obtains $\mathfrak{M}$ from $\mathfrak{P}$.

Now by the Schoenflies theorem there is an extension h^* of h which maps P homeomorphically onto $\mathfrak{P}$. The homeomorphism h^* of the theorem is simply the above h^* considered as a mapping from M onto $\mathfrak{M}$.

PROOF OF THEOREM 2. Assuming that $h(J_i)=\mathfrak{J}_i$, $i=1, \cdots, n$, let M^* and $\mathfrak{M}^*$ be the closed non-orientable 2-manifolds obtained by adjoining 2-cells to the boundary curves $J_1, \cdots, J_n$ and $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$. These manifolds are homeomorphic; suppose that their 1-dimensional Betti number is $(k-1)$, $k>0$. By a suitable cutting of M^* ($\mathfrak{M}^*$) one obtains the fundamental polygon $P^*:A_1A_1 \cdots A_kA_k$ ($\mathfrak{P}^*:\mathfrak{A}_1\mathfrak{A}_1 \cdots \mathfrak{A}_k\mathfrak{A}_k$) and the Jordan curves $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$) are interior to P^* ($\mathfrak{P}^*$). (See Seifert-Trelfall [3, chap. VI].)

Consider a fixed orientation on P^*. This determines an orientation on J_i, and the homeomorphism $h|J_i$ induces an orientation on $\mathfrak{J}_i$, which, in turn, determines an orientation on $\mathfrak{P}^*$, $i=1, \cdots, n$. If this orientation on $\mathfrak{P}^*$ is independent of $i=1, \cdots, n$, then it is readily seen that the proof can be completed as in Theorem 1. Suppose, therefore, that the orientations on $\mathfrak{J}_1, \cdots, \mathfrak{J}_m$ $(m<n)$ determine one orientation on $\mathfrak{P}^*$ while those on $\mathfrak{J}_{m+1}, \cdots, \mathfrak{J}_n$ determine the other. There is a cross cut $\mathfrak{A}_1'$ of $\mathfrak{P}^*$ joining the first point of $\mathfrak{A}_1$ to the last point of $\mathfrak{A}_1$ and separating $\mathfrak{J}_1 \cup \cdots \cup \mathfrak{J}_m$ from $\mathfrak{J}_{m+1} \cup \cdots \cup \mathfrak{J}_n$. (See Fig. 1.) Cut $\mathfrak{P}^*$ along $\mathfrak{A}_1'$ and identify the points of the two arcs labelled $\mathfrak{A}_1$ to obtain Fig. 2. Notice that in doing this one obtains the fundamental polygon $\mathfrak{P}':\mathfrak{A}_1'\mathfrak{A}_1'\mathfrak{A}_2\mathfrak{A}_2 \cdots \mathfrak{A}_k\mathfrak{A}_k$ of $\mathfrak{M}^*$ and $\mathfrak{J}_i$ now determines an orientation on $\mathfrak{P}'$ which is independent of $i=1, \cdots, n$. The proof is completed as in Theorem 1.

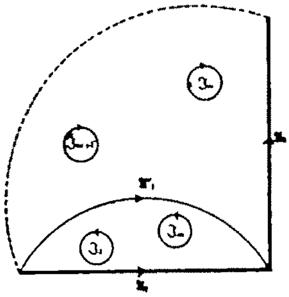

FIG. 1

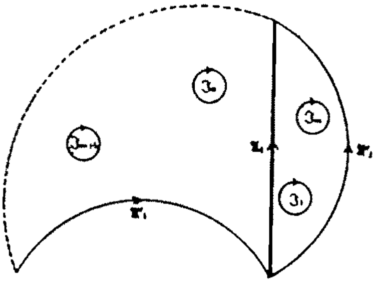

FIG. 2

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TOPOLOGICAL RINGS

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I. INTRODUCTION

1. The literature. Topological rings were apparently first defined in van Dantzig's thesis [13], [14].[1] Of course innumerable instances of topological rings had been studied earlier: notably Hensel's p-adic numbers and their abstraction by Kürschak [54] to fields with valuations. A very considerable literature has since grown up around valuation theory and its applications to algebraic geometry, algebraic functions, and algebraic number theory. Van Dantzig's thesis inaugurated a second stream of investigation: locally compact rings. During the present decade much study has been devoted to Banach algebras (normed rings) in work begun by Gelfand and his colleagues, though foreshadowed by Stone [75], Nagumo [60], and Hebroni [32]. This address is divided into three parts more or less reflecting these three lines of investigation: topological division rings, locally compact rings, and normed algebras.

Besides papers actually referred to in the address, the bibliography includes most of the relevant contributions of the last decade. In certain respects there is no pretense of completeness: for example, there are only a few of the papers on valuations. Particular attention should perhaps be called to the reports of Köthe [52] and Lorch [55], and to the somewhat less closely related reports of Dunford [19], Taylor [78], and Hyers [34].

2. Definitions. By a topological ring we mean a ring and a Hausdorff space in which $a-b$ and ab are jointly continuous in a and b. This is substantially the definition of van Dantzig, except that he imposed a countability restriction. A brief account of the fundamentals is presented in Bourbaki [10].

By a normed algebra we mean a normed linear space which is at the same time an algebra satisfying

$$\|ab\| \leq \|a\| \|b\|. \tag{1}$$

If one merely assumes joint continuity of ab, one can pass to an equivalent norm satisfying (1). A Banach algebra is a complete

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

normed algebra. It was noted by Gelfand [22, Theorem 1] that in the presence of completeness, separate continuity of ab implies joint continuity. Arens [7, Theorem 5] extended this result to the case of any topological ring whose underlying space is complete metric. In both accounts the proof rests ultimately on a category argument.

To discuss inverses without assuming a unit element we introduce the operation $a \circ b = a+b+ab$, and call b a right quasi-inverse of a if $a \circ b = 0$, a quasi-inverse if also $b \circ a = 0$. An element is quasi-regular if it has a quasi-inverse. Under the operation $\circ$ the quasi-regular elements form a group, which will be a topological group if quasi-inversion is continuous. For a normed algebra this continuity can be proved in much the same way as in elementary analysis (cf. [7, p. 626]). The continuity of the quasi-inverse is also valid in locally compact rings without divisors of 0 [44, Theorem 8]. However, local compactness alone does not suffice, as is shown by an example in [44]; nor does it suffice to have a complete metric space [7, p. 630].

To treat maximal ideals in rings without unit it is convenient, following Segal [69, p. 74] to introduce regular ideals. A (right, two-sided) ideal I is regular if there exists a (left, two-sided) unit modulo I. In a Banach algebra all regular maximal ideals are closed. More generally this is true in any ring in which the right quasi-regular elements form an open set; such rings are called Q_r-rings in [40]. If the quasi-regular elements are open (a formally stronger condition), the ring is called a Q-ring.

We shall use the Perlis-Jacobson radical, defined in [38] to be the union of the right quasi-regular ideals; it may be shown to be the intersection of the right (or left) regular maximal ideals. Hence in a Q_r-ring the radical is closed. The radical is also closed if the right quasi-regular elements form a closed set, which is true for example in a compact ring. But the radical is not always closed, as is shown by an example in [40]. In certain respects the following is a simpler example. Let P be the ring of p-adic integers and A the ring of all infinite matrices over P, with only a finite number of nonzero elements in each row. We topologize A with the "finite" topology: the general neighborhood of 0 consists of all matrices with first n rows 0. (The p-adic topology in P is ignored; in fact any ring with a non-nil radical will do in place of P.) Then it is easily checked that the radical R of A contains all matrices with only a finite number of nonzero entries, and these all divisible by p. The matrix $p(e_{12}+e_{23}+e_{34}+ \cdots)$ therefore lies in the closure of R, but it is not itself in R, since it is not even quasi-regular.

A useful concept in the study of topological rings is that of boundedness, first introduced by Shafarevich [72], and presumably inspired by the analogous concept for topological linear spaces. A set S in a topological ring is right bounded if for any neighborhood U of 0 there exists a neighborhood V of 0 such that $SV\subset U$. Left boundedness is analogously defined, and a set is bounded if it is both right and left bounded. Compact sets are bounded, and for algebraic purposes bounded sets may often be regarded as the appropriate generalization of compact sets. For example, a polynomial is uniformly continuous on any bounded set; and if the quasi-inverse is continuous, then it is uniformly continuous on any bounded set.

II. TOPOLOGICAL DIVISION RINGS

3. Normed division algebras. Mazur announced in [58] the theorem that the only (complex) normed division algebra is the field of complex numbers itself. Gelfand proved this in [22], asserting that his proof was different from Mazur's. Gelfand's proof rests on an application of Liouville's theorem and, as Lorch has remarked, substantially the same result had been given earlier by Taylor [77]; the same sort of argument occurs in many older references.

We present this Taylor-Gelfand argument in a somewhat generalized form substantially due to Arens [7]. Let A be a topological algebra over the complex numbers. We assume that A has a unit, that the inverse in A is continuous, and that A admits a total set of functionals (the latter is true for example if A, as a topological linear space, is convex). The spectrum of x is the set of scalars λ such that $(x-\lambda)^{-1}$ does not exist. We assert that *every element of A has a nonvoid spectrum*. For if not, we form for an arbitrary functional f the function $g(\lambda)=f[(x-\lambda)^{-1}]$, and verify that $g(\lambda)$ is entire and approaches 0 as $\lambda\rightarrow\infty$. By Liouville's theorem, $g\equiv 0$, whence $(x-\lambda)^{-1}\equiv 0$, a contradiction. If we specialize A to be a division algebra, we deduce the Gelfand-Mazur theorem: every element of A is a scalar multiple of the unit. By a standard embedding device one obtains a corresponding theorem in the real case; here the reals, complexes or quaternions may arise.

While the above argument appears to have squeezed everything possible out of Liouville's theorem, it seems probable that stronger theorems are true, and perhaps even that no nontrivial topological division algebras over the complexes exist at all. The question may be rephrased thus: is it possible to convert the field of rational functions $f(z)$ of a complex variable z into a topological algebra? An easily proved result in this connection is that it cannot be done if $f(z)$ is to be continuous jointly in f and z.

Returning to normed algebras, we mention two generalizations of the Gelfand-Mazur theorem which are given in [46] and [45] respectively. (1) Call x a topological divisor of 0 if there exists a non-null sequence y_n such that xy_n or $y_nx \to 0$. The following theorem is a refinement of results in [73], [7] and [68]: a normed algebra without any nonzero topological divisors of 0 is the reals, complexes, or quaternions. It is to be observed that neither a unit element nor completeness is assumed. (2) A Banach algebra which is regular in the sense of von Neumann (for any a there exists an x such that $axa = a$) is finite-dimensional. Here completeness is indispensable.

Topological linear spaces are of course necessarily connected, and it is perhaps natural, in connection with the above results, to ask what connected fields exist. In a somewhat different context, much the same question was raised by Baer and Hasse [9]. Some light has been thrown on this question recently, by Dieudonné [17] who constructed a connected subfield of the complex numbers, and Kapuano [47] who constructed a one-dimensional subfield of the complex numbers (in both cases a field other than the reals, of course).

4. Valuations. Kürschak [54] defined a valuation of a division ring to be a real-valued function satisfying $|a| \geqq 0$, $|a| = 0$ if and only if $a = 0$, $|ab| = |a|\,|b|$, $|a+b| \leqq |a| + |b|$. He was inspired by two outstanding examples: complex numbers and p-adic numbers. Ostrowski [61] showed that these two examples are typical. A field with a valuation is either (archimedean case) a subfield of the complex numbers with essentially its ordinary absolute value, or (nonarchimedean case) it satisfies with the p-adic numbers the strong triangle inequalty

$$|a + b| \leqq \max(|a|, |b|). \tag{2}$$

Thus structure theory is of interest only in the non-archimedean case, although in applications (notably algebraic number theory) it remains important to consider all valuations. Krull [53] took the next step of observing that in (2), addition of real numbers has disappeared; only the multiplicative group of positive real numbers is relevant, and one may generalize by replacing it by an arbitrary ordered abelian group Γ. The classification of fields with valuations was successfully studied in case Γ is the additive group of integers; cf. Teichmüller [79] and MacLane [56]. For an arbitrary group Γ, definite but more complicated results were obtained in [39]. A similar study has been made for local rings by Cohen [12].

Examination of some of the proofs in valuation theory reveals that full use is not made of the existence of a valuation. In many cases one can get by with a topological field having the following property: if a set S is bounded away from 0; then S^{-1} is bounded (in the sense of §2). In [41], division rings with this property are said to be of "type V." The same class of division rings is briefly studied in an exercise in Bourbaki [10, p. 57, Ex. 13].

The following are examples of theorems which carry over to the case of type V. (1) A finite-dimensional topological linear space over a complete division ring of type V necessarily carries the Cartesian product topology. (2) The completion of an algebraically closed field of type V is algebraically closed. It has yet to be determined whether a finite algebraic extension of a field of type V can be topologized so as to be of type V. This is proved for quadratic extensions in [41].

Fields with an (ordinary real-valued) valuation can be characterized as being of type V and having a neighborhood of 0 consisting of topologically nilpotent elements [41, Theorem 3]. Zelinsky [83] has similarly characterized fields with a non-archimedean valuation. As a test problem for further progress, one may propose the following question: how can the field of rational numbers be topologized so as to be of type V? Are there any such topologies other than those given by valuations?

We mention finally [42], in which there is presented a topological discussion of polynomials in fields of type V. In particular a recent theorem of Habicht [31] on real closed fields is generalized. (It is to be observed that any ordered field in its order topology is of type V.)

5. **Locally compact division rings.** With the examples of the real numbers and p-acid numbers before one, it is natural to ask what locally compact fields exist, and in particular if any exist essentially different from these two examples. Such an investigation was undertaken independently by van Dantzig [13] and Pontrjagin [64]. The latter showed that the only locally compact connected division rings are the reals, complexes, and quaternions. (In his book [65, pp. 171–178] he assumes the second axiom of countability, but no countability assumption is made in the paper.) Van Dantzig confined himself to the commutative case and assumed the second axiom of countability. In the connected case he obtained the result cited above (excluding of course the quaternions). He also treated the totally disconnected case, obtaining either a finite extension of the p-adic numbers (a "π-adic field") or the field of formal power series over a finite field (a "z-adic field"). Jacobson [36] completed the problem by treating the non-commutative totally disconnected case, showing that one gets an algebra of finite order over a p-adic or z-adic field; known struc-

ture theory shows that the algebra is cyclic. Jacobson assumed the first axiom of countability in his proof; however the opening lines in Pontrjagin's paper constitute a proof that any locally compact division ring satisfies the first axiom of countability. Otobe [62] proved again this superfluity of a countability assumption. It may be remarked that slightly later in the argument one of course gets the second axiom of countability free of charge (always assuming, as was tacitly done above. that the topology is not discrete).

Thus the structure of locally compact division rings was fully determined. The elegance of the result has attracted further study since then. Shafarevich [72] made the following contribution: he gave a characterization of those topological fields which admit a valuation giving the same topology. The conditions in question are readily verifiable in a locally compact field, and so the problem is reduced to one in valuation theory. The known theorems on valuations readily complete the solution. In the non-commutative case there are some additional difficulties which were overcome in [41]. A noteworthy feature of this solution of the problem is that it treats the connected and totally disconnected cases on precisely the same footing.

The latter merit is also shared by the interesting proof announced by Braconnier [11]. He observes that the mapping $a \rightarrow ax$ $(x \neq 0)$ is a bicontinuous automorphism of the additive group of a locally compact division ring. The uniqueness of Haar measure shows that this mapping multiplies all measures by a positive real number $v(x)$. One then verifies that $v(x)$ is in fact a valuation. From this point on, valuation theory is presumably to be applied.

Various generalizations of the Pontrjagin-van Dantzig-Jacobson theory might be proposed. Locally compact rings which are not necessarily division rings will be discussed in Part III of this address. The hypothesis of local compactness might be weakened, perhaps most plausibly to local boundedness. Finally we shall mention the weakening or dropping of the associative law. This has some interest from the point of view of the foundation of geometry—cf. Kolmogoroff [50] and Köthe [52]; and, in view of the work of Ruth Moufang, particular interest is attached to alternative division rings. The connected case falls under the results discussed in §6 below. It appears that nothing is known in the totally disconnected case; but one may venture the conjecture that the only totally disconnected locally compact alternative division rings are Cayley-Dickson algebras over a p-adic or $\mathfrak{p}$-adic field. The methods that work in the associative case do not seem to apply without very considerable modification, though perhaps Braconnier's is the most hopeful.

III. LOCALLY COMPACT RINGS

6. The component of 0. A great deal is known about the structure of locally compact abelian groups, mainly through the work of Pontrjagin and van Kampen. Expositions can be found in [65] and [80]. Thus the study of locally compact rings begins with the observation that the structure of the underlying additive group is substantially known. But even if the additive group is completely known, this does not mean that there is nothing left to study in the ring. To cite an extreme case: finite abelian groups are completely known, but it can hardly be said that there are no unsolved questions about finite rings.

Nevertheless the additive theory has provided strong tools for ring investigations. This is particularly notable in the connected case, where fairly decisive results were obtained by Jacobson and Taussky [35]. The heart of the argument is given in [44] in the following general form: let A be a locally compact ring, C the component of 0, and B a right bounded additive subgroup of A; then $CB=0$. This is a fairly immediate consequence of the existence of sufficiently many characters of the additive group of A, which in turn rests ultimately on the Peter-Weyl theorem. Indeed the cited result is true under the mere hypothesis (in lieu of local compactness) of the existence of sufficiently many characters, or even the still weaker hypothesis that there exist sufficiently many continuous homomorphisms into a group with no arbitrarily small subgroups.

We quote next the structure theorem which asserts that A is the direct sum of a vector group N and a group in which the component P of 0 is compact [80, p. 110], and we deduce $P^2=PN=NP=0$. In particular if A is compact and connected, $A^2=0$. In any event, if A is connected, the hypothesis that there is no total divisor of 0 will entail $P=0$, $A=N$, and it follows readily that A is an algebra of finite order over the reals. If it is an associative division algebra we get the theorem of Pontrjagin, and if alternative, A may further be the Cayley numbers. We may note at this point that these four algebras have also been characterized by Albert [1] as being precisely those that admit valuations which are homogeneous with respect to real scalars.

Something further can be said about locally compact connected rings without the assumption that there are no total divisors of 0. We in any event have the additive decomposition $A=N+P$, with $PA=AP=0$. We may thus describe the structure of A as follows: N is an algebra of finite order over the reals, P a compact abelian group, and multiplication takes place in A according to the rule

$$(n_1 + p_1)(n_2 + p_2) = n_1 n_2 + f(n_1, n_2)$$

where $n_i \in N$, $p_i \in P$, and f is a continuous homomorphism of the Kronecker product $N \times N$ into P. To some extent one can prescribe f more fully in terms of the structure of N, but we shall not enter into details. The following remark may however be of interest. If A is locally connected, then P is a direct sum of circle groups, and f is a direct sum of characters. If we write $\{u_i\}$ for a basis of N, then the characters in question take the form

$$(\sum x_i u_i, \sum y_j u_j) \to \sum t_{ij} x_i y_j \pmod 1,$$

t_{ij} being real numbers. If finally A is locally Euclidean, this remark shows that a local coordinate system is introducible in which ring multiplication is a bilinear function. The analogue of Hilbert's fifth problem for rings thus has a strongly affirmative answer.

If we are given an arbitrary locally compact ring A, then the preceding discussion may be applied to its component C of 0. To some extent, questions on the structure of A may be reduced to the analogous ones for the connected ring C and the totally disconnected ring $A - C$; details are given in [44]. In the remainder of our discussion of locally compact rings we shall confine ourselves to the totally disconnected case.

7. **The compact case.** The fundamental feature of the totally disconnected case is the existence of group neighborhoods of 0, a result due to van Kampen. For a compact ring A we may pass at once to ideal neighborhoods of 0; if U is a group neighborhood we can find V such that the open ideal $V + AV + VA + AVA$ is contained in U. Thus a compact totally disconnected ring is b_r-adic in the sense of van Dantzig [15]. This fact makes possible a workable theory of the radical and semi-simplicity, as presented in [40].

Call an ideal I topologically nilpotent if for any neighborhood U of 0 there exists an integer n such that $I^n \subset U$. We begin with the theorem that the radical of A is topologically nilpotent; this is an immediate consequence of the ideal neighborhoods and the nilpotence of the radical of a finite ring. We can then proceed to a complete determination of compact semi-simple rings. Theorem: they are Cartesian direct sums of (any number of) finite simple rings. We shall sketch a proof which is slightly simpler than that in [40], the simplification resting on the use of regular ideals. First we show that in a compact semi-simple ring A there exist open regular maximal ideals, and that their intersection is 0; the proof in [40], for the corresponding statement with the word "regular" deleted, is valid almost unchanged. This already proves that A is a subdirect sum of finite simple rings. The structure theory of finite rings shows moreover that any finite combination of coordinates occurs; because of the compactness we readily pass to the complete direct sum.

This determination of compact semi-simple rings makes possible some results on the non-semi-simple case. These are obtained by the classical procedure of constructing in A idempotents which map on the idempotents of $A-R$, where R is the radical. It is appropriate to cite at this point the following unpublished remark due to Jacobson. The construction of idempotents in question can be carried through in a ring A if its radical R satisfies the following condition: for any $y \in R$ the equation $x^2+x=y$ can be solved for x in R. The solvability of this equation can be established by a series expansion which is valid if y is nilpotent in the ordinary sense. In our topological context, nilpotence of y in the topological sense will do, provided there is some further assurance of the convergence of the series. For compact rings, this assurance is provided by the group neighborhoods; in Banach algebras the convergence can be established by estimations of the norm. It may be well to remark that there exist rings where $x^2+x=y$ is not always solvable in the radical—an example is furnished by the ring of all rational numbers with odd denominators. In a sense we may ascribe the failure of this ring to the fact that it is not complete in its natural (2-adic) topology.

The ability to construct idempotents carries with it certain structure theorems for compact rings, notably an additive decomposition into primary rings, and the fact that a primary ring is the ring of matrices over a completely primary ring. Besides the light they shed on compact rings, these results are useful in the locally compact case as we shall see in §8.

This comparatively complete array of structure theorems runs of course parallel to the classical theory of rings with descending chain condition. It is natural to ask whether one can formulate a theory covering both cases at once. In [44] it is shown that a suitable set of axioms for this purpose is given by: (1) local compactness, (2) boundedness, (3) the descending chain condition for right ideals containing a fixed open two-sided ideal.

8. Compact subrings. Let A be a locally compact totally disconnected ring. We know that A has a system of group neighborhoods of 0; but we cannot pass to ideal neighborhoods of 0 as we did in the compact case. However we can achieve the next best thing: subring neighborhoods of 0. The argument for this is implicit in the work of Jacobson [36] and runs as follows. For a compact open subgroup U select an open subgroup V such that $V\subset U$, $VU\subset U$. Then $W=V+V^2+V^3+\cdots$ is a compact open subring contained in U.

The existence of these compact open subrings provides a foothold from which an invasion of locally compact rings becomes possible to some extent, and is launched in [44]. The spirit of the investigation is as follows. Let B be a compact open subring of A. We subdivide into two cases. (1) The radical of B is open: in this case B, and hence also A, is a Q-ring. (2) The radical of B is not open: in this case there exist idempotents arbitrarily near zero, by the compact ring theory. In either eventuality we have something from which we may hope to proceed further.

We mention two theorems which are proved in this way in [44]. (a) If A has no divisors of 0, or under any of several alternative hypotheses, we can rule out case (2) and conclude unreservedly that A is a Q-ring. (b) If A is not a radical ring it contains at least one closed regular maximal right ideal. Some contributions are also made in [44] to the structure theory of simple and semi-simple locally compact rings, but a reasonably complete theory has yet to be constructed.

IV. NORMED ALGEBRAS

9. The radical. The fundamental result on the radical of a Banach algebra is due to Gelfand [22]; cf. also Jacobson [38]. We present a somewhat generalized version. Let A be a complex Banach algebra. As is appropriate if A does not have a unit, we define the spectrum of $x\in A$ to consist of all scalars λ such that $-\lambda^{-1}x$ does not have a quasi-inverse. Then

$$\sup \left| \text{spectrum } (x) \right| = \lim_{n\to\infty} \|x^n\|^{1/n}. \tag{3}$$

The proof rests on the expansion of the quasi-inverse of $-\lambda^{-1}x$ as a power series in λ and examination of the radius of convergence; by taking functionals one can reduce the problem to the case of ordinary analytic functions. With a suitable definition of the spectrum, (3) can also be extended to a real Banach algebra.

If in particular x is in the radical, then its spectrum contains only 0, and it follows that $x^n\to 0$. One easily deduces that the radical is the union of all nil ideals, a nil ideal being one consisting entirely of topologically nilpotent elements. It seems remarkable that this nilpotence of the elements of the radical is valid both for compact rings and Banach algebras, but must be established by such different methods.

10. **Semi-simple algebras.** The analogy between Banach algebras and rings with a finiteness condition persists to some extent in the structure theory of semi-simple algebras; that is, in certain cases we obtain a (suitably generalized) direct sum of (suitably generalized) matrix algebras. Of course in the commutative case the matrix algebras collapse. We mention at this point that great simplifications also occur in the case of completely continuous Banach algebras (that is, algebras in which the mappings $x \rightarrow ax$ and $x \rightarrow xa$ are completely continuous). Such algebras are studied in [21] and [46].

Probably the most satisfactory structure theorem for commutative Banach algebras is due to Gelfand and Neumark [26]. Let A be a complex commutative Banach algebra with unit, and suppose there is an involution $x \rightarrow x^*$ which is conjugate linear and satisfies $\|xx^*\| = \|x\|\|x^*\|$. Then A is isomorphic to the algebra of all continuous complex functions on a compact Hausdorff space, with the norm identified as sup, and the * identified as conjugate. An obscure point in the Gelfand-Neumark proof was cleared up by Arens [5], who also [6] extended the theorem to the case where no unit is assumed. Analogous characterizations of the algebra of all real functions are given in [6] and [71]; and in [8] the real and complex cases are united.

There are important applications of such a characterization of the ring of continuous functions. We mention two in particular: the Stone-Čech compactification of a completely regular space, and the spectral theorem for a bounded normal operator on Hilbert space. Various accounts of these two applications can be found in the recent literature. Another interesting example on which to test the power of such a characterization is the theorem proved by Stone in [75] and Eidelheit in [20]: a closed sub-algebra of the algebra of continuous functions on a compact Hausdorff space is itself an algebra of all continuous functions on a compact Hausdorff space.

The main theorem of Gelfand and Neumark in [26] concerned noncommutative Banach algebras. To the axioms listed above one adds: $\|x^*\| = \|x\|$ and $1 + xx^*$ has an inverse (it is not known whether these two axioms are independent of the preceding ones). It is then shown that the algebra is isomorphic to a closed self-adjoint subalgebra of the algebra of all bounded operators on a Hilbert space. This is an important and elegant theorem, but it should be pointed out that the result merely identifies the algebra as being a subset of something, and for many purposes this is a serious drawback.

11. **H^*-algebras.** There is a structure theorem for noncommutative Banach algebras which does effect a complete identification of the algebras in question as being a full direct sum of full matrix algebras, in remarkable analogy to the finite-dimensional case. The theorem applies to the H^*-algebras of Ambrose [2]. The axioms are as follows: A is a Hilbert space and a Banach algebra; there is an involution which is conjugate linear and satisfies $(xy, z) = (y, x^*z)$, $(yx, z) = (y, zx^*)$ where the parentheses denote the Hilbert space inner product; and finally we add an axiom assuring nontriviality: $xA = 0$ implies $x = 0$. The structure theorem asserts that A is a direct sum of simple H^*-algebras, the direct sum being meant in the Hilbert space sense; and a simple H^*-algebra is the set of all matrices $a = (a_{ij})$ of complex numbers with $\sum |a_{ij}|^2 < \infty$ under ordinary matrix multiplication, with

$$(4) \qquad (a, b) = \alpha \sum a_{ij} b_{ij}^{c}$$

and $a^* = (a_{ji})^c$ (the superscript denoting complex conjugate). In [43] a somewhat simplified proof is given, and the theorem is extended to real H^*-algebras.

The outstanding application of this theorem is to the L_2-algebra of a compact group. Let G be a compact group and A the algebra of complex functions square-summable with respect to the Haar measure of G. We use the L_2-norm, define multiplication by convolution

$$(5) \qquad fg(x) = \int f(y) g(y^{-1}x) dy,$$

and set $f^*(x) = f(x^{-1})^c$. Then it follows directly from elementary properties of Haar measure and convolution that A is an H^*-algebra, and so the above structure theorem is applicable. It should be remarked that this highly algebraic approach to the study of compact groups was initiated by Köthe [51]; however, he based his work on the algebra of continuous functions instead of on L_2, and his results are not quite as satisfactory as those of Ambrose.

The fact that A is an H^*-algebra contains the Peter-Weyl theorem and related facts. In order to see this, one has to show (among other things) that these particular H^*-algebras have simple components that are finite-dimensional. As indicated by Ambrose, this can be done by appealing to the fact that A is a completely continuous algebra. However it is possible to deduce the finite-dimensionality without using any further information from the group, except for some rudimentary integration, and thus we can arrive at the Peter-Weyl theorem without mentioning the words "completely continuous."

The computation at the same time establishes the connection between the algebra A and the representations of G, the orthogonality relations, and so on.

One proceeds as follows. Let e_{ij} denote matric units for one of the simple components of A. In (5) set $f = e_{ik}$, $g = e_{kj}$, $x = 1 =$ the group identity. Then

$$e_{ij}(1) = \int e_{ik}(y) e_{kj}(y^{-1}) dy = (e_{ik}, e_{jk}) = \alpha \delta_{ij},$$

the last equality following from (4). Next define h by $h(x) = e_{ij}(xy)$; it is easy to see that h is in the right ideal generated by e_{ij}, say $h = \sum_k a_k e_{ik}$. On setting $x = 1$ we obtain $e_{ij}(y) = \alpha \sum a_k \delta_{ik} = \alpha a_i$. Hence

$$e_{ij}(xy) = \alpha^{-1} \sum_k e_{ik}(x) e_{kj}(y). \tag{6}$$

Set $y = x^{-1}$, $i = j = 1$ in (6) and integrate over the group (the total measure is assumed to be unity).

$$\alpha = \alpha^{-1} \sum_k (e_{1k}, e_{1k}). \tag{7}$$

From (7) it follows that the sum on k must be finite, and hence that the matrix algebra must be finite-dimensional; in fact α is precisely the degree of the matrices. Equation (6) shows that $x \to \alpha^{-1} e_{ij}(x)$ is a unitary representation of G, which one proves to be irreducible by standard methods. This substantially establishes the Peter-Weyl theorem and its attendant facts.

12. **Group algebras.** Let G be a locally compact group and A the L_1-algebra of G, that is, the algebra of all complex functions summable with respect to the left Haar measure of G, with convolution as multiplication. The investigation of these Banach algebras is a subject of considerable current interest, which promises to produce results of first-rate importance.

It is appropriate to begin the discussion with the theorem that A is semi-simple. Segal [69] proves this neatly as follows. He remarks that any self-adjoint algebra of bounded operators on Hilbert space is semi-simple; this is a direct corollary of the spectral theorem, or rather of that fragment of it which asserts that a bounded self-adjoint operator with only 0 in its spectrum is necessarily 0. The result then follows from the fact that A is faithfully represented by operators on $L_2(G)$. In certain cases Segal proves that the intersection of the regular maximal two-sided ideals is 0; this is a statement which is stronger than semi-simplicity. Raikov [66] has given a different proof of the semi-simplicity of A.

This semi-simplicity can be used to extablish the existence of a complete set of irreducible representations of G by operators on Banach spaces, called B-representations in [69]. If G is compact or abelian, these B-representations are finite-dimensional and hence similar to unitary representations. In the general case it is not known whether the B-representations are similar to unitary representations on Hilbert space; lack of information on this point is one of the obstacles impeding further progress on the structure of the L_1-algebra. It has been proved [28], [70] that G has a complete set of irreducible unitary representations. In the latter reference the proof is carried out by regarding A as an algebra of operators on $L_2(G)$ and taking the uniform closure.

We turn now to the case where A is commutative (which is true if and only if G is), and we shall briefly discuss ideal theory in A. Let $I \neq A$ be a closed ideal in A. We first raise the question: is I contained in a regular maximal ideal? This is immediate if A has a unit—which is true if and only if G is discrete. This easy result, plus the information that the maximal ideals correspond precisely to the characters of G, constitutes a generalization of Wiener's theorem on the reciprocal of an absolutely convergent trigonometric series. Even if A does not have a unit, it is still true that I can be embedded in a regular maximal ideal, although it requires a fair amount of proof—cf. [69] and [30]. It should be noted that the standard process of adjoining a unit does not resolve the difficulty, and in fact it is technically preferable not to adjoin a unit. In the case where G is the group of real numbers, this is the theorem of Wiener on spanning A by translates of a function whose Fourier transform does not vanish.

The question just raised might have been paraphrased as follows: is $A-I$ a radical ring (for only then would it be impossible to embed I in a regular maximal ideal)? A more far-reaching question suggests itself: is $A-I$ even semi-simple? Even for specific groups like the integers or reals, this is still an open question. For special choices of the ideal I, it has been answered affirmatively in [18] and [69], and these partial results are generalizations of Tauberian theorems of Wiener and Pitt.

Such is the state of our knowledge in the commutative case. In the noncommutative case there are as yet few results concerning $L_1(G)$ worth quoting. An attack in a different direction has been made by Ambrose [3] who proposes to study $L_2(G)$. This is not an algebra since it is not closed under convolution, but in compensation for this difficulty one has the availability of Hilbert space techniques. A somewhat similar program has been announced by Mautner [57]. Under the assumption that G is separable, he decomposes unitary representations of G, and in particular the regular representation, obtaining an analogue of Plancherel's theorem. The decomposition is based on a theorem of von Neumann concerning the expression of a ring of operators as a direct integral of irreducible rings.

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UNIVERSITY OF CHICAGO

THE JUNE MEETING IN VANCOUVER

The four hundred thirty-seventh meeting of the American Mathematical Society was held at the University of British Columbia, Vancouver, Canada on Saturday, June 19, 1948. The attendance was approximately fifty, including the following 42 members of the society:

S. P. Avann, C. R. Ballantine, J. P. Ballantine, Samuel Beatty, R. A. Beaumont, E. F. Beckenbach, A. L. Blakers, Gertrude Blanch, J. L. Brenner, Daniel Buchanan, Albert Cahn, J. W. Campbell, W. B. Caton, C. M. Cramlet, Harold Davenport, Douglas Derry, S. P. Diliberto, D. G. Duncan, N. S. Free, W. H. Gage, J. W. Green, Olaf Helmer, M. G. Humphreys, R. D. James, S. A. Jennings, D. H. Lehmer, L. H. McFarlan, B. N. Moyls, D. C. Murdock, R. E. O'Connor, T. G. Ostrom, Edmund Pinney, Peter Scherk, W. H. Simons, Fritz Steinhardt, Otto Szász, Olga Taussky-Todd, John Todd, W. L. G. Williams, Wilfrid Wilson, R. M. Winger, Clement Winston.

In the morning, there was a short session for research papers, at which Dean Daniel Buchanan presided. This was followed by the hour address, *Recent progress in the Goldbach problem*, by Professor R. D. James, of the University of British Columbia. Professor James was introduced by Professor Harold Davenport. In the afternoon, additional papers were presented in two sections, at which Professor R. M. Winger and Dean Samuel Beatty presided.

In the evening, those attending the meetings were the guests of the University of British Columbia at a dinner and social gathering at the student union. In addition, those who were fortunate enough to be able to spend several days in Vancouver were most hospitably entertained in many other ways by the University members and townspeople.

Abstracts of all papers presented at the meeting are given below. Papers read by title are indicated by the letter "*t*." Paper number 367 was presented by Mr. Todd, number 379 by Professor Jennings, and number 402 by Mr. Carter. Professor Volkoff and Mr. Carter were introduced by Professor R. D. James.

ALGEBRA AND THEORY OF NUMBERS

364*t*. Richard Bellman: *On the number of squarefrees of the form* $p+2$.

It is demonstrated that the number of squarefree numbers less than x of the form $p+2$, where p is a prime, is asymptotic to $cx/\log x$, where c is a determined constant. The method also furnishes an error term. (Received April 29, 1948.)

365. J. L. Brenner: *Equivalence of pencils of hermitian matrices under unitary transformations.* Preliminary report.

Necessary and sufficient conditions are obtained for the equivalence of pencils of hermitian matrices under unitary transformation. (Received June 18, 1948.)

366t. Harold Chatland: *On the Euclidean algorithm in quadratic number fields.*

The problem of determining whether or not the Euclidean algorithm exists in quadratic number fields has been worked out in all cases except for primes of the form $24n+1$. L. Rédei has shown that the algorithm exists in $R(73^{1/2})$, $R(97^{1/2})$. Recently H. Davenport showed that the Euclidean algorithm does not exist for quadratic fields whose discriminant exceeds 16384. In this paper, using a method due to Erdős and Ko, it is shown that for primes of the form $24n+1$ greater than 97 the algorithm does not exist in $R(P^{1/2})$ except possibly for $p=193, 241, 313, 337, 457, 601$. (Received May 14, 1948.)

367. S. D. Chowla and John Todd: *The density of reducible integers.*

An integer n is said to be reducible if the greatest prime factor of $1+n^2$ is less than $2n$. It has been shown (John Todd, Amer. Math. Monthly, in the press) that reducibility of n is a condition necessary and sufficient for the existence of a relation of the form $\arctan n=\sum f_i \arctan n_i$, where the f_i are integers and the n_i are positive integers less than n. It was noticed that the density of reducible integers (in the range $1\leqq n \leqq 5000$) was always very close to .3 . . . . No result of this kind has yet been established but it has been shown that the density of those integers n whose largest prime factor is less than $2n^{1/2}$ is $1-\log 2=.3069$ . . . . (Received May 11, 1948.)

368. Harold Davenport: *A divisor problem.*

Many of the results proved in the analytic theory of numbers in the period up to 1937 can be improved by making use of the powerful inequalities for exponential sums devised by Vinogradov. As an example, consider the problem of the error term $R(x)$ in the formula $\sum_{n\leqq x}\sigma(n)/n=(1/6)\pi^2 x-(1/2)\log x+R(x)$, where $\sigma(n)$ denotes the sum of the divisors of n. The classical estimate is $R(x)=O(\log x)$, and Walfisz (Math. Zeit. vol. 26 (1927) pp. 66–88) proved by means of Weyl's inequality that $R(x)=O[(\log x)/(\log\log x)]$. By using one of Vinogradov's inequalities, in the form given by Titchmarsh (Quart. J. Math. vol. 9 (1938) pp. 97–108), it is possible to prove that $R(x)=O(\log x)^{4/5+\epsilon}$ for any $\epsilon>0$. (Received May 17, 1948.)

369t. W. M. Gilbert: *Associative elements of quasi-groups.*

An element a of a quasi-group S is called left, center, or right associative if $a(xy)=(ax)y$, $x(ay)=(xa)y$, or $x(ya)=(xy)a$ for all x and y in S. G. N. Garrison (Ann. of Math. vol. 41 (1940) p. 479) has shown that in a commutative quasi-group left associative elements are right associative, and vice versa, and that left and right associative elements are center associative. An eighth order quasi-group is exhibited which shows that center associative elements need not be left or right associative. D. C. Murdoch has termed a quasi-group S abelian if $(ab)(cd)=(ac)(bd)$, for all a, b, c, and d in S. As a trivial extension of a lemma due to R. H. Bruck (Trans. Amer. Math. Soc. 55 (1944) p. 45), it is shown that if an abelian quasi-group contains a center associative element, or a left associative element and a right associative element, then it is an abelian group. (Received May 11, 1948.)

370*t*. Franklin Haimo: *Subgroups with division properties in Abelian groups.*

Generalizations of the serving subgroups of Prüfer (Math. Zeit. vol. 20 (1924) pp. 165–187) and of the convex subgroups of Riss (C. R. (Doklady) Akad. Sci. URSS. N.S. vol. 13 (1947) pp. 987–988) are defined. If a subgroup has a generalized Riss property and a generalized Prüfer property of the types discussed, it is also convex. If a group G contains a torsion-free cyclic subgroup with a generalized Prüfer property, then all the cyclic subgroups of G have this property. A group is torsion-free if, and only if, all of its subgroups have some generalized Riss property or the convex property; and if a group contains a torsion-free subgroup with such a property, then the group itself is torsion-free. (Received May 14, 1948.)

371*t*. J. L. Hodges and Alfred Horn: *On Maharam's conditions for measure.*

In a recent paper (Ann. of Math. vol. 48 (1947) pp. 154–167), Dorothy Maharam gave a set of necessary and sufficient conditions that a countably complete Boolean algebra have a strictly positive, countably additive measure. The purpose of this note is to point out that one of these conditions (Condition I) is a consequence of two of the others (Conditions II and IIIa). The proof uses Theorem 2 of Maharam's paper. (Received May 14, 1948.)

372*t*. Eri Jabotinsky: *The arithmetical properties of the coefficients A_n in the expansion* $(e^x-1)/(e^x-x)=\sum_0^\infty A_nx^n/n!$.

The A_n satisfy: $A_0=0$ and $1+A_{n-1}=\sum_{k=0}^{n}(C_{n,k})A_k$, and A_n is integral. Explicitly: $A_n=(n+1)!\sum_{k=0}^{n}(-n-1+k)^{(k-1)}/k!$, $n>0$. Therefore, $A_n\equiv(-1)^{n-1} \bmod n$ and for $n>0$, no A_n is 0. Also $A_n\equiv -n! \bmod (n+1)$; whence, by Wilson's theorem, $A_n\equiv 1 \bmod (n+1)$ *if and only if* $(n+1)$ is prime. If $n-m=p$ is prime, then, by Fermat's theorem, $A_n\equiv A'_m \bmod p$, where $A'_m=(m+1)!\sum_{k=0}^{m}(-m-1+k)^k/k!$. More generally, for integral $l\geqq 0$, let $A^l_m=(m+1)!\sum_{k=0}^{m}(-m-1+k)^{(l-1+k)}/k!$. The A^l_m are integral and $A^{l+1}_m=(m+1)\cdot(A^{l+1}_{p-1}-A^l_m)$ if $n-m=p$ is prime, then $A^l_n\equiv A^{l+1}_m \bmod (n-m)$. Hence $A_n=A^0_n\equiv A^l_{n-pl} \bmod p$. Examples: for $m\geqq 1$, $A_{2m}\equiv 1 \bmod 2$ and $A_{2m+1}\equiv 0 \bmod 2$, and for $m\geqq 2$, $A_{3m-1}\equiv 0 \bmod 3$, $A_{3m}\equiv(-1)^{m-1} \bmod 3$, and $A_{3m+1}\equiv -1+(-1)^{m-1} \bmod 3$. Also $\sum_{k=0}^{l}(-1)^k(C_{l,k})A^k_{l-k}=0$, so that for prime p, $\sum_{k=0}^{l}(-1)^k(C_{k,l})A_{l+k(p-1)}\equiv 0 \bmod p$. Consider the function $F(w,x)=(e^x-w)/(e^x-wx)$ and the operator $\alpha=-w(d/dw)$. Then $\alpha^lF(1,x)=\sum_{n=0}^{l}A^l_nx^n/n!$. Let $\sum b^{-n}$ designate the sum over all roots b of the equation $b=e^b$. Then $\sum b^{-n}=-A_{n-1}$, which permits to extend the definition of A_n to all real n greater than zero. (Received April 20, 1948.)

373. D. H. Lehmer: *Six multiplicative functions associated with Ramanujan's $\tau(n)$.*

Let $\sigma_k(n)$ denote the sum of the kth powers of the divisors of n and let $\tau(n)=\tau_1(n)$ be the coefficient of x^{n-1} in the expansion of the 24th power of the product $(1-x)(1-x^2)(1-x^3)\cdots$. We consider six functions $\tau_\lambda(n)$ defined for $\lambda=1,2,\cdots,6$ by $\tau_\lambda(n)=\tau_1(n)+A\sum_{m=1}^{n}\sigma_k(m)\tau_1(n-m)$ where $(A,k)=(A_\lambda,k_\lambda)$ have the six values $(0,-1)$, $(240,3)$, $(-504,5)$, $(480,7)$, $(-264,9)$ and $(-24,13)$. These functions, already noticed by Ramanujan and Hecke, are of interest in connection with a question recently raised by S. Chowla concerning the sum $\sum\sigma_1(a_1)\sigma_1(a_2)\cdots\sigma_1(a_m)$ extending over all positive integers a_i whose sum is n. Besides the functions $\sigma_k(n)$, these six are the only multiplicative functions occurring in the expression for this sum. They all have properties similar to those of Ramanujan's $\tau_1(n)$. For $n=p^a$, p a fixed prime, $\tau(n)$ form a recurring series of the second order whose scale is $x^2-\tau_\lambda(p)x+p^{k+12}$. There are numerous congruence properties. For example $\tau_\lambda(n)$ is congruent to $\sigma_{k+13}(n)$ modulo the numerator of the $(k+13)$th Bernoulli number. The Fourier series, Dirichlet series, and certain Bessel series involving $K_\nu(x)$, whose coefficients are $\tau_\lambda(n)$, all have characteristic properties. (Received May 13, 1948.)

374. B. N. Moyls: *The structure of valuations of the rational function field $K(x)$.*

Let V_0 be a rank one valuation of a field K with value group G_0 and residue class field F_0; and let V be a rank one extension of V_0 to the field $K(x)$ with value group G and residue field F. By a method due to S. MacLane (cf. Trans. Amer. Math. Soc. vol. 40 (1936) pp. 363–395, for a treatment of the case in which V_0 is discrete) it is shown that the sum of the transcendence degree $T[F/F_0]$ of F over F_0 and the rational rank of the factor group G/G_0 must be less than or equal to one. If the equality holds, F and G can be finitely generated over F_0 and G_0, respectively. If $T[F/F_0]=1$, F is a rational function field over a finite algebraic extension of F_0. In every case, F and G are at most denumerable extensions of F_0 and G_0, respectively. Conversely it is shown that extensions of V_0 to $K(x)$ with prescribed value groups and residue class fields conforming to the above restrictions can be constructed in all cases except perhaps where G/G_0 is finite and F is a finite algebraic extension of F_0. (Received May 13, 1948.)

375. Ivan Niven: *On the location of the roots of a polynomial and its derivative.*

It is proved that, for a polynomial with real roots, the average distance between the roots is not less than the corresponding average for the derivative. It has been conjectured by Paul Erdös and the writer that this result holds for any polynomial with complex coefficients. (Received May 17, 1948.)

376t. H. E. Salzer: *Table expressing every square up to one million as a sum of four non-negative tetrahedral numbers.*

The author's empirical theorem that every square integer is the sum of four non-negative tetrahedral numbers, $n(n+1)(n+2)/6$, $n\geqq 0$ (whose verification was announced previously for the first 300 squares), has now been verified as far as the first 1000 squares. A manuscript table establishing this fact is in the author's possession. Thus to disprove this conjecture would necessitate finding a square greater than 1,000,000 (that is, after the 1000th case) which requires five tetrahedral numbers. (Received April 9, 1948.)

377. W. H. Simons: *Modular functions of stufe 2.*

Convergent series are obtained for the Fourier coefficients of $k^2=\lambda(\tau)$, the fundamental modular invariant of stufe 2, and also for the reciprocal of $\lambda(\tau)$. Moreover, the series obtained for the Fourier coefficients of $\lambda(\tau)$ and $\mu(\tau)=1/\lambda(\tau)$ are subseries of that obtained by Rademacher for Klein's absolute invariant $J(\tau)$. The terms whose subscripts are of the form $4n+2$ in the series for the Fourier coefficients of $J(\tau)$ give the terms of the series for $\lambda(\tau)$, while the terms whose subscripts are of the form $4n$ give those for $\mu(\tau)$. (Received May 13, 1948.)

378. Olga Taussky: *Bounds for characteristic roots of matrices.*

A. Brauer (Duke Math. J. vol. 13 (1946)) proved that all the characteristic roots of the $n\times n$ matrix (a_{ij}) with complex elements lie inside or on the boundary of the circles with centres a_{ii} and radii $\sum_{j\neq i}|a_{ij}|$. Several generalizations are made; for example, the boundary points of the domain formed by the circles cannot, in general, be roots. If $n=2$, inner points in common to both circles cannot be roots. Results concerning multiple roots are obtained. (Received May 13, 1948.)

379. R. M. Thrall and S. A. Jennings: *Some remarks on quasi-cyclic algebras.*

In a recent paper (Bull. Amer. Math. Soc. 53 (1947) pp. 369–377) G. Hochschild has introduced the notion of quasi-cyclic algebra, and has shown that to every separable algebra B with radical $R\neq 0$ there exists a unique maximal quasi-cyclic extension. The present note is concerned with some of the properties of quasi-cyclic algebras. In particular, examples are given to show that no *minimal* quasi-cyclic extension exists in general. (Received May 15, 1948.)

ANALYSIS

380. E. F. Beckenbach: *On harmonic, subharmonic, and linear functions of several variables.*

Equalities and inequalities between area and circumferential means of positive functions are investigated. It is shown, for instance, that if $f(x, y)$ is positive and of class C'' in a domain D, then its harmonic area mean is equal to its geometric circumferential mean for all circular discs in D if and only if $f(x, y)$ is a linear function. (Received May 15, 1948.)

381*t*. Richard Bellman: *On analytic point transformations and interpolation of iterates.*

It is shown that if $f(x)=\sum a_n x^n$, $1\leq n<\infty$, is an analytic point transformation satisfying the conditions that $\sum|a_n|<1$, $a_1\neq 0$, then $f(x)$ is the value at $t=1$ of the solution of an analytic differential equation of the form $y'=\sum b_n y^n$, $1\leq n<\infty$, $y(0)=x$, provided that $|x|$ is sufficiently small. This shows that $f(x)$ and its iterates may be imbedded in a continuous semi-group of functions, $f(x, t)$, having the properties that $f(f(x, t), s)=f(x, s+t)$, and that $f(x, n)$ is the nth iterate of $f(x)$, for s and $t\geq 0$. (Received April 29, 1948.)

382*t*. Richard Bellman: *On the asymptotic behavior of solutions of linear differential equations.* Preliminary report.

The behavior of solutions of the differential equations, $a(t)y''+y'+y=0$, $y''+b(t)y'+y=0$, is studied as $t\to+\infty$, under the assumptions that $a(t)\to 0$ and $b(t)\to+\infty$ as $t\to+\infty$. It is shown that all solutions of the second differential equation do not necessarily $\to 0$ as $t\to+\infty$. (Received April 29, 1948.)

383*t*. Richard Bellman: *On the summability of solutions of linear integral equations.*

The summability of the Liouville-Neumann solution of the Fredholm integral equation is studied. It is shown that for the case of a continuous symmetric kernel, the Liouville-Neumann solution is summable to the Fredholm solution for all values of the parameter not equal to characteristic values by any summation method which sums the series $\sum z^n$ to $1/(1-z)$ for all values of $z\neq 1$. Other methods of summability are also discussed. (Received April 29, 1948.)

384. W. B. Caton: *On asymptotic formulas for Laguerre polynomials.*

In this note the general asymptotic methods of Langer are used to obtain asymptotic formulas of the Plancherel-Rotach type for Laguerre polynomials. (Received May 4, 1948.)

385*t*. Harvey Cohn: *Diophantine aspects of modular functions.* I.

The author investigates the relationship between modular functions and real irrationals by considering the behavior of certain modular functions at their singularities. This process must reveal properties of real numbers invariant under the modular group $z'=(az+b)/(cz+d)$, where a, b, c, d are integers which satisfy the condition $ad-bc=1$. For instance, when the complex variable z approaches the rational number p/q in a Stolz (angular) region, the Eisenstein series $\sum(az+b)^{-2\alpha}$ behaves, as expected from the individual terms, like const. $\cdot\,(z-p/q)^{-2\alpha}$. Likewise, this series of the Poincaré theta type: $G_{2\alpha}(z)=\sum(cz^2+(a+d)z+b)^{-2\alpha}$, summed over $ad-bc=1$, behaves like $O(z-\theta)^{-2\alpha}$ as $z\rightarrow\theta$ in a Stolz region at θ, where θ is a quadratic irrational, or more generally, any irrational number with bounded partial denominators in its simple continued fraction. It is further shown that $G_{2\alpha}(z)$ accounts for all Poincaré theta functions for which $(z-\theta)^{-2\alpha}$ is the exact order of magnitude of irrational singularities with bounded denominators, approached in any Stolz region. The method used is to express these functions in terms of the elliptic modular function and to keep transforming the plane as z approaches θ, so as to magnify the region about the point θ. (Received March 19, 1948.)

386. C. M. Cramlet: *A generalization of a theorem of Jacobi.*

A system of k contravariant vectors $A_{(i)}$, $i=1, 2, \cdots, k$, $\alpha=1, 2, \cdots, n$, is said to be Jacobian if the associated contravariant vectors $T^{\beta}_{(ij)}=A^{\alpha}_{(i)}\partial A^{\beta}_{(j)}/\partial x^{\alpha}-A^{\alpha}_{(j)}\partial A^{\beta}_{(i)}/\partial x^{\alpha}$ vanish identically. For such systems a class of *multipliers* M exist, which are relative invariants of weight one and have the property that the tensor densities $\partial(MA^{\alpha}_{(i)})/\partial x^{\alpha}$ vanish. In particular, when $k=n-1$ and λ_{α} is the vector product of the $n-1$ vectors, $M\lambda_{\alpha}$ is a gradient. The contravariant vectors determine a Jacobian system of k linear partial differential equations, the covariant vector, an associated Pfaffian with M as its integrating factor. (Received June 9, 1948.)

387*t*. R. F. Deniston: *Existence of integrals in the Pollard-Moore sense.* Preliminary report.

Let N and σ denote for integrals the norm and Pollard-Moore sense respectively, following the notation of Hildebrandt (Amer. Math. Monthly vol. 45 (1938) pp. 265–278). The Cauchy left σ-integral, to be denoted by $\sigma^{(-)}\int_a^b f dg$, has been studied for the case f bounded with only simple discontinuities and g nondecreasing in $\langle a, b\rangle$ by

Price (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 625–630). The following theorems give conditions for the existence of the specified integrals for the case of f bounded, g of bounded variation in $\langle a, b\rangle$. In Theorems 1–3 conditions given are necessary and sufficient. Theorem 1. For the existence of $\sigma^{(-)}\int_a^b f dg$: (a). In every point of the interval (a, b) in which g is discontinuous on the left the function f has a left limit, $f(x-0)$. (b). The set of the points in (a, b) which are left-sided discontinuities of the function f is a null set with respect to the continuity function of g. (A similar theorem holds for the σ-right Cauchy integral.) Theorem 2. The condition that the interval function on which $\sigma^{(-)}\int_a^b f dg$ is based be pseudo-additive in the sense of Getchell is that in every point of (a, b) in which g is discontinuous on the right the function f be continuous on the left. Getchell (Bull. Amer. Math. Soc. vol. 49 (1943) pp. 413–418) has given a theorem which together with Theorems 1 and 2 proves Theorem 3. For the existence of $N^{(-)}\int_a^b f dg$: (a′). Same as (a); (b′). The set of the points in $\langle a, b\rangle$ which are left-sided discontinuities of the function f is a null set with respect to the left-sided continuity function g_l of g. This theorem has been given by Schaerf (Portugaliae Mathematica vol. 4 (1943–1944) pp. 73–118). (Received May 14, 1948.)

388*t*. P. R. Garabedian: *Distortion of length in conformal mapping.* Preliminary report.

Let D be a domain of the z-plane of finite connectivity bounded by analytic curves C. A method is developed for finding extremal functions $f_0(z)$ minimizing the length integral $\int_C |f'(z)| |dz|$ for various different normalizations of the conformal mappings $f(z)$. The idea consists in solving extremal problems of the Schwarz lemma type in the normal, compact class of functions $F(z)$ mapping each countour of C into a countour of diameter not greater than 2. One poses such a problem in the subclass of maps on Riemann surfaces S bounded by non-concentric unit circles. A method of variation of the branch points of the extremal surface S_0 is developed which leads to a quadratic differential $R(z)dz^2$ with zeros at the branch points of S_0. Let F_0 be the map of D on S_0. Then the relation $f_0'(z)F_0'(z)dz^2 = R(z)dz^2$ yields the extremal function $f_0(z)$ for the minimum length problem. The proof of extremal properties is carried out by the method of contour integration, using the relation $R(z)dz^2 \leqq 0$ on C. Variational formulas of the Schiffer type are developed for the branch point variations, and the results are generally applicable throughout conformal mapping. Loewner's differential equation can be derived as a special case. (Received May 6, 1948.)

389*t*. M. O. Gonzalez: *On a certain class of transcendental functions.*

Let $P(x, y) = \sum_{r=0}^{n} a_{n-r,r}x^{n-r}y^r = 1$ $(a_{n,0} = 1)$ be the equation of the algebraic curve C. Let $u = 2 \cdot \text{area } OAM$, where O is the origin, $A = (1, 0)$, and $M = (x, y)$ a point on the branch of C through A. Then by definition Cos $u = x$, Sin $u = y$, Tan $u = y/x$, and so on. Some general properties of these generalized trigonometric functions are proved, in particular, the differentiation formulas $D(\text{Sin } u) = P_x/n$, $D(\text{Cos } u) = -P_y/n$, $D(\text{Tan } u) = 1/\cos^2 u = \sec^2 u$. The author considers in detail the case $P(x, y) = x^4 + 2\lambda x^2y^2 + y^4 = 1$, for which $D(\text{Sin } u) = \text{Cos } u(\text{Cos}^2 u + \lambda \text{ Sin}^2 u)$, $D(\text{Cos } u) = -\text{Sin } u(\lambda \text{ Cos}^2 u + \text{Sin}^2 u)$, thus furnishing a comprehensive account of the circular $(\lambda = 1)$, hyperbolic $(\lambda = -1)$ and elliptic functions $(|\lambda| \neq 1)$. The addition theorems, duplication formulas, and so on, and several formulas connecting Tan u with the elliptic functions of Jacobi and Weierstrass are given. For complex values of the argument Tan u is defined by the formula Tan $u = \vartheta_1(v)\vartheta_3(v)/\vartheta_4(v)\vartheta_2(v)$, $u = \pi v\vartheta_3^2$, and its zeros, poles, periods, expansions and other properties are studied. (Received May 11, 1948.)

390*t*. J. P. LaSalle: *Uniqueness theorems and successive approximations.*

Two cases where the method of successive approximations provides a sequence of functions which converge to a solution of (1) $y_i' = f_i(t, y_1, \cdots, y_n)$, $i = 1, \cdots, n$, are considered. In the first case the solution need not be unique but by starting with *suitable* zero approximations the method applies. A theorem which gives bounds on the solutions of (1) is obtained and leads to a uniqueness theorem which includes those of Osgood, Montel, and Nagumo. Under the conditions of this theorem the solution can be successively approximated. The interval of convergence and the class of zero approximations are as in Picard's classical theorem with the Lipschitz condition. This is an extension of Wintner's recent result for Osgood's uniqueness condition and of a much earlier theorem by A. Rosenblatt (1909). (Received April 27, 1948.)

391. Otto Szász: *Summation of slowly convergent series.*

Given a series $\sum_0^\infty U_n$ or the sequence of its partial sums $s_n = \sum_{\nu=0}^n U_\nu$; the author considers transforms t_n which yield more rapid convergence for certain classes of series, such as convergent series with positive or alternating terms. A case in point is the transform $t_n = s_n + anU_n$, where a is a constant. If for instance $U_n > 0$ and for some constant $b > 1$, $U_n/U_{n-1} = 1 - b/n + O(n^{-2})$, then the best choice for a is $a = 1/(b-1)$. The subclass of series with positive terms for which t_n converges more rapidly than s_n can be characterized, and $s - t_n$ can be estimated. For $a = 1$, t_n is the inverse of the $(C, 1)$ transform: $s_n = (\sum_0^n t_\nu)/(n+1)$. Discussion of other transforms and their inverses, such as Euler's method, leads to further results. Some applications to numerical series are given. (Received May 13, 1948.)

392*t*. Cengiz Uluçay: *On the values of the Bloch-Landau constants* $\mathfrak{B}$, $\mathfrak{L}$. I.

Fundamental Lemma I. Let G be the total group of the inner automorphisms g which leave fixed the unit circle $|z| < 1$. Let R be the fundamental region corresponding to the properly discontinuous subgroup $H \subset G$. Consider all the conjugate subgroups gHg^{-1}, $g \not\subset H$. Then there exists a one-to-one continuous correspondence between the totality of the elements $\{gHg^{-1}\}$ and the points M of a normal polygonal region R_τ lying in the half τ-plane of Poincaré and which is the transform of R by means of a linear substitution. R_τ may be referred to as the space of the conjugate subgroups of the given properly discontinuous subgroup H or simply the space of parameters or the functional space. It follows immediately that (i) any two such spaces are topologically equivalent, that is, homeomorphic; (ii) the space R_τ is reduced if symmetric, for example, the functional space of the group H generated by a zero-angled circular triangle T inscribed to $|z| < 1$ and its repeated symmetries with respect to the sides ad infinitum and of any of its subgroups $\overline{H} \subset H$ is the modular triangle T_τ. (Received May 11, 1948.)

393*t*. Cengiz Uluçay: *On the values of the Bloch-Landau constants* $\mathfrak{B}$, $\mathfrak{L}$. II.

Fundamental Lemma II. A theorem of uniqueness. Let $\rho w = f(z; M)$, where $|\rho|$ is a positive scale constant, be the analytic function of modular parabolic type which maps a zero-angled circular triangle T conformally on to a straight equilateral triangle $(T_w)_{\rm I}$. Normalizing the above function and calling $(L_M)_{\rm I}$ the radius of the "assumed" circle circumscribed to $(T_w)_{\rm I}$ one has (i) clearly $\mathfrak{L} \leqq (L_M)_{\rm I}$, (ii) let T_τ be the space of parameters bounded by the two positive half rays $R_\tau = 0$, $R_\tau = 1$ and the upper semicircle $|2\tau - 1| = 1$. The arcs of circle of radius unity centered at $\tau = 0$, $\tau = 1$ and the positive half ray $R_\tau = 1/2$ intersect at $\mathfrak{F} = 1/2 + i2^{(1/2)}/2$. The theorem of uniqueness is that $(L_{\mathfrak{F}})_{\rm I} < (L_M)_{\rm I}$ for all points M which do not coincide with $\mathfrak{F}$. If instead of $(T_w)_{\rm I}$ one would consider straight triangles $(T_w)_{\rm II}$ and $(T_w)_{\rm III}$ with the angles $\pi/2$, $\pi/4$, $\pi/4$ and $\pi/2$, $\pi/3$, $\pi/6$ respectively, then by Lemma I, (i) $(L_{\mathfrak{F}})_{\rm II} < (L_M)_{\rm II}$, $(L_{\mathfrak{F}})_{\rm III} < (L_M)_{\rm III}$ for all $M \neq \mathfrak{F}$, (ii) $(L_{\mathfrak{F}})_{\rm I} = 0.54326 < (L_{\mathfrak{F}})_{\rm II} < (L_{\mathfrak{F}})_{\rm III}$ (this estimate has been first obtained by Rademacher: *On the Bloch-Landau constant*, Amer. J. Math. (1943)). (Received May 11, 1948.)

394t. Cengiz Uluçay: *On the values of the Bloch-Landau constants* $\mathfrak{B}$, $\mathfrak{L}$. III.

Let $T^{l,m,n}$, where the superscripts shall be dropped whenever $l = m = n = 1$, be a circular triangle lying in $|z| < 1$ and bounded by circular arcs orthogonal to $|z| = 1$. Let $\pi/6l$, $\pi/6m$, $\pi/6n$, where l, m, n are positive integers subject to the condition $1/l + 1/m + 1/n < 6$, denote the angles of the circular triangle. Let $\rho w = f^{l,m,n}(z; M)$ be the analytic function of elliptic type which maps $T^{l,m,n}$ conformally onto $(T_w)_{\rm I}$. By Lemma I one may assume M generating the functional space T_τ. Then (i) clearly $\mathfrak{B} \leqq (B_M^{l,m,n})_{\rm I}$, (ii) every point M determines uniquely $B_M^{l,m,n}$ for every set of integers l, m, n and such that $(B_M)_I \leqq (B_M^{l,m,n})_{\rm I}$ where the equality holds if and only if $l = m = n = 1$ (the proof of this theorem requires the use of Schwarz' lemma). Moreover the correspondence between $(B_M)_{\rm I}$ and $(B_M^{l,m,n})_{\rm I}$ is monotonic increasing (by the use of Lemma I and Lemma II). Consequently $(B_{\mathfrak{F}})_{\rm I} < (B_M)_{\rm I}$ for all $M \neq \mathfrak{F}$. Similar considerations hold in the case of $(T_w)_{\rm II}$ and $(T_w)_{\rm III}$, whereby $(B_{\mathfrak{F}})_{\rm I} = 0.4719 < (B_{\mathfrak{F}})_{\rm II} < (B_{\mathfrak{F}})_{\rm III}$ (this estimate has been first obtained by Ahlfors and Grunsky: *Über die Blochsche Konstante*, Math. Zeit. (1936)). (Received May 11, 1948.)

395t. Cengiz Uluçay: *On the values of the Bloch-Landau constants* $\mathfrak{B}$, $\mathfrak{L}$. IV.

Consider the hexagon T: $GAG'EBE'FCF'$ where $G'E$, $E'F$, $F'G$ are equal arcs of the circle orthogonal to the unit circle $|z| = 1$ whereas GAG', EBE', FCF' are equal arcs of the circle belonging to the circumference of $|z| < 1$. Thus T is symmetric with respect to the axes OA, OB, OC which make with each other an angle of $2\pi/3$ and T is circumscribed to the concentric circle of radius $s_0 < s < 1$ where s_0 is the radius of the circle inscribed in the zero-angled circular triangle ABC. Then $w = z + \cdots$ is such that it maps T conformally onto a straight equilateral triangle T_w: efg, with the cuts ga, eb, fc issuing from the vertices e, f, g and extending halfway along the radii og, oe, of where o is the centre of T_w. The length of these cuts uniquely determines s which need not be calculated explicitly (in any case it is not difficult to determine a priori the corresponding value of s, in fact it is $s = 3^{1/2}/3$). It is clear that $w = z + \cdots$ cannot "assume" a circle of size greater than the circle inscribed in T_w. If $\mathfrak{L}^*$ denotes the radius of this circle than it is found that $\mathfrak{L}^* = 0.64$. If $\rho w = f(z; M)$ is the family of analytic functions of hyperbolic type regular in $|z| < 1$ and such that $w = z + \cdots$ is contained in it then $\mathfrak{L}^*$ is again uniquely determined within the family. (Received May 11, 1948.)

396*t*. Cengiz Uluçay: *On the values of the Bloch-Landau constants* $\mathfrak{B}$, $\mathfrak{L}$. V.

Let $w=f(z)=z+\cdots$ be a Bloch function $\mathfrak{C}$ where $\mathfrak{C}$ stands for $\mathfrak{B}$ or $\mathfrak{L}$ according to cases (R. M. Robinson: *Bloch functions*, Duke Math. J. (1936)). As usual let $\rho w=f(z;M)$ be the family of analytic functions regular in $|z|<1$ and containing $w=f(z)$. To each function of the family is associated its group $H'=gHg^{-1}$, $g\notin H$. The totality of these functions can thus be represented by points M in the functional space R_r which is quasi-compact or compact according as M can or cannot reach the real axis. As M varies continuously in R_r the open circle "assumed univalently" or "assumed", according to cases, by the corresponding normalized function varies also continuously. By Lemma I and II the correspondence between the size of these circles and the size of the circles $(L_M)_{\mathrm{I}}$ is monotonic increasing. With the help of this uniqueness theorem it follows readily that H is derived from a fundamental triangle and its repeated symmetries ad infinitum. Hence $\mathfrak{B}=(B_{\mathfrak{B}})_{\mathrm{I}}$, $\mathfrak{L}=(L_{\mathfrak{B}})_{\mathrm{I}}<\mathfrak{L}^*$ as conjectured by Ahlfors-Grunsky and Rademacher respectively. This last step presupposes essentially that H does not reduce to the identity element which is the case. Hence it fails in the case of a Bloch function $\mathfrak{U}$ since the latter is univalent in $|z|<1$. Finally it is conjectured that $\mathfrak{U}\geqq\mathfrak{L}^*$. (Received May 11, 1948.)

397*t*. J. G. Wendel: *A degenerate van der Pol equation*. Preliminary report.

The equation (1): $k\ddot{x}+f(x)\dot{x}+kg(x)=e(t)$, k small and positive, is considered, relative to the degenerate equation (2): $f(y)\dot{y}=e(t)$. $F(x)$ and $E(t)$ denote fixed indefinite integrals of f and e respectively; $F(x)\operatorname{sgn} x\to\infty$ as $|x|\to\infty$. If F and E are piecewise strictly monotone then solutions of (2) can be defined, which exist for all $t\geqq t_0$, but which are in general discontinuous. The two chief results are I: Solutions $x(t)$ of (1) for which $x(t_0)=x_0$, $f(x_0)>0$, $|\dot{x}(t_0)|\leqq k^{-1/2}$, tend as $k\to 0$ to the solution $y(t;x_0,t_0)$ of (2) for which $y(t_0)=x_0$; the convergence is uniform in x_0 and t if $t_0\leqq t\leqq t_0+T$ and the discontinuities of $y(t)$ have been isolated. II: If $E(t)$ is periodic with period p; if $g(x)$ is strictly monotone increasing; if x_0 and t_0 exist such that $y(t;x_0,t_0)$ has period p, $g(y(t;x_0,t_0))$ has mean value zero, and $F(x)=F(x_0)$ implies $x=x_0$—then for any δ there exists a k_0 such that for $0<k\leqq k_0$ any solution $x(t)$ of (1) satisfies $|x(t_0+np)-x_0|\leqq\delta$ for all sufficiently large n. In the equation of Cartwright and Littlewood $k\ddot{x}+(x^2-1)\dot{x}+kx=b\lambda\cos(\lambda t+\alpha)$ (J. London Math. Soc. vol. 20 (1945) pp. 180–189) the condition $b>2/3$ is necessary and sufficient for the hypotheses of II to be satisfied. (Received April 27, 1948.)

Applied Mathematics

398. J. P. Ballantine: *The electrical resistance of certain infinite nets*.

N is an infinite screen each segment of which has a resistance of 1 ohm. P and Q are two certain vertices of N. The net resistance between P and Q is then found. In another problem, N has infinite dimensions one way, and finite dimensions the other. When the finite dimension is limited to 8 segments, the computation of the resistance between P and Q is reduced to the solution of 4 quadratic equations, and 4 simultaneous linear equations in 4 unknowns. (Received May 10, 1948.)

399. Gertrude Blanch: *On the normalization of Mathieu functions*.

The solutions of period π and 2π of Mathieu's differential equation $y''+(b-s\cos^2 x)y=0$ have been widely studied. Once the characteristic values b, corresponding to the given parameter s, have been obtained, these solutions are completely determined, apart from an arbitrary constant factor which can be fixed by suitably normalizing the solutions. Two normalizations are in current use: (1) The normalization determined by setting $y(0)=1$ or $y'(0)=1$, depending on whether the periodic solution is different from zero or not, at $x=0$. (2) The normalization determined by setting the integral from zero to 2π of y^2, with respect to x, equal to π. It is shown that although the normalization (2) has one desirable property, namely that the solution ranges numerically between 0 and 10 practically everywhere, these solutions cannot now be readily determined for large values of s, when the trigonometric series becomes too cumbersome to use. Moreover, the proponents of the normalization (2) have failed to point out that for large s, the desirable range of the solution does not necessarily give any useful information about the *order of magnitude* of the solutions. In contrast to this, the solutions as normalized according to (1) can all be determined asymptotically with relative ease when s is large, and their order of magnitude can be obtained everywhere from the asymptotic expansions. It would therefore seem that with the current knowledge about the asymptotic developments for these solutions, the normalization (1) is the better one to adopt. (Received May 13, 1948.)

400. Albert Cahn: *Some formulae for representing* $\int_0^x J_n(t)dt$ *and* $\int_0^x Y_n(t)dt$.

From elementary properties of Bessel functions follow the relationships: $\int_x^\infty C_\nu t^{-q}dt = C_{\nu-1}x^{-q}+(\nu-q-1)\int_x^\infty C_{\nu-1}t^{-(q+1)}dt$; $\int_x^\infty C_\nu t^{-q}dt=(\nu+q-1)^{-1}\left[C_\nu x^{-(q-1)}+\int_x^\infty C_{\nu-1}t^{-(q-1)}dt\right]$; where C_ν represents either J_ν or Y_ν. These equations may be used to obtain a representation of $\int_x^\infty C_n(t)dt$ in terms of Bessel functions of order less than n and the integral $\int_0^x C_0(t)dt$. Such representations are useful in checking the results of numerical integrations of Bessel functions. (Received May 15, 1948.)

401. Edmund Pinney: *The floating elastic sphere.*

The stresses and displacements are obtained for an isotropic elastic sphere floating in a fluid and acted upon by the force of gravity. In the particular case in which the sphere is completely submerged, the answer is in closed form. In this case the sphere is compressed and slightly flattened in the vertical direction. (Received May 17, 1948.)

402. G. M. Volkoff and D. S. Carter: *On the kinetic theory of the viscosity of gases.*

The traditional theory of Maxwell for the transport of momentum in a gas as usually presented in standard texts (cf. Page, *Introduction to theoretical physics*) fails to account for the fact that the stress tensor is symmetric. An alternative elementary kinetic theory derivation of the coefficient of viscosity is given which yields the usual value and, at the same time, demonstrates by a purely geometric argument that the stress tensor is symmetric. (Received May 11, 1948.)

GEOMETRY

403t. W. L. Chow: *On the genus of curves of an algebraic system.*

A simple, purely algebraic proof is given for the well known theorem that the genus of any curve of an algebraic system of algebraic curves is not greater than the genus of the generic curve of the system. The method is based on the idea of the associated form of an algebraic variety, which was introduced by van der Waerden and the author some years ago. (Received May 10, 1948.)

404t. David Ellis: *Superposability properties of metric groups.*

Let G be an additively written Abelian group and denote by $|G|$ the set of pairs $(a, -a)$ of inverse elements of G. A metric group in the sense of Menger is obtained by attaching to each order pair a, b of elements of G the element $(b-a, a-b)$ of $|G|$ as distance. In the paper introducing this notion (Math. Zeit. vol. 33 (1931) pp. 396–418) Menger showed that a metric group has two-point superposability; that is, corresponding to each two congruent pairs of elements of G there is a congruence of G with itself that maps one pair of elements onto the other. In this note two-point superposability is given a two-fold extension. It is shown that (1) any two congruent subsets of a metric group are superposable, and (2) any congruence between any two subsets is extendable to the entire group. (Received March 30, 1948.)

405. J. W. Green: *On the centroids of convex areas.*

The region of variation of the centroid of a convex area in the plane is computed, where the convex area is subjected to certain conditions; for example, if the area is required to contain a line segment as boundary and to have prescribed supporting lines at the end of the segment. Several applications are made, and the results include the following: If lines are drawn through the centroid of a convex area dividing it in two areas, the locus of the centroids of the areas is a convex curve. (Received May 15, 1948.)

406t. C. C. Hsiung: *Rectilinear congruences.*

In his prize memoir (*Sur la théorie générale des congruences*, Mémoires publiés par la classes des sciences de l'Académie Royale de Belgique (2) vol. 3 (1911)) Wilczynski has established the theory of a rectilinear congruence in ordinary three-dimensional projective space by using a system of linear partial differential equations. However, his method of deriving the system of the differential equations is not completely geometric. The author proposes to remedy this lack of geometric content in the present paper. By purely geometric determinations, a completely integrable system of linear homogeneous partial differential equations in canonical form is introduced, defining a rectilinear congruence $\mathcal{L}$ in ordinary space except for a projective transformation. Some invariants and covariants of these equations and a completely invariant coordinate system are obtained. The local power series expansions for the two focal surfaces and the curves of the two focal nets of the congruence $\mathcal{L}$ are calculated and expressed in terms of the invariants. The author also studies some covariant congruences associated with the congruence $\mathcal{L}$ and presents simple geometric characterizations of W congruences, Segre-Darboux congruences and some other special congruences. (Received May 14, 1948.)

407. Peter Scherk: *On the minimal spaces of curves of order $n+1$ in projective n-space.* Preliminary report.

A K^{n+1} is a differentiable closed curve in real projective n-space that has a maximum number of $n+1$ points in common with any hyperplane. If an m-space is spanned by osculating spaces it is called normal [abundant] if it meets the K^{n+1} $(m+1)$ times [$(m+2)$ times]. A minimal space is an abundant space that contains no other abundant subspaces. The following results are proved: (1) Each m-space contains at most one minimal subspace. A space spanned by osculating spaces contains a minimal subspace if and only if it is abundant [$m<n$]. (2) Suppose the m-space E is normal and the point s lies on the K^{n+1} but not on E. Then the projection of s from E is q-times singular if and only if either s is $(q+m+1)$-times singular or if E meets the osculating $(n-m-q)$-space but not the osculating $(n-m-q-1)$-space of s [$n\geqq 2$; $0\leqq m\leqq n-2$; $1\leqq q\leqq n-m-1$]. (Received May 10, 1948.)

408*t*. Peter Scherk: *On surfaces of constant curvature.*

By a well known theorem, the Gaussian curvature of a surface is constant if all the geodesic distance circles on the surface have constant geodesic curvatures (cf., for example, Blaschke: *Vorlesungen über Differentialgeometrie*, vol. 1, 3d ed., 1930, pp. 154–155). This theorem is a corollary of the following observation: Suppose that the geodesic distance circles about one point have constant geodesic curvatures. Then the Gaussian curvature at a point on these circles depends only on its geodesic distance from the first point. So the following stronger result may be stated: Suppose there exists a finite or infinite set of points on the surface so that the geodesic curvatures of the geodesic distance circles about these points are constant, and suppose any two points of the surface can be joined by a polygon whose sides are arcs of geodesic distance circles about points of the set. Then every geodesic distance circle has constant geodesic curvature and the Gaussian curvature of the surface is constant. (Received May 10, 1948.)

409*t*. Peter Scherk: *On the minimal osculating spaces of curves of order $n+1$ in projective n-space.* I. Preliminary report.

The minimal spaces of a closed curve K^{n+1} of order $n+1$ in real projective n-space are defined in abstract 54-9-407. Let $N^n_{p_1,\dots,p_j}$ denote the number of minimal spaces through j different points of the curve that contain the osculating p_1-space of one of them, the osculating p_2-space of another, etc. Put $A_n=\sum_0^{n-1}(n-p)N^n_p$, $B_n=\sum_1^{n-2}(n-p-1)N^n_{p0}$. The author conjectures $A_n+B_n\leqq n+1$. He proved before $A_n\leqq n+1$; $A_n+B_n=O(n^2)$ [Ann. of Math. vol. 46 (1945) pp. 68–82, 175–181]. He now proves (1) $2A_n+B_n\leqq 2(n+1)$, (2) N^n_{p11} is bounded for a given n [$0\leqq p\leqq n-4$]; N^n_{p10} is finite [$0\leqq p\leqq n-5$]. (3) If $N^n_{p0}=0$ for $p\not\equiv n \pmod 2$, then $N^n_{n-1,1}=\sum_1^{n-2}N^n_{p0}$ [$n>4$]. (4) The K^{n+1} with $N^n_{10}+N^n_{000}>0$ are classified. (Received May 10, 1948.)

LOGIC AND FOUNDATIONS

410*t*. A. P. Morse: *Squares are normal.*

Two plane sets are *finitely equivalent* if and only if they can be split respectively into sets $a_1, a_2, \cdots, a_n$ and into sets $a_1', a_2', \cdots, a_n'$ in such a way that the corresponding subdivisions are congruent. A plane set S is *paradoxical* if and only if it can be split into two sets each of which is finitely equivalent to S. A plane set which is not paradoxical is *normal*. It has been known for some time that squares and a variety of other sets are normal. However, all known verifications of the normality of squares depend in an essential way on the axiom of choice. In the present paper, which will appear in Fundamenta Mathematicae, it is found that it is indeed possible to show, without the axiom of choice, that any bounded plane set with inner points is normal. (Received April 21, 1948.)

STATISTICS AND PROBABILITY

411*t*. Waclaw Kozakiewicz: *On the necessary and sufficient conditions for the convergence of a sequence of moment generating functions.* Preliminary report.

Let $\{F_n(x_1, x_2)\}$ be a sequence of distribution functions, and let $\{\phi_n(t_1, t_2)\}$ be the corresponding sequence of moment generating functions all of which are assumed to exist for $|t_i| < \alpha_i$, $\alpha_i > 0$ $(i = 1, 2)$. Assuming $s_i \geqq 0$ $(i = 1, 2)$, let $M_n(s_1, s_2) = \iint_R dF_n(x_1, x_2)$, where R is the region (dependent on s_1, s_2) for which the inequalities $|x_i| \geqq s_i$ $(i = 1, 2)$ are satisfied. Let $M(s_1, s_2)$ be the least upper bound of the sequence $\{M_n(s_1, s_2)\}$. It may be shown that the necessary and sufficient conditions for the convergence of the sequence $\{\phi_n(t_1, t_2)\}$ for $|t_i| < \alpha_i$ $(i = 1, 2)$ are: (a) for any fixed t_i satisfying the inequality $|t_i| < \alpha_i$ $(i = 1, 2)$, there exists a number K (dependent only on t_1 and t_2) such that $M(s_1, s_2) \leqq K \exp(-|t_1|s_1 - |t_2|s_2)$ for all $s_i \geqq 0$ $(i = 1, 2)$; (b) $\{F_n(x_1, x_2)\}$ converges to a distribution function $F(x_1, x_2)$ at each point of continuity of $F(x_1, x_2)$. Further, it may be proved that when the conditions (a) and (b) are satisfied, the moment generating function of $F(x_1, x_2)$ exists for $|t_i| < \alpha_i$ $(i = 1, 2)$, and is equal to the limit of the sequence $\{\phi_n(t_1, t_2)\}$ for $|t_i| < \alpha_i$ $(i = 1, 2)$. All results can be easily extended to the case when distribution functions and moment generating functions are defined in the Euclidean space of any finite number of dimensions. (Received May 14, 1948.)

TOPOLOGY

412*t*. R. D. Anderson: *Concerning upper semi-continuous collections of continua.*

The author proves that if M is any compact continuous curve and ϵ is any positive number, there exists a finite collection G of continuous curves filling up M each of diameter less than ϵ such that if H is any subcollection of G the common part of all the continua of H, if it exists, is the sum of a finite number of continuous curves. With the aid of this result, he shows that if M is any locally compact completely separable metric *continuous curve* there exist a one-dimensional *continuous curve* K in three-dimensional Euclidean space and an upper semi-continuous collection of mutually exclusive compact continua filling up K which with respect to its elements as points is topologically equivalent to M. (Received May 4, 1948.)

413. A. L. Blakers: *Obstructions to deformations of mappings.*

Let K be a simplicial complex, X and A arcwise connected topological spaces, with $A \subset X$. Assume also that the fundamental group $\pi_1(A)$ operates simply on the relative homotopy group $\pi_n(X, A)$. Let f be a mapping $f: K \rightarrow X$ which maps the n-skeleton $K^{(n)}$ into A. Let σ^{n+1} be an oriented $(n+1)$-simplex of K. Then the mapping $f|\sigma^{n+1}: (\sigma^{n+1}, \dot{\sigma}^{n+1}) \rightarrow (X, A)$ determines uniquely an element $\alpha \in \pi_{n+1}(X, A)$, and the function defined by $c^{n+1}(f; \pi_{n+1}(X, A))(\sigma^{n+1}) = \alpha$ is an $(n+1)$-cochain of K with coefficients in $\pi_{n+1}(X, A)$. It is shown by homotopy addition theorems that $c^{n+1}(f; \pi_{n+1}(X, A))$ is a cocycle and is trivial if and only if the mapping f can be deformed by a homotopy which is constant over $K^{(n)}$ to a mapping which takes $K^{(n+1)}$ into A. It is also shown that $c^{n+1}(f; \pi_{n+1}(X, A))$ is a coboundary if and only if f can be deformed by a homotopy which is constant over $K^{(n-1)}$ to a mapping which takes $K^{(n+1)}$ into A. Thus the cocycle $c^{n+1}(f; \pi_{n+1}(X, A))$ is related to deformations of mappings in the same way that the obstruction cocycle of Eilenberg (Ann. of Math. vol. 41 (1940) pp. 231–251) is related to extensions of mappings. (Received April 27, 1948.)

414. S. P. Diliberto: *Poincaré's geometric theorem.* Preliminary report.

The writer gives a new proof of Poincaré's geometric theorem which simplifies Birkhoff's proof (*An extension* · · · , Acta Math. vol. 47 (1926)), and which removes the requirement that one boundary be a circle and the other star shaped: *If A is a closed annular region bounded by a pair of nested, simple, Jordan curves C_i $(i=1, 2)$ and T an area preserving homeomorphism of A onto itself, the C_i having rotation numbers of opposite sign, then T must have at least two distinct fixed points.* A is replaced by A^*, its universal covering space: an infinite strip bounded by a pair of parallel lines, and T by a periodic map T^* of A^* onto itself. The mapping vectors of T^* are shown to have a singular point of index one by a device using Brouwer's translation arcs (Kerekjarto, Acta Univ. Szeged. vol. 4 (1928)). The second fixed point follows from the index sum theorem. (Received May 17, 1948.)

415*t*. M. H. A. Newman: *Local connection in locally compact spaces.*

It was proved by Hurewicz in 1935 (Fund. Math. vol. 25 (1935) pp. 467–485) that a compact space which is both LC^1 and lc^n is LC^n. In the present paper the corresponding result for locally compact spaces is proved, (a) for uniform local connection, and (b) for relative local connection, thereby settling affirmatively a question posed by Eilenberg and Wilder (Amer. J. Math. vol. 64 (1942) pp. 613–622). The main lemma used by Hurewicz, on the passage from ϵ-homotopy to true homotopy, cannot be carried over to locally compact spaces without substantial modification. A stronger form of the lc^p and LC^p conditions is therefore used, namely (for lc^p) the existence of a function $\phi(\delta, x)$ such that, given a compact set F in the neighborhood $U(x, \phi)(\delta, x))$ of any point x, there is a compact subset F' of $U(x, \delta)$ such that every q-cycle in F bounds in F', for $0 \leqq q \leqq p$; and analogously for LC^p. These conditions are proved equivalent to the usual definitions in locally compact (metric) spaces. (Received May 12, 1948.)

J. W. GREEN,
Associate Secretary

SECOND SYMPOSIUM IN APPLIED MATHEMATICS

The Second Annual Symposium in Applied Mathematics of the American Mathematical Society was held at The Massachusetts Institute of Technology, Cambridge, Massachusetts, Thursday to Saturday, July 29–31, 1948. The subject of the Symposium was *Electromagnetic theory*. The following 232 persons, including 133 members of the Society, registered:

J. C. Abbott, Charles Adams, P. L. Alger, W. P. Allis, Willis Altar, W. S. Ament, Richard Arnowitt, N. N. Aronszajn, H. A. Baerwald, E. W. Banhagel, Shepard Bartnoff, P. F. Bartunek, R. H. Battin, R. E. Beam, E. G. Begle, D. L. Benedict, H. S. Bennett, W. R. Bennett, Stefan Bergman, C. A. Bergren, Madeleine Berthaud, E. W. Beth, M. L. Boas, R. P. Boas, H. W. Bode, E. M. Boggs, F. W. Boggs, F. E. Bothwell, George Bott, J. G. Brainerd, F. R. Britton, H. K. Brown, C. T. Bumer, E. R. Caianiello, E. B. Callahan, P. G. Carlson, J. W. Carr, G. F. Carrier, Nicholas Chako, L. G. Chelius, Jack Chernick, Peter Chiarulli, R. A. Chipman, R. V. Churchill, S. B. Cohn, Robert Corbó, S. H. Crandall, P. D. Crout, E. H. Cutler, P. B. Daitch, D. A Darling, H. C. Davis, W. S. Dawkins, J. J. Devaney, L. T. Devore, J. B. Díaz, R. D. Douglass, R. J. Duffin, J. E. Eaton, F. E. Ehlers, H. J. Ettlinger, B. G. Farley, Eugene Feenberg, Louis Fein, Herman Feshbach, Morton Finston, W. T. Fishback, F. G. Fisher, Alan Fletcher, R. M. Foster, F. H. Fowler, A. H. Fox, Philip Franklin, J. C. Freeman, E. J. Frey, D. L. Fuller, T. S. George, H. A. Giddings, Peter Gilmer, H. E. Goheen, Lawrence Goldmuntz, C. H. Gordon, W. M. Gottschalk, R. E. Graves, L. F. Green, Harry Greenberg, F. T. Haddock, M. E. Haller, J. M. Ham, L. B. Ham, W. H. Hamilton, H. V. Hance, Duncan Harkin, G. G. Harvey, C. M. Hebbert, A. E. Heins, M. J. Herzberger, F. B. Hildebrand, J. A. Hofmann, C. W. Horton, B. E. Howard, G. B. Huff, L. C. Hutchinson, M. D. Indjoudjian, Leopold Infeld, W. J. Jacobi, Jørgen Jensen, Miroslav Joachim, Mark Kac, Robert Kahal, M. L. Kales, E. L. Kaplan, Meyer Karlin, E. R. Keown, T. J. Killian, J. R. Kline, Walter Kohn, S. H. Lachenbruch, J. H. Laning, J. P. LaSalle, Y. W. Lee, Alfred Leitner, D. C. Lewis, C. C. Lin, A. N. Lowan, R. B. Lowe, S. C. Lowell, Jerome Lurye, Ingo Maddaus, P. M. Marcus, A. E. Marston, W. T. Martin, L. L. Merrill, David Middleton, M. G. Morgan, S. P. Morgan, H. H. Mostafa, C. C. Nash, Philip Newman, A. A. Oliner, C. H. Papas, N. G. Parke, Philip Parzen, E. K. Paxton, C. L. Pekeris, A. J. Perlis, Arnold Peterson, Hillel Poritsky, R. E. Porter, M. H. Protter, E. G. Ramberg, Robert Rawhouser, C. J. Rees, C. F. Rehberg, Edgar Reich, Eric Reissner, Daniel Resch, R. R. Reynolds, S. O. Rice, J. K. Riess, W. L. Root, J. H. Rosenbloom, V. H. Rumsey, J. P. Russell, Charles Saltzer, M. M. Schiffer, Harold Schutz, H. M. Schwartz, J. F. Scully, C. H. W. Sedgewick, W. W. Seifert, Samuel Sensiper, H. S. Sharp, P. C. Shedd, Herbert Sherman, S. S. Shu, K. M. Siegel, D. N. Silver, T. M. Simpson, George Sinclair, David Slepian, B. D. Smith, J. J. Smith, A. B. Soble, William Sollfrey, R. D. Spence, George Springer, W. D. Stahlman, C. W. Steeg, A. F. Stevenson, S. W. Stewart, J. A. Stratton, D. J. Struik, W. L. Sullivan, W. H. Surber, L. W. Swanson, J. L. Synge, C. T. Tai, A. H. Taub, J. G. C. Templeton, G. B. Thomas, D. L. Thomsen, H. I. Treiber, H. M. Trent, W. J. Trjitzinsky, Rohn Truell, J. W. Tukey, W. G. Tuller, L. A. Van Dyke, A. H. Van Tuyl, A. D. Wallace, Henry Wallman, S. R. Warren, W. H. Watson, G. C. Webber, Herschel Weil, W. G. Welchman, David Wellinger, C. P. Wells, L. R. White, R. M. Whitmer, Norbert

Wiener, J. E. Wilkins, Henry Wolf, R. A. Woodson, A. W. Wundheiler, W. J. Yost, L. A. Zadeh, M. E. Zaret, D. E. Zilmer, J. W. Zimmer, F. J. Zucker.

The dormitories of The Massachusetts Institute of Technology and the cafeteria and Litchfield Lounge of Walker Memorial Building were available to those attending the Symposium and to their families and their guests.

The Symposium was cosponsored by the American Institute of Electrical Engineers, the American Institute of Physics, and the Institute of Radio Engineers.

Professor J. L. Synge, Chairman of the Committee on Applied Mathematics, presided at the opening session and gave a brief review of the plans for Symposia in Applied Mathematics.

Tea was served by the ladies of the Department of Mathematics of Massachusetts Institute of Technology on Thursday afternoon in Walker Memorial Building.

There were six forty-minute addresses and eleven twenty-minute addresses, all presented by invitation of the Committee on Program and Arrangements. The Symposium was divided into three morning and two afternoon sessions and one evening session. Each paper was followed by a discussion. Owing to the illness of the authors, the paper by Professor R. K. Luneburg of New York University, entitled *Asymptotic development of steady solutions of Maxwell's equations based upon discontinuities of pulse solutions*, and the paper by Dr. C. E. Shannon of Bell Telephone Laboratories, entitled *Theory of the transmission of information*, could not be presented.

The seventeen papers were presented in the following order:

1. S. O. Rice, Bell Telephone Laboratories: *Reflections from bends and corners in electromagnetic wave guides.* (40 minutes)
2. A. E. Heins, Carnegie Institute of Technology: *Systems of simultaneous Wiener-Hopf integral equations and their application to some boundary value problems in electromagnetic theory.* (20 minutes)
3. J. L. Synge, Carnegie Institute of Technology: *Electromagnetism without metric.* (40 minutes)
4. E. G. Ramberg, Radio Corporation of America: *Aberration correction with electron mirrors.* (20 minutes)
5. A. F. Stevenson, University of Toronto: *Wave propagation in electromagnetic horns.* (20 minutes)
6. Leopold Infeld, University of Toronto: *The factorisation method and its application to differential equations in theoretical physics.* (40 minutes)
7. Eugene Feenberg, Washington University: *A method of analyti-*

cal continuation in the eigen-value and scattering problems of quantum theory. (20 minutes)

8. Rohn Truell, Brown University: *Problems related to measuring the field strength of high frequency electromagnetic fields.* (20 minutes)

9. Herman Feshbach, Massachusetts Institute of Technology: *The new quantum electrodynamics.* (40 minutes)

10. A. H. Taub, Institute for Advanced Study: *Orbits of charged particles.* (20 minutes)

11. C. L. Pekeris, Weizmann Institute, Rehovoth, Palestine, and Institute for Advanced Study: *Ray theory versus normal mode theory in wave propagation problems.* (20 minutes)

12. R. J. Duffin, Carnegie Institute of Technology: *Non-linear networks.* (20 minutes)

13. Mark Kac, Cornell University: *Distribution problems in the theory of random noise.* (40 minutes)

14. W. H. Watson, National Research Council of Canada: *Discontinuity in electromagnetism.* (20 minutes)

15. Henry Wallman, Massachusetts Institute of Technology: *Transient response and the central limit theory of probability.* (20 minutes)

16. Norbert Wiener, Massachusetts Institute of Technology: *Entropy information.* (40 minutes)

17. Y. W. Lee, Massachusetts Institute of Technology: *The statistical theory of message transmission.* (20 minutes)

The presiding officers at the sessions were, respectively: Professors J. L. Synge, J. A. Stratton, P. D. Crout, R. V. Churchill, Dr. H. W. Bode, and Professor R. M. Foster.

At the close of the Saturday afternoon session, a resolution of thanks and appreciation to Massachusetts Institute of Technology for its hospitality was read by Professor J. L. Synge and unanimously approved.

The addresses and papers presented at the Symposium are to be published by the Society in a volume of the Proceedings of the Second Symposium in Applied Mathematics.

W. T. MARTIN
Chairman of the Committee on Program and Arrangements
T. R. HOLLCROFT
Associate Secretary of the Society

THE LOS ANGELES MEETING ON COMPUTING MACHINES

The Institute for Numerical Analysis of the National Bureau of Standards and the Departments of Astronomy, Engineering, and Mathematics of the University of California, Los Angeles, sponsored on July 29, 30, and 31, 1948, at the University of California, Los Angeles, a series of symposia on modern calculating machinery and numerical methods. The American Mathematical Society was included among the numerous cooperating scientific and learned societies. The attendance at the symposia was approximately five hundred.

The principal address, *Electronic methods of computation*, was given by Professor John von Neumann, of the Institute for Advanced Study. A list of sessions on special topics, together with the papers presented to each, follows.

Present status of computing machinery developments, J. H. Curtiss, Chairman. *General survey of current British developments*, D. R. Hartree; *General survey of current American developments*, Perry Crawford.

Progress reports from principal academic research centers, Paul Morton, Chairman. *The electric analog computer*, G. D. McCann; *Recent developments at the Harvard Computation Laboratory*, H. H. Aiken; *Project Whirlwind at the Massachusetts Institute of Technology*, J. W. Forrester; *Recent developments at project EDVAC*, R. L. Snyder; *Recent developments at the Institute for Advanced Study*, H. H. Goldstine; *Recent developments at the Illinois Institute of Technology*, T. J. Higgins.

Progress reports from principal commercial research laboratories, N. E. Edelfsen, Chairman. *Recent developments—UNIVAC*, J. W. Mauchly; *Recent developments—REEVAC*, H. J. Zagar; *Recent developments—E.R.A.*, C. B. Tompkins; *Recent developments in electronic computers—I.B.M.*, R. R. Seeber; *Recent developments, Bell Telephone Laboratories*, Brockway McMillan; *Recent developments—Raytheon Laboratories*, R. V. D. Campbell.

Programming for automatic digital computing machinery, John Todd, Chairman. *Programming for the Dahlgren machine*, C. C. Bramble; *Programming for the Aberdeen machine*, Franz Alt; *Programming for the I.B.M. Selective Sequence Electronic Calculator*, R. R. Seeber; *Programming for machines under development*, H. D. Huskey; *Programming for machines under construction*, Ida Rhodes.

The future of numerical analysis, E. F. Beckenbach, Chairman. *Some unsolved problems in numerical analysis*, D. R. Hartree; *Numer-*

ical methods in pure mathematics, I, D. H. Lehmer; *Numerical methods in pure mathematics*, II, Hans Rademacher; *Problems in probability and combinatorial analysis*, S. M. Ulam.

Numerical analysis in applied mathematics. J. L. Barnes, Chairman. *Numerical calculations in non-linear mechanics*, Solomon Lefschetz; *Programming in a linear structure*, G. B. Dantzig; *Wave propagation in hydrodynamics and electrodynamics*, Bernard Friedman; *Eigenvalues and eigenvectors for symmetric matrices*, H. H. Goldstine.

Reports of a number of the symposia and individual papers are to be published. Enquiries and requests for copies may be addressed to the Institute for Numerical Analysis, National Bureau of Standards, University of California, Los Angeles 24, California

J. W. GREEN,
Associate Secretary

DUNHAM JACKSON
1888–1946

WILLIAM L. HART

1. Introduction. My first meeting with Dunham Jackson occurred in September, 1916, at Cambridge, Massachusetts, on the day when I arrived to begin an interesting period as a Benjamin Peirce Instructor in the Department of Mathematics at Harvard University. At that time, Jackson was one of the youngest permanent members of Harvard's mathematical staff. I recall that he immediately made me feel as if we were old friends, and I never lost that reaction to him. Later, he and I served together briefly in the United States Army, and thereafter we were colleagues in the Department of Mathematics at the University of Minnesota from 1919 until his death. I am confident that my warm personal feelings for the man were typical of those held by all members of the Society who knew him intimately.

2. Biographical remarks. Dunham Jackson was born on July 24, 1888, son of William Dunham Jackson and Mary Vose Jackson, in Bridgewater, Massachusetts, where his father was a teacher in a normal school. He attended the elementary and secondary schools in Bridgewater and entered Harvard University in 1904, where he obtained his bachelor's degree in 1908 and his master's degree in mathematics in 1909. At the various stages in his academic training at Harvard University, he received all possible honors and scholarships due to the excellence of his work. In 1909, he was awarded a Sheldon Fellowship from Harvard University for graduate study in Europe, where he attended the University of Göttingen in 1909–1911, and the University of Bonn, briefly, in 1911. He received his Ph.D. in mathematics at Göttingen in 1911, with Professor Landau as his adviser. He returned to Harvard University as an instructor in mathematics in 1911 and became an assistant professor there in 1916. He performed valuable mathematical work for the U. S. Army during 1918–1919 as a captain in the Ordnance Department, where he served in the Ballistic Unit organized in Washington, D. C. by Dr. F. R. Moulton (then a major in the Ordnance Department). On June 20, 1918, Jackson married Harriet Spratt Hulley, whom he had met in 1917 while she was a graduate student in English at Radcliffe College. They had two daughters, Anne Hulley and Mary Eloise (Mrs. William J. Thorpe). He went to the University of Minnesota as a professor of mathematics in 1919 and held that position until his death on

November 6, 1946, at the age of 58.

Jackson experienced a serious heart attack early in 1940. Thereafter, although he was able to return to duty for various periods at the University of Minnesota, his health steadily deteriorated. In 1940, and after 1943, due to recurring illness he was frequently confined for long periods to his room either at home or in a hospital, and this was necessary continuously during the last 18 months of his life. Nevertheless, during his last year, he gave approval to four Ph.D. dissertations, the final one only two weeks before his death; also, a research paper which he finished in 1946 has been published posthumously. It was evident that, in spite of his long and trying illness, he maintained creative ability, interest in the progress of his advanced students and the affairs of the Society, and normal reactions to his family and friends to the end of his life.

The special abilities possessed by Jackson and his many services to the field of mathematics were appropriately recognized during his lifetime. He was a member of the Council of the Society from 1918 to 1920, Vice-President in 1921, and a member of the Transactions Editorial Committee from 1926 to 1931. He gave the colloquium lectures at the Summer Meeting of the Society in 1925. He was a charter member of the Mathematical Association of America, a member of its Board of Governors from 1923 to 1929, Vice-President in 1924 and 1925, and President in 1926; he was awarded the Chauvenet Prize of the Association for the period 1932–1934 in 1935 (see §4). He was a Vice-President of the American Association for the Advancement of Science in 1927, and Secretary of Section A (Mathematics) of that Association for many years. He was a fellow of the American Academy of Arts and Sciences and a member of the National Academy of Sciences. He always maintained lively interest in mathematical affairs at the secondary level, and frequently played an active role in joint actions of the fields of college and secondary mathematics.

3. Personal qualities. Jackson was a man of high ideals, extremely unselfish, and very conscious of his responsibilities, not only in strictly personal affairs but also with respect to his profession as an educator and his duties as a citizen. He was intensely devoted to the field of mathematics, had thought deeply of its place in advancing human welfare, and was fluent in expressing his convictions on such matters. As an outstanding characteristic, he enjoyed his associations with people. In particular, his congenial wife and family were a continuing source of pleasure and inspiration to him. It was typical of the man that, after relatively few contacts, he was likely to call any new acquaintance by his given name as a sincere implication of permanent regard. Until his final illness restricted his activities, he was a regular attendant at meetings of the Society, where his cheerful manner and cordiality, in addition to his active interest in the mathematical programs, made him a great asset. One of his major regrets during his last few years was his inability to attend the meetings of the Society to renew old friendships and to make new ones. He always preferred to have a substantial part of his teaching schedule at the undergraduate level because he enjoyed personal contacts with young people and the opportunity to influence them scientifically at an early age. His students will remember him as a skillful and inspiring teacher who also was a sincere friend. His former associates in various activities will recall him not only because of his scholarly attainments and his ability as a counselor, but also because of his warm personality, his lively sense of humor, and the breadth of his general interests. In his untimely death, the field of mathematics experienced a very great loss.

4. **Preliminary summary of Jackson's mathematical work.** He presented his first paper to the Society in September, 1909, before going to Göttingen for his final graduate study; this paper [1][1] dealt with a topic in algebra, but practically his entire remaining research was in the field of analysis. His last paper [63] was presented to the Society posthumously on December 26, 1946. In all, he published approximately 75 mathematical papers involving novelty in the results, or in the methods and organization. He was the author of two books, his colloquium lectures [38] and a volume [58] in the series of Carus Monographs. Also, he published various articles of general mathematical interest intended for teachers at the secondary or college level, or for the general public. While he was in the Army, he wrote a textbook in pamphlet form for the Ordnance Department on the method of numerical integration as applied in exterior ballistics. Jackson was noted for his ability as an expositor of mathematics in his published work, in lectures, and in teaching at all levels. The excellence of his expository papers [42], [45], and [46], on series of orthogonal polynomials, orthogonal trigonometric sums, and the convergence of Fourier series, was the basis for the award to him of the Chauvenet Prize of the Mathematical Association of America for the period 1932–1934.

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[1] Numbers in brackets refer to the bibliography of Jackson's publications at the end of the article.

Jackson's choice of a subject for his Göttingen dissertation permanently channeled his main research into the field of approximation theory and orthogonal functions, which he enriched by many new results and outstanding expositions of fundamental content. In the preface to his colloquium lectures [38], he made the following remarks concerning this choice on his part.

"Guided partly by natural inclinations, perhaps, and partly by recollection of a course on methods of approximation which I had taken with Professor Bôcher a few years earlier, I committed myself to one of the topics which Landau had proposed (for a thesis), an investigation of the degree of approximation with which a given continuous function can be represented by a polynomial of given degree.[2] When I reported my choice he said meditatively . . . *Das ist ein schönes Thema, ich beneide Sie um das Thema . . . Nein, ich beneide Sie nicht, aber es ist ein wunderschönes Thema.*"

5. **Miscellaneous research.** Before discussing the nature of Jackson's contributions in his major field, it is pertinent to mention that his published papers exhibit a reasonable diversity of content in other directions. Thus, for a few years while he was a member of the faculty of Harvard University, and occasionally later, he turned his attention to boundary value problems. One extensive paper [9] dealt with an ordinary linear differential equation and associated irregular boundary conditions, where he established the divergence of corresponding expansions of functions in series of characteristic functions, in contrast to convergence for the case of regular conditions. He discussed a system of boundary conditions associated with an ordinary linear differential equation of degree n and obtained a matrix condition for the system to be self-adjoint [12]. He applied the contour integral method for a boundary value problem to obtain the customary formula for the partial sum of a Fourier series [13].

In an early period, he also investigated certain aspects of the theory of functions of several complex variables. Thus, he gave conditions under which such a function is the quotient of two polynomials [10]. He introduced two definitions of a non-essential singularity for a function of several complex variables which led to the conclusion that, in the neighborhood of the singularity, except at certain points, the function is expressible as a quotient of analytic functions [11]. In a study of semi-analytic functions $f(x, y)$, he obtained a theorem concluding with an expression for $f(x, y)$ as a product of two functions of the same type, one factor being a polynomial in one variable and continuous in the other, in the neighborhood of a zero of $f(x, y)$; also, he obtained an expression for $f(x, y)$ as a quotient of semi-analytic functions in the neighborhood of any singularity, of the types he defined, except at certain points [14].

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[2] In considering this topic, Jackson enlarged it to include approximation by trigonometric sums as well as by polynomials.

During the last two-thirds of Jackson's life at the University of Minnesota, he developed considerable interest in mathematical statistics and related topics although, simultaneously, he remained active in his main field. He presented a very useful new definition of percentiles designed to improve their mathematical formulation [23]. He proposed a novel symmetric coefficient of correlation for several variables [28]. He discussed various features of the geometry of function space designed as an aid to the appreciation of certain features of the theory of correlation [32], [34], [38, Chap. V]. In this connection, his exposition of some elements of sampling theory [48], and his clear presentation of contacts between Pearson frequency functions and various types of orthogonal polynomials [58; Chap. VI and later] deserve special mention.

In 1921, Jackson published an interesting brief paper [21] on the Picard method for establishing the existence of a solution of a system $dy_i/dx = f_i(x, y_1, y_2, \cdots, y_n)$, with $y_i = b_i$ when $x = a$, for $i = 1, 2, \cdots, n$. By simply extending the functions f_i outside their original region of definition R, he proved in a few words, under the usual hypotheses, that the approximations of the method converge to the unique solution as long as it remains in R, if R is of reasonably convenient shape. This important point had been open to question previously.

6. Doctoral dissertation and early related investigations. Jackson's thesis [2] was never published except as a separate pamphlet in German. This relative inaccessibility, and the importance of the content make it appropriate to refer to some of the details of the thesis at this point. His primary result was as follows:

There exists an absolute constant K_1 *with the following property: if* $f(x)$ *has the period* 2π *and satisfies a Lipschitz condition,* $|f(x_1) - f(x_2)| \leq \lambda |x_1 - x_2|$*, then, for every positive integer* n*, there exists a trigonometric sum*[3] $T_n(x)$ *of order* n *at most so that*

$$|f(x) - T_n(x)| \leq K_1\lambda/n \tag{1}$$

for all values of x*, and thus* $\max |f(x) - T_n(x)| = O(1/n)$.

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[3] Throughout the article, T_n and P_n will have the meanings indicated in this section.

In his demonstration of this result, $T_n(x)$ was explicitly exhibited. Under similar hypotheses on a function $f(x)$ defined on an interval (a, b), omitting reference to periodicity, he exhibited polynomials $P_n(x)$ of degree n at most so that $|f(x)-P_n(x)| \leqq G_1\lambda l/n$, where $l=b-a$ and G_1 is an absolute constant. Moreover, if $T_n(x)$ now represents the trigonometric sum of order n at most for which max $|f(x) - T_n(x)| = \phi(n)$ has its *least value*, Jackson proved the important result that, under his hypotheses, in general $\phi(n)$ is *not of lower order than* $1/n$, with a similar result for the polynomial case. If $f(x)$ possesses an $(k-1)$th derivative satisfying a Lipschitz condition, Jackson obtained $K_k\lambda/n^k$ in place of the right-hand side of (1), and a parallel result for polynomials. He proved similar theorems for a function $f(x, y)$ with respect to approximation by polynomials $P_n(x, y)$. The key detail in Jackson's thesis was an expression for $f(x)$ by means of a new type of singular integral, probably suggested by those which had been employed previously in proofs of Weierstrass's theorem about polynomial approximation. Jackson's thesis answered a prize question which had been proposed earlier by the Göttingen faculty: "*Would it be possible to improve on results* (on approximation by polynomials) *previously obtained by de la Vallée Poussin and Lebesgue?*" In a complimentary statement awarding Jackson the prize, we find the remark: "*The author . . . has enriched the science with valuable results, all in competition with mathematicians of the first rank.*"

Jackson's thesis provided powerful assistance for studying the degree of convergence of Fourier series, or series in orthogonal polynomials. After returning to the United States from Germany, he quickly used these new tools. First, he discussed the degree of convergence of Legendre series [3]. Then, he published considerably simplified proofs of those results of his thesis concerning upper bounds of the order of approximation; he obtained close *lower* bounds for the important absolute constants, such as K_1 in (1); he investigated the degree of convergence of Fourier series under various hypotheses [4], [5]. He extended his methods to a study of approximation by trigonometric interpolation formulas [6], [7], and rounded out this theory six years later [15]. In the direction of boundary value problems, he obtained results, similar to those for Fourier series, on the degree of convergence of an interesting class of Sturm-Liouville series [8]. He gave a simple proof of a Fejér theorem on the uniform summability, in the sense of the first arithmetic mean, of the Fourier series for a periodic continuous function $f(x)$ [16]. Jackson's method employed the Weierstrass theorem on the approximation of $f(x)$ by means of trigonometric sums, and a modification of a standard method which had been used in discussing the ordinary convergence of Fourier series.

7. **Approximation in the sense of least mth powers; 1921–1925.** A new stage of Jackson's main work was initiated by his pioneer investigations of "best" approximating trigonometric sums $T_{n,m}(x)$, or polynomials $P_{n,m}(x)$, where the measure of the error (stating it for the trigonometric case) of $T_{n,m}(x)$ as an approximation to $f(x)$ was taken as $\int_0^{2\pi}|f(x)-T_{n,m}(x)|^m dx$, with fixed $m>0$.[4] The objective here was to generalize the familiar least-square case ($m=2$), where $T_{n,m}$ becomes the nth partial sum of the Fourier series, and $P_{n,m}$ is the nth partial sum of the Legendre series for $f(x)$. In a group of papers [17], [18], [20], [25] he established the existence of such best approximations $T_{n,m}$, or $P_{n,m}$, and their convergence to $f(x)$ as n becomes infinite, under the various assumptions $m\geqq 1$ (unique determination of $T_{n,m}$ and $P_{n,m}$) and $0<m<1$ (possibly ambiguous case). His treatment was phrased for a family of approximating functions including, as special cases, the family of all trigonometric sums, or all polynomials. An interesting by-product was his proof that $P_{n,m}(x)$ tends to the Tchebichef polynomial of degree n as m becomes infinite [17]. Then, he widened the scope of the treatment by the introduction of a non-negative weight function $\rho(x)$ as a factor under the integral sign in the expression for the measure of error [24], [27], as found in earlier literature for the case $m=2$.

In connection with the research just mentioned, we observe various related investigations by Jackson. He applied a method of "least mth powers" in the solution of a system of linear equations [29]. He gave a treatment of approximation with a general function $E(y)$, subject only to liberal hypotheses, in place of $|y|^m$ as the essential analytic feature of the integrand in the expression for the error of approximation [30]. Also, he considered trigonometric approximation in the sense of least mth powers, without a weight function, to an "ill-defined" function $y=f(x)$, where the many-valued function y has values distributed (statistically) in accordance with a frequency function $w(x, y)$; he reduced this problem to one of ordinary type for a new single-valued function $F(x)$, to be approximated in the sense of least mth powers with an appropriate weight function [31].

8. **Comprehensive research reports to the Society.** In 1921, at a symposium of the Society in Chicago, and at the Summer meeting

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[4] The polynomial case had previously been treated to a certain extent by Pólya in a note in C. R. Acad. Sci. Paris vol. 157 (1913) pp. 840–843. Jackson's original work in this direction was done without knowledge of Pólya's note.

of the Society in 1925, Jackson had opportunities to present extensive discussions concerning his major field. His symposium lecture was essentially a synoptic treatment of his own main research up to 1921 and related literature [19]. His colloquium lectures of 1925 on approximation theory, as published in 1930 [38], include complete presentations of essential parts of his own research, associated new results and refinements of existing literature, and collateral classical material to give a rounded view of the field. A final chapter, loosely connected with the other content through the notions of least-square approximation and orthogonality, deals with the geometry of function space. This chapter leads to geometrical viewpoints about certain features of correlation theory in statistics, and had been published previously by Jackson.

9. Expository and detached publications in his main field; 1926–1946. Jackson's mathematical publications after 1925 involved several excursions (already mentioned) into the theory of statistics, particularly before 1930. In his field of major interest, he devoted considerable time to expositions of fundamental content, written at the level of readers satisfying only minimum mathematical prerequisites, in order to make the material more accessible in its fields of application. His principal work of this type was his Carus Monograph [58], 1941, on Fourier series and orthogonal polynomials, which requires only a substantial first course in calculus, and active interest, as the prerequisites for intelligent appreciation of most of the content. His other outstanding expository publications are the three related papers (loc. cit.) for which he was awarded the Chauvenet Prize in 1935. There is considerable overlapping between these papers and his Carus Monograph.

In this period, Jackson published various results somewhat apart from his main program (which I will mention later). Thus, I first note a few papers on problems of approximation in the complex domain. In [37], Jackson discussed the existence of $P_n(z)$ to minimize $\int_C \rho(s)|f(z)-P_n(z)|^m ds$, where C is the boundary of a closed region S, and considered the convergence of $P_n(z)$ to $f(z)$ in S; C was subject to restrictions designed to permit the application of Bernstein's theorem on the maximum of the derivative of a polynomial in z. This gave results similar to some previously obtained otherwise by J. L. Walsh, cited in [37]. In [40], Jackson relaxed the preceding condition on C and, in his proofs, replaced Bernstein's theorem by one of Markoff, with somewhat more stringent hypotheses about $f(z)$. In [26] at an earlier period, [36], and [43], Jackson considered various "nonlinear" problems of approximation (returning to the real domain), where in each case the approximating functions belong to a nonlinear family. He investigated problems where the approximating functions are required to satisfy auxiliary conditions, point-wise [33], [35], or expressed by means of integrals [41].

10. **Foundation papers for a major program.** The remaining research by Jackson to which I wish to refer appeared in 1933–1946, and is roughly connected in regard to methods and viewpoints. These publications start with four papers (two are expository) which establish patterns for later proofs and concentrate essential reference material from many sources in a few convenient locations. Thus, in his expositions concerning systems of polynomials [42] and trigonometric sums [45] orthonormal with respect to a weight function $\rho(x)$, he emphasized the role played by the property of boundedness for the orthonormal functions, in conjunction with the corresponding Christoffel-Darboux identity, in proving the convergence of related expansions. In a third paper [44], he obtained various upper bounds for the order of magnitude of $|P_n(x)|$ as a function of n in case $P_n(x)$ is normalized, $\int_a^b \rho(x)|P_n(x)|^2 dx=1$. Also, as an independent part of [44], if P_n minimizes $\int_a^b \rho(x)|f(x)-P_n(x)|^p dx$, he developed standard devices for establishing an upper bound for the error of approximation, $|f(x)-P_n(x)|$, in terms of an upper bound, ϵ_n, of the error of approximation to $f(x)$ attainable by use of *other polynomials* of degree n at most. The hypotheses of theorems on the convergence of $P_n(x)$ to $f(x)$ are then stated in terms relating to the rapidity with which ϵ_n tends to zero as n becomes infinite, where such facts are considered as known under various hypotheses about $f(x)$ from Jackson's earliest research. The key detail of [44] is an expression for an upper bound of $|P_n(x)|$ in terms of the value of $\int_a^b |P(x)|^p dx$, and the proofs make frequent use of Bernstein's and Markoff's theorems on the magnitude of the derivative of $P_n(x)$. Parallel results are developed in [44] for trigonometric sums $T_n(x)$, with integration from $x=0$ to $x=2\pi$. In a fourth foundation paper [49], he extended the preceding results with respect to T_n to the inconvenient case where all integration involved is from $x=c$ to $x=d$, with $0<c<d<2\pi$. In convergence results, [44] and [49] are noteworthy because of the relative liberality of the hypotheses with regard to the troublesome feature of the vanishing of the weight function $\rho(x)$.

Adaptations of methods from some of the preceding papers were used by Jackson to obtain theorems on the summability of expansions in series of orthonormal polynomials or trigonometric sums [47].

Also, in [53], he applied methods like those of [44] and [49] to the two-dimensional case, where he considered approximation to $f(x, y)$ over a region R by use of polynomials $P_n(x, y)$ or trigonometric sums $T_n(x, y)$. In this discussion, an essential preliminary detail was the development of analogs, for two dimensions, of the standard Bernstein and Markoff theorems; Jackson obtained these generalizations on the basis of a moderate restriction on the region R.

11. **Orthonormal polynomials in two or more variables; orthonormal systems on algebraic loci; 1936–1946.** In a sequence of publications commencing with [50] in 1936 and ending with Jackson's final research paper, he concerned himself with systems of polynomials (and, in a few instances, trigonometric sums) in two or three variables which are orthogonal and normalized, for a given weight function ρ, with respect to integration over a specified closed point set R of one, two, or three dimensions. In [50], he considered the system of polynomials $[p_{nk}(x, y)]$, where p_{nk} is of degree n and $n=1, 2, \cdots$, which are orthonormal over a region R; the orthonormality condition is that $\int\int_R \rho(x, y)p_{nk}(x, y)p_{gh}(x, y)dx$ equals 1 or zero according as $|n-g|+|k-h|$ equals zero, or is not equal to zero. For this system, he developed a recursion formula and an analog of the Christoffel-Darboux identity, and discussed features arising under transformation of the variables. He paralleled these considerations in [55] for a system $[p_{nk}(x, y, z)]$, orthonormal with respect to integration over a point set R which might be a region, a surface, or a curve. If a function f, defined in R, is expanded formally in a series of such polynomials, he sketched least-square reasoning in various situations to prove that the series converges to f in the mean, weighted by ρ, if suitable restrictions are placed on ρ and the given function [55, p. 445]. He then turned his attention primarily to cases of the interesting type where a system $[p_{nk}]$ is orthonormal on a locus which is *algebraic*. The characteristic feature here is that, due to whatever algebraic identity or identities may be associated with the locus, the number h_n of polynomials of degree n in the orthonormal system is smaller than for a *general* system. With each type of algebraic locus which he considered later, a discussion of the order of magnitude of h_n was an essential detail. Thus, for an algebraic plane curve, he found that there exists a constant N so that $h_n=N$, if $n \geqq N$ [51]. He discussed h_n and formal properties of the orthonormal system for an algebraic plane curve [51], and for an algebraic surface or an algebraic space curve [55].

On the basis of results just mentioned, Jackson investigated the convergence of the expansion of a function $f(x, y)$, defined on an algebraic plane curve C, in series of polynomials $p_{nk}(x, y)$ orthonormal with respect to integration on C. In a group of papers, his apparent object was to illustrate characteristic difficulties, methods, and results by means of special examples, since a general treatment appeared to be an unfruitful venture. One of his methods was to develop a modification of Bernstein's theorem applicable on the special curve C involved; then, he could conclude by merely remarking that standard methods of the foundation paper [44] applied in treating convergence. As another method, Jackson emphasized the classical use of the Christoffel-Darboux identity, obtained in [51], to establish convergence in case $p_{nk}(x, y)$ is uniformly bounded, for all n and k. As an aid in proving this property for specific cases, he established a theorem, in [59], stating hypotheses on C and a function $\rho(x, y)$ so that, if the orthonormal polynomials $p_{nk}(x, y)$ on C are known to be uniformly bounded for a given weight function $\sigma(x, y)$, we may infer that the orthonormal polynomials $q_{nk}(x, y)$ for the new weight function $\rho\sigma$ also will be uniformly bounded (generalization of a theorem of Korous). Then, Jackson's procedure with a special curve C was to produce *one* weight function $\sigma(x, y)$, sometimes $\sigma \equiv 1$, for which the polynomials $p_{nk}(x, y)$ were uniformly bounded. From this, he would then conclude that a similar boundedness property, and hence convergence of expansions, held for a wide class of weight functions of the type $\rho\sigma$. With these remarks as to Jackson's methods, his papers [52], [60], [62], [63] on orthonormal polynomials on special algebraic curves are sufficiently described by reading their titles in the bibliography. Special results of a related character, under a less descriptive title, are given in [61] for certain curves of rational type in two and three dimensions. In regard to polynomials $p_{nk}(x, y, z)$ orthonormal on an algebraic surface R, Jackson treated the convergence of the associated expansion of a function $f(x, y, z)$ only for one illustration, where R is the unit sphere and the weight function is unity [55].

Jackson also considered notions related to the preceding content, extended in two directions. In [54], he discussed a system of orthonormal polynomials $[p_{nk}(z_1, z_2)]$ of the complex variables z_1 and z_2, where the relation of orthogonality is of the form

$$\int_E \rho(z_1, z_2) p_{nk}(z_1, z_2) p_{gh}(z_1, z_2) dE = 0, \quad |n - g| + |k - h| \neq 0,$$

in which $\rho \geqq 0$, and E is a one-, two-, three-, or four-dimensional spread in the space of the corresponding four real coordinates (x_1, y_1, x_2, y_2). In regard to the expansion of a function $f(z_1, z_2)$ in series of the polynomials $p_{nk}(z_1, z_2)$, he treated convergence in the mean for any range

E, and ordinary convergence for illustrative simple ranges E. In [56], he investigated systems of "mixed sums" $u_k(x, y)$, orthonormal with respect to integration on a plane curve C, where $u_k(x, y)$ is a trigonometric sum in x with coefficients which are polynomials in y. Also, in [56] he touched lightly on systems where u_k is a trigonometric sum in each variable. To illustrate possibilities, he discussed the convergence of expansions in series of mixed sums u_k for a special curve C defined by an identity which involves a mixed sum.

12. **Conclusion.** A discussion of the mathematical work of any member of a graduate faculty is incomplete without comment on his activities in connection with doctoral dissertations. Jackson directed the thesis research of eighteen successful candidates for the Ph.D. degree at the University of Minnesota, and of two candidates at Harvard University during the first few years of his career as a teacher. The topics of these theses, as is to be expected, present a cross section of his research interests during the corresponding periods.

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BOOK REVIEWS

Length and area. By Tibor Radó. American Mathematical Society Colloquium Publications, vol. 30. New York, American Mathematical Society, 1948. 6+572 pp. $6.75.

Since as long ago as the days of the early Greek mathematicians, men have been interested in the areas of surfaces. Special examples were studied, and with the invention of the calculus a formula was found expressing the area of a sufficiently smooth surface by means of a well known double integral. Nevertheless, the careful study of areas of continuous surfaces has been almost entirely a product of the preceding half-century. During this time about a dozen major definitions of area have been proposed, all being in agreement when applied to sufficiently well-behaved surfaces, but often having widely different properties when considered for all continuous surfaces. Of these various definitions, the one due to Lebesgue has been most thoroughly investigated, and upon it there now rests a theory possessing a high degree of completeness. It is this theory which constitutes the principal subject matter of Radó's *Length and area.* The "length" is of course an interesting subject; but its placidity, as contrasted with area, permits a very complete discussion in much less than half the book. The pattern of this discussion furnishes a model for developing the study of the area.

The theory presented in this book is the outstanding application of analytic topology to a problem in analysis. Since both the topology and the analysis are essential, the author devotes the first chapter to "background material." First the ideas of curve and surface are briefly discussed (a full discussion occurs later) and the distinction is drawn between definitions based on measure theory and those (like Lebesgue's) based on semi-continuity. Examples show that even in elementary situations there is need for care and precision. Next there are 17 pages of résumé of important theorems of topology, and 20 pages of résumé of analysis, particularly set-functions and integrals.

Part II is largely concerned with the topology of curves and surfaces. A transformation on a Peano space to a Peano space is monotone if the inverse image of each point is connected; it is light if the inverse image of each point is totally disconnected. The "factorization theorem" is demonstrated: if $T(P)=P^*$ is continuous, there is a monotone transformation $M(P)=\mathfrak{M}$ of P onto a set $\mathfrak{M}$ and a light transformation $L(\mathfrak{M})=P^*$ such that $T=LM$. If the mappings $T_1(P_1)=P^*$, $T_2(P_2)=P^*$ are Fréchet equivalent they are also "K-

equivalent," which means that T_1 and T_2 have simultaneous factorizations $T_1=LM_1$, $T_2=LM_2$ with the same middle-space $\mathfrak{M}$ and the same light factor L. In certain cases adequate for the sequel (such as when P_1, P_2 and $\mathfrak{M}$ are all spheres) K-equivalence implies Fréchet equivalence. Hereby the analysis is linked with the topology. The decomposition of $\mathfrak{M}$ into cyclic elements lets T be studied as though analyzed into light transformations on spheres or two-cells or arcs or simple closed curves.

Part III begins with a study of interval functions (in two dimensions), their Burkill integrals and their extensions to functions of Borel sets. Next follows a chapter on functions (of one variable) of bounded variation, and absolutely continuous functions. These are used in developing the theory of the lengths of continuous curves, and of the concepts and consequences of convergence in length and in variation. Here again we see the excellent matching of the definitions of length, bounded variation, absolute continuity, Lebesgue integral and derivative.

In the next part we turn to the far more subtle theory of plane transformations and the substitution theory of double integrals. Of crucial importance is the concept of "essential multiplicity," developed by Radó from a beginning made by Geöcze. Let T: $z=t(w)$ map a Jordan region R in the complex w-plane continuously into the z-plane. The multiplicity $N(z, T)$ of a point z is the number of distinct points w which map on z. Its *essential* multiplicity $\kappa(z, R)$ is the greatest number k (possibly ∞) with the following property: if $k'<k$, for all continuous mappings T' of R which differ everywhere from T by less than a certain positive $\epsilon=\epsilon(k')$ the multiplicity $N(z, T')$ exceeds k'. If $\mathfrak{D}$ is an arbitrary domain, $\kappa(z, \mathfrak{D})$ is defined as the limit of the essential multiplicities with respect to Jordan regions which expand and fill $\mathfrak{D}$. The transformation T is essentially of bounded variation (eBV) if $\kappa(z, \mathfrak{D})$ is summable over the z-plane; a concept of essential absolute continuity (eAC) is also defined. We obtain a function of rectangles r in $\mathfrak{D}$ by mapping r by T into the z-plane and retaining only the image-points at which $\kappa(z, r)\neq 0$; the measure of this set is a function of r which has a derivative $D_\kappa(w)$ for almost all w in $\mathfrak{D}$. This is, roughly, a "local area-magnification ratio," nonessential images being discarded. It is thus analogous to the absolute value of the Jacobian. A local topological index $i_e(w)$ of the mapping is defined, and the "essential generalized Jacobian" is defined to be $\mathfrak{J}_e(w) = i_e(w)D_\kappa(w)$. If the ordinary Jacobian exists almost everywhere, it is equal to $\mathfrak{J}_e$ almost everywhere. For all finite-valued measurable functions $H(z)$ the formula

$$\iint_{\mathfrak{D}} H(\mathbf{t}(\mathbf{w}))\,|\,\mathfrak{J}_e(\mathbf{w})\,| = \iint H(\mathbf{z})\kappa(\mathbf{z}, \mathfrak{D})$$

holds whenever either integral exists. Also, if $\nu(\mathbf{z})$ is the sum of $\mathbf{i}_e(\mathbf{w})$ for all inverse images $\mathbf{w}$ of $\mathbf{z}$, then

$$\iint_{\mathfrak{D}} H(\mathbf{t}(\mathbf{w}))\mathfrak{J}_e(\mathbf{w}) = \iint H(\mathbf{z})\nu(\mathbf{z})$$

if the left member exists.

The family of eAC transformations is not merely larger than the corresponding families in earlier theories of plane transformations; it also has closure properties which simplify the task of establishing that various special types of transformations are actually eAC.

The theory culminates in Part V, devoted to areas of surfaces. The strength of the results attained can be shown by quoting two theorems. The functions $x(u, v)$, $y(u, v)$, $z(u, v)$ $(0 \leq u \leq 1,\ 0 \leq v \leq 1)$ defining S furnish three plane transformations, by projection. The square root of the sum of the squares of the three essential generalized Jacobians will be denoted by $W_e(u, v)$; if the ordinary Jacobians exist, the square root of the sum of their squares is $W(u, v)$. Then:

(A) If $A(S) < \infty$, the three plane transformations are eBV, and W_e is defined almost everywhere in the unit square and is summable; and its integral is at most $A(S)$, being equal to $A(S)$ if and only if the three plane transformations are eAC.

(B) If $A(S) < \infty$ and $W(u, v)$ is defined almost everywhere in the unit square, it is summable, and its integral is at most $A(S)$, being equal to $A(S)$ if and only if the three plane transformations are eAC.

By his choice of methods of proof and by clarity of exposition, the author has provided a well-engineered road into a difficult territory. However, in the multitude of theorems a reader might well be puzzled about the interrelations, motivations and origins of the mathematical objects he encounters. The author has therefore provided a final chapter to each part except the first, in which he casts a backward look over the preceding chapters, coordinates their contents, and indicates directions in which further research is needed.

E. J. McShane

The advanced theory of statistics. Vol. 2. By M. G. Kendall. London, Griffin, and Philadelphia, Lippincott. 1st ed., 1946, 2d ed., 1948. 8+521 pp. 50s.

The great treatise, of which the first volume was reviewed in Bull. Amer. Math. Soc. vol. 51 (1945) p. 214, is now complete. For the first time a connected account is available of the main outlines of the whole theory of statistics, from elementary probability and the fundamentals of estimation and testing of hypotheses to recent advanced researches and techniques, with attention to many unsolved problems. A debt of gratitude is due the author for the prodigious task of working through thousands of papers scattered through a multitudinous variety of journals devoted to statistics, mathematics, economics, psychology, actuarial work, physics, biology of many kinds, and other fields, and assembling, arranging and rewriting their contents with original additions and comments.

The second volume covers many of the more difficult and controversial subjects concerning which knowledge is still incomplete. This was apparently one object in the ordering of the chapters of the complete treatise, though both historical and logical bases for the arrangement are also discernible, and the actual arrangement seems to be a compromise. Volume 2 begins with Chapters 17 on the method of maximum likelihood, 18 on estimation in general, 19 on confidence intervals, 20 on fiducial inference, and 21 on some common tests of significance. All these constitute chiefly an expansion of R. A. Fisher's work, with more mathematical details than Fisher himself gives, and with accounts of the extensions of Fisher's ideas by Doob, Welch, Pitman, Bartlett, Geary, Koopman, Dugué, Wilks, Wald, Daly, E. S. Pearson, Yates, Sukhatme and others. Chapter 22, on regression, includes an account of the orthogonal polynomials used for fitting with equidistant intervals of the argument. The next two chapters, on the analysis of variance, and Chapter 25 on the design of sampling inquiries, again reflect strongly Fisher's influence, though Mr. Kendall's interest as an economic statistician leads to a greater emphasis on social-economic sampling than on the agricultural and biological problems which originally led to the modern theory of experimental design as formulated by Fisher.

The work of Neyman and Pearson on the fundamentals of testing hypotheses is postponed until after all this, with the result that its assimilation with Fisher's work into a rounded theory is incomplete. However the treatment is excellent, and many a reader will get his first clear idea of the subject from these two chapters. The difficult problems of testing hypotheses involving two or more parameters are expounded as adequately as the present stage of knowledge permits.

The chapter on multivariate analysis covers most of the fundamental work in this field, including Wishart's distribution, the T distribution, discriminatory analysis, and canonical correlations. It is not, however, made clear that the distribution found for the roots of a determinantal equation is applicable to the comparison of two samples as well as to canonical correlations; and a fuller account would be desirable of the contributions of Wilks and Hsu to the generalized analysis of variance.

The two final chapters deal with time series. This vexing but urgent subject is still not in a satisfactorily complete state. Mr. Kendall has given the principal available material a thorough working over, using a variety of series of actual observations and a series of random numbers as examples for the application of weighted and unweighted moving averages, correlograms, and other devices. Interlaced parabolas, the variate difference method, autoregression equations (that is, stochastic difference equations), serial correlation, lag correlation and periodogram analysis are all treated, and outstanding unsolved problems are indicated.

Mr. Kendall amplifies R. A. Fisher's discussion of fiducial probability, including the Behrens test for the difference between two means. In attempting to understand these ideas and connect them with other statistical tests one encounters some fundamental difficulties. Probability is in the conception of Fisher and nearly all other present-day statisticians an idealization of frequency-ratios in large samples, and is thus subject to operational measure and verification. This concept should be distinguished sharply from the "degrees of rational belief" of writers such as Keynes who have used the word "probability" in this sense, a sense to which the reviewer would prefer, in the interest of clarity, to attach the word "credibility." In this review "probability" will be used with the former "objective" or statisticians' meaning. When we speak of probability distributions, and of probability levels for tests of significance, we ordinarily understand probability in this sense. Thus, when the Student distribution is used in a way now familiar to test the hypothesis that a sample has been drawn from a population of mean value 37 and the .05 probability level is employed, with the certain knowledge that the sample is a random one from some normal population, we can make the following statement: In the class of cases in which the mean of the population is 37, this hypothesis will be erroneously rejected with probability exactly .05. This statement implies an anticipation that in a long run of such trials the erroneous rejection will take place in almost exactly one-twentieth of the cases. Experiments of just this type have in fact been made, and have yielded close agreement with theory. Further consequences of routine application of the Student test have been elucidated through the power function and certain theorems on optimum properties of tests. The growing popularity of such tests is based on considerations of this kind. "Belief" in the hypothesis that the true mean is 37 may be determined by the outcome of the Student test in a way wholly consistent with the pragmatic philosophy of William James, according to which the truth is that which works, and the goodness of a belief is to be judged by its total consequences, in this case its probable consequences with measurable probabilities.

The "fiducial probability" and "fiducial distributions" introduced by Fisher and now expounded by Kendall do not refer to probability in the sense indicated above. This fact, which has been lost sight of in much discussion, is pointed out by Kendall on pages 90 and 112. Furthermore, if one tries to read into the derivation of the Behrens-Fisher test on pages 91 and 92 the objective or frequency meaning of probability, the result is nonsense. Thus the distribution of

$$\epsilon = t_1 \cos \psi - t_2 \sin \psi$$

is obtained by convolution of two independent Student distributions of t_1 and t_2, with ψ regarded as a constant. But ψ is defined as the arctangent of the ratio of the two sample standard deviations, and if these are held constant t_1 and t_2 do not have the Student distribution but normal distributions, whence the conditional probability distribution of ϵ is normal.

Thus the Behrens distribution does not give probabilities in the ordinary sense, but fiducial probabilities. The application of this test for the difference of two means does not yield definite proportions of fallacious rejections which can be expected to be verified in long series of experiments, but instead a specially defined measure of the degree of belief imputed to the result. What may be the consequences of routine application of the test remain to be elucidated through study of its power function. The underlying principle is in contrast to the pragmatic interpretation of the simpler tests. Thus we have what appears to be a dangerous and objectionable dualism. To escape from such dualism the available alternatives seem to be (a) to modify or abandon the Behrens test, (b) to derive it from some different principle, such for example as the comparison of the sample in hand with a sub-population of possible samples defined by constancy of some function of the sample standard deviations other than their ratio, but using the standard notion of probability, or (c) to insist that the proper justification of all statistical tests is to be obtained through fiducial probability. The last would be a drastic step hard to justify on pragmatic or other established principles, and in any postulational treatment would require an independent axiom not yet precisely stated and whose full consequences have not been explored.

In trying to understand the literature of modern mathematical statistics it is essential to remember that, wherever written, it is fundamentally British, and derives historically from the British tradition of applied mathematics much more than from the meticulous continental tradition of rigorous pure mathematics. Karl Pearson began as a mathematical physicist working in elasticity, R. A. Fisher as an astronomer, "Student" as a chemist, and all became biological statisticians. Like Newton, Stokes, Green and the physicists, English statisticians have been more concerned with getting useful formulae than in delimiting the exact conditions of their validity, and have frequently used loose mathematical arguments. The proofs in the present volume are largely carried over from the original sources, often with arguments in forms subject to criticism. Thus on p. 7, following the formula for a two-dimensional normal distribution regarded as a limiting form, the statement is made and proved that the correlation is unity, meaning that it approaches unity, but this contradicts the postulated two-dimensional density. A more serious difficulty is with Fisher's famous original proof of the efficiency of maximum likelihood estimates, which became the point of departure of much new work and is here reproduced on pages 18 and 19. The free interchange of various limiting processes in the absence of any proof of validity, and the omission from (17.66) of terms involving the derivative of v, are faults in this proof which should be repaired at the first opportunity, even though similar results have recently been obtained by Cramér and Rao using other methods. An error due to the present reviewer, who in 1930 gave a proof of consistency and asymptotic normality of maximum likelihood estimates for discrete distributions but used insufficient care in passing over to continuous distributions, is here reproduced. The essential correctness of the result was established in 1936 by Doob by a different method.

There are some difficulties due in part to oversimplification of notation without sufficient indication as to what arguments enter into particular functions. Thus in various uses of the formula $\phi' = A + B\phi$ in Chap. 27, the reader will do well to remember that A and B may involve the variable parameter. On p. 333 there is a slight error in the statement of the region of variation of the a_{ij}; the matrix must be positive definite. Also, it is still not clear what the statistician is supposed to do with ancillary statistics.

These and similar shortcomings are far more than compensated by the accomplishment of the immense task. The 62-page bibliography of something like 1800 entries (which the author states constitute only half his collection of references) is in itself enough to make this an indespensable work of reference. With the companion Volume 1 (of which a slightly revised third edition has now appeared) it forms a contribution far beyond the range of ordinary textbooks, suitable for reference and for serious students.

HAROLD HOTELLING

The methods of plane projective geometry based on the use of general homogeneous coordinates. By E. A. Maxwell. Cambridge University Press, 1946. 19+230 pages. $2.75.

This is an introductory treatise on algebraic projective geometry. As the author states in the preface, "It is written as a study of *methods* and not as a catalogue of theorems." Synthetic methods are used occasionally when deemed preferable. Although it is stated in the introduction, "No knowledge of geometry is assumed explicitly," it is implied that the student should have had plane analytical geometry, including the properties of conic sections, before beginning the study of this book.

The first ten chapters deal with non-metric concepts. The principal topics and the order in which they occur are: homogeneous coordinates, equations of lines, duality, one-to-one correspondence cross ratio, conics treated parametrically, projective properties of conics, the quadrangle and its dual, the generation of conics by pairs of projective pencils or ranges, pencils of conics, harmonic properties, reciprocation, poristic systems, theorems of Pascal and Brianchon.

In the last two chapters, eleven and twelve, relations between projective and Euclidean geometry are studied both algebraically and synthetically.

The book is well written; the explanations are clear and ample. The printing is excellent and the formulas well displayed. The caption of each section describes pithily the contents. The book has an excellent index and a very detailed table of contents. The total absence of figures. however, is to be deplored; illustrative drawings are especially helpful to a beginner in projective geometry.

The many excellent and carefully chosen problems constitute one of the best features of the book. These problems are chiefly taken from various examinations. At the end of each problem is a symbol referring to the source given in the preface. Answers to the more difficult problems are given in the back of the book.

T. R. HOLLCROFT

NOTES

The Society is translating selected papers from languages which are not widely read by American mathematicians. Funds for this project have been made available under a contract with the Office of Naval Research. The project is under the direction of a committee whose members are R. P. Boas, Samuel Eilenberg, D. H. Lehmer, William Prager, and G. Y. Rainich. The committee will be largely guided in its selection of papers for translation by suggestions from mathematicians. Suggestions may be sent to any member of the committee.

Professor L. E. J. Brouwer of the University of Amsterdam has been elected to membership in the Royal Society of London.

Professor Marston Morse of the Institute for Advanced Study has been elected a member of the Accademia delle Scienze dell'Istituto di Bologna. Professor Morse has recently received an honorary doctorate of science from Kenyon College.

Dr. R. G. D. Richardson who has recently retired as dean emeritus and professor emeritus from Brown University has been awarded the honorary degree of Doctor of Laws by Brown University.

Professor Oswald Veblen of the Institute for Advanced Study has been elected to membership in the Accademia Nazionale dei Lincei.

Professor Hermann Weyl of the Institute for Advanced Study has been made a correspondent of the French Academy of Sciences and has been elected to honorary membership in the Calcutta Mathematical Society.

Professor J. W. Bradshaw of the University of Michigan has retired with the title emeritus.

Professor W. H. Bussey of the University of Minnesota has retired with the title emeritus.

Professor B. H. Camp of Wesleyan University has retired with the title emeritus.

Professor W. B. Carver of Cornell University has retired with the title emeritus.

Miss Alice C. Dean of Rice Institute has retired with the title librarian emerita.

Professor G. W. Mullins of Columbia University has retired with the title emeritus.

Dean J. R. Overman of Bowling Green State University has retired with the title emeritus.

Professor R. R. Shumway of the University of Minnesota has retired with the title emeritus.

Dean H. L. Slobin of the University of New Hampshire has retired with the title emeritus.

Professor Beno Eckmann of the University of Lausanne has been appointed to a professorship at the Swiss Federal School of Technology in Zurich.

Dr. Erling Følner of the Institute for Advanced Study has been appointed to an assistant professorship at the Mathematical Institute in Copenhagen.

Mr. I. B. Perrott of the College of Technology in Leicester has been appointed lecturer at the University of Leeds.

Mr. Hans Rundberg of the Iron Refining Company, Haelleforsnaes, Sweden, has been promoted to a vice presidency.

Dr. A. H. Stone of Trinity College has been appointed a lecturer at the University of Manchester.

Dr. P. H. Anderson of the Commerce Department has been appointed to an associate professorship of marketing at Loyola University, New Orleans, Louisiana.

Mr. L. U. Albers of the University of Michigan has been appointed to an assistant professorship at Western Reserve University.

Mr. A. B. Alexander of Purdue University has been appointed to an assistant professorship at Ripon College.

Mr. K. J. Arrow of the Cowles Commission for Research in Economics has been appointed to an assistant professorship at the University of Chicago.

Professor H. C. Ayres of the United States Naval Academy has been appointed to a professorship at Jersey City Junior College.

Assistant Professor J. W. Beach of Iowa State College of Agriculture and Mechanic Arts has been appointed to an associate professorship at the New Mexico School of Mines.

Assistant Professor Truman Botts of the University of Delaware has been appointed to an acting assistant professorship at the University of Virginia.

Assistant Professor N. A. Brigham of Pennsylvania State College has been appointed to an assistant professorship at the University of Maryland.

Mr. Paul Brock of Hunter College has accepted a position as mathematician with Reeves Instrument Company of New York City.

Mr. R. G. Brown has accepted a position with an Operations Evaluation Group, Navy Department.

Dean Daniel Buchanan of the University of British Columbia has retired.

Dr. C. T. Bumer of the Division of Industrial Cooperation, Massachusetts Institute of Technology, has been appointed to a professorship at Clark University.

Mr. G. P. Burns of the Naval Research Laboratory, Washington, D. C., has been appointed to an assistant professorship of physics at Mary Washington College of the University of Virginia.

Professor S. S. Cairns of Syracuse University has been appointed to a professorship at the University of Illinois.

Mr. R. C. Campbell of the University of Pennsylvania has been appointed to an assistant professorship at the U. S. Naval Postgraduate School.

Mr. F. D. Carpenter of Johns Hopkins University has accepted the position of vibration engineer with Glenn L. Martin Company in Baltimore, Maryland.

Dr. R. E. Chamberlin has been appointed to an assistant professorship at the University of Utah.

Dr. Abraham Charnes of the University of Illinois has been appointed to an assistant professorship at Carnegie Institute of Technology.

Professor W. L. Chow of National Tung Chi University, Shanghai, has been appointed to an associate professorship at Johns Hopkins University.

Mr. C. J. Clark of Oklahoma Agricultural and Mechanical College has accepted a position as research engineer with the Curtiss-Wright Aircraft Corporation, Columbus, Ohio.

Dr. Louise H. Chin of the University of California has been appointed to an assistant professorship at the University of Arizona.

Mr. Harvey Cohn has been appointed to an assistant professorship at Wayne University.

Associate Professor W. J. Combellack of Northeastern University has been appointed to a professorship at Colby College.

Assistant Professor J. C. Currie of Louisiana State University has been appointed to an associate professorship at Alabama Polytechnic Institute.

Dr. D. A. Darling of Cornell University has been appointed to an assistant professorship at Rutgers University.

Mr. R. A. Deutsch of the Triborough Bridge and Tunnel Authority has accepted a position as electrical engineer with the Kellex Corporation of New York City.

Assistant Professor Roy Dubisch of Triple Cities College, Syracuse University, has been appointed to an associate professorship at Fresno State College.

Dr. James Dugundji has been appointed to an assistant professorship at the University of Southern California.

Dr. W. F. Eberlein of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Wisconsin.

Mr. R. P. Eddy has accepted a position as mathematician in the Mechanics Division of the Naval Ordnance Laboratory at Silver Spring, Maryland.

Dr. Benjamin Epstein of the Carnegie Institute of Technology has been appointed to an associate professorship at Wayne University.

Miss Frances E. Falvey of Hollins College has been appointed dean of women and assistant professor at James Millikin University, Decatur, Illinois.

Dr. B. G. Farley of Yale University has accepted a position with the Bell Telephone Laboratories.

Mr. A. L. Fass has been appointed a lecturer at Columbia University.

Professor H. F. Fehr of Montclair State Teachers College, Montclair, New Jersey, has been appointed to a professorship at Teachers College, Columbia University.

Associate Professor W. W. Flexner of Cornell University has accepted a position as senior statistician in charge of transport statistics in the Department of Economic Affairs of the United Nations.

Dr. M. M. Flood has been appointed Executive Secretary of the American Statistical Association.

Dr. W. H. Fuchs of the University of Liverpool has been appointed to a visiting associate professorship at Cornell University.

Professor W. H. Gage of the University of British Columbia has been appointed dean of administrative and inter-faculty affairs.

Dr. H. L. Garabedian of the Oak Ridge National Laboratories has been appointed Chief of Research Reactors Section, U. S. Atomic Energy Commission.

Dr. B. R. Gelbaum has been appointed to an assistant professorship at the University of Minnesota.

Professor F. C. Gentry of Louisiana Polytechnic Institute has been appointed to an associate professorship at Arizona State College.

Mr. E. N. Gilbert of the Massachusetts Institute of Technology has been appointed a member of the technical staff of the Bell Telephone Laboratories.

Dr. M. A. Girshick of the Douglas Aircraft Corporation has been appointed to a professorship in statistics at Stanford University.

Assistant Professor H. E. Goheen of the University of Delaware has been appointed to an assistant professorship at Syracuse University.

Dr. Ruth E. Goodman of Syracuse University has been appointed to an assistant professorship at Duquesne University.

Dr. R. D. Gordon of Indiana University has been appointed to an assistant professorship at the University of Buffalo.

Mr. G. E. Gourrich of the University of California at Los Angeles has accepted a position as electrical engineer with the National Bureau of Standards.

Professor E. H. Hadlock of Hastings College has been appointed to an associate professorship at the University of Florida.

Mr. A. E. Halteman of the University of Wisconsin has been appointed to an assistant professorship at the University of Idaho.

Assistant Professor G. H. Handelman of Brown University has been appointed to an assistant professorship at Carnegie Institute of Technology.

Dr. Gerald Harrison of Sperry Gyroscope Company has been appointed to an assistant professorship at Wayne University.

Assistant Professor E. E. Haskins of the United States Air Force Institute of Technology, Wright Field, Dayton, Ohio, has been appointed to an associate professorship at Fenn College, Cleveland, Ohio.

Dr. I. R. Hershner of the University of Chicago has been apointed to an assistant professorship at the University of North Carolina.

Assistant Professor D. M. Hester of Northwest Missouri State Teachers College, Maryville, Missouri, has accepted a position at Baker University, Baldwin City, Kansas.

Dr. Edwin Hewitt of the University of Chicago has been appointed to an assistant professorship at the University of Washington.

Mr. J. L. Hildebrand has been appointed an associate at the University of Washington.

Dr. G. P. Hochschild of Harvard University has been appointed to an assistant professorship at the University of Illinois.

Assistant Professor Carl Holtom of Purdue University has been appointed to an associate professorship at the United States Air Force Institute of Technology, Wright Field, Dayton, Ohio.

Mr. E. A. Hoy has been made Chief of the Research and Evaluation Section of the Naval Reserve Training Publications Project, Naval Gun Factory.

Dr. Ralph Hull of Boeing Aircraft Corporation has been appointed to a professorship at Purdue University.

Professor T. R. Humphreys of Bergen Junior College has been appointed to an assistant professorship at the New Jersey State Teachers College, Upper Montclair, New Jersey.

Dr. N. C. Hunsaker of Rice Institute has been appointed to an associate professorship at Utah State Agricultural College.

Mr. R. Y. Iwanchuk of St. Basil's College, Stamford, Connecticut, has been appointed to an assistant professorship at Kent State University.

Dr. L. H. Kanter of the University of Wisconsin has been appointed to an assistant professorship at the University of Arkansas.

Dr. Samuel Kaplan of Princeton University has been appointed to an assistant professorship at Wayne University.

Mr. H. P. Kean has been appointed to an associate professorship at McMurry College, Abilene, Texas.

Dr. L. M. Kelly of the University of Missouri has been appointed to an assistant professorship at Michigan State College.

Assistant Professor W. W. Kester of Yakima Valley Junior College, Yakima, Washington, has been appointed to an assistant professorship at Southwestern University, Georgetown, Texas.

Assistant Professor G. L. Kreezer of Cornell University has been appointed to an associate professorship at Washington University.

Professor J. P. LaSalle of the University of Notre Dame has been appointed lecturer at Princeton University.

Mr. Benjamin Lepson has been appointed a lecturer at Columbia University.

Dr. B. J. Lockhart of the University of Michigan has been appointed to an assistant professorship at the U. S. Naval Postgraduate School.

Assistant Professor A. T. Lonseth of Northwestern University has been appointed to an associate professorship at Oregon State College.

Professor E. R. Lorch is on leave from Barnard College, Columbia University, and will be a visiting professor at Carnegie Institute of Technology for the second semester.

Professor A. W. McGaughey of Westminister College, New Wilmington, Pennsylvania, has been appointed to an associate professorship at Bradley University, Peoria, Illinois.

Professor H. M. MacNeille of Kenyon College has accepted the position of Chief of the Fundamental Research Branch of the United States Atomic Energy Commission.

Professor M. S. Macphail of Acadia University, Wolfville, Nova Scotia, Canada, has been appointed to an associate professorship at Carleton College, Ottawa, Ontario, Canada.

Mr. A. G. Makarov of the University of Pennsylvania has been appointed to an assistant professorship at Rutgers University.

Dr. C. G. Maple of Carnegie Institute of Technology has been appointed to an associate professorship at the North Texas State College.

Assistant Professor F. J. Massey of the University of Maryland has been appointed to an assistant professorship at the University of Oregon.

Mr. C. C. Miesse of Brown University has accepted a position as Research Engineer at the Battelle Memorial Institute, Columbus, Ohio.

Dr. S. H. Min of the Institute for Advanced Study has been appointed to an assistant professorship at the Tsing-Hua University, Peiping, China.

Professor L. I. Mishoe of Delaware State College has been appointed to an associate professorship of physics at Morgan State College, Baltimore, Maryland.

Assistant Professor Josephine M. Mitchell of Oklahoma Agricultural and Mechanical College has been appointed to an assistant professorship at the University of Illinois.

Associate Professor Deane Montgomery of Yale University has been appointed a permanent member in the Institute for Advanced Study.

Dr. F. M. Morgan former Headmaster of Clark School, Hanover, New Hampshire, has been appointed Treasurer of Clark School Foundation.

Assistant Professor W. B. Morgan of Bluffton College, Bluffton, Ohio, has been appointed to an assistant professorship at Wilmington College, Wilmington, Ohio.

Rev. T. F. Mulcrone, S.J. has been appointed to an assistant professorship at Spring Hill College, Spring Hill, Alabama.

Professor F. D. Murnaghan is on leave from Johns Hopkins University and will be a visiting professor at Carnegie Institute of Technology for the first semester.

Dr. E. D. Nering of Rutgers University has been appointed to an assistant professorship at the University of Minnesota.

Professor C. V. Newsom of Oberlin College has accepted the position of Assistant Commissioner for Higher Education for the State of New York.

Dr. E. N. Nilson of the United Aircraft Corporation has been appointed to an assistant professorship at Trinity College, Hartford, Connecticut.

Mr. Lawrence Norwood of Yale University has accepted a position as mathematician with the Evans Signal Corps Laboratory, Fort Monmouth, New Jersey.

Dr. T. E. Oberbeck of the California Institute of Technology has accepted a position as a scientific analyst with the Operations Evaluation Group at the Massachusetts Institute of Technology.

Assistant Professor Izaak Opatowski of the University of Michigan has been appointed to an assistant professorship at the University of Chicago.

Dr. Mary H. Payne of the University of Detroit has been appointed to an assistant professorship at Michigan State College.

Dr. L. G. Peck of New York University has been appointed to an assistant professorship at the Johns Hopkins University.

Mr. R. P. Peterson of the University of California at Los Angeles has accepted a position as mathematician at the Institute for Numerical Analysis, University of California at Los Angeles.

Associate Professor J. A. Pierce of Wayne University has resigned.

Rev. Albeni Poitras of St. Joseph's University, St. Joseph's, New Brunswick, Canada has been made dean of studies.

Mr. F. H. Pott of the University of Wisconsin has accepted a position as Aeronautical Research Scientist with the National Advisory Committee for Aeronautics, Cleveland, Ohio.

Mr. H. B. Prendergast of the University of Chicago has been appointed lecturer at the University of Manitoba.

Professor J. F. Randolph of Oberlin College has been appointed to a professorship at the University of Rochester.

Professor W. D. Reeve of Teachers College, Columbia University, has retired.

Dr. Olav Reiersol of the University of California has been appointed lecturer in statistics at Purdue University.

Dr. Irving Reiner of the Institute for Advanced Study has been appointed to an assistant professorship at the University of Illinois.

Mr. J. D. Riley of the University of Kansas has accepted a position as mathematician with the Naval Research Laboratory, Washington, D. C.

Professor R. F. Reinhart of Case Institute of Technology has accepted a position as director of the planning division of the Research and Development Board, Washington, D. C.

Mr. I. H. Rose of Pennsylvania State College has been appointed to an assistant professorship at the University of Massachusetts.

Mr. Ira Rosenbaum of Brooklyn College has been appointed to an assistant professorship at the University of Miami.

Dr. M. F. Rosskopf of John Burroughs School, Clayton, Missouri, has been appointed to an associate professorship at Syracuse University.

Assistant Professor C. W. Saalfrank of Franklin and Marshall College has been appointed to an assistant professorship at Rutgers University.

Mr. Max Sasuly has accepted a position as research associate with the Robinson Foundation, New York, New York.

Mr. C. H. Savit of the California Institute of Technology has accepted a position as mathematician with the Western Geophysical Company, Los Angeles, California.

Associate Professor Henry Scheffé of the University of California at Los Angeles has been appointed to an associate professorship at Columbia University.

Assistant Professor A. E. Schild of Carnegie Institute of Technology has been appointed to an assistant professorship at the University of Toronto.

Dr. Lowell Schoenfeld of Harvard University has been appointed to an assistant professorship at the University of Illinois.

Dr. Atle Selberg of the Institute for Advanced Study has been appointed to an associate professorship at Syracuse University.

Associate Professor M. F. Smiley of Northwestern University has been appointed to a professorship at the State University of Iowa.

Mr. W. N. Smith of the University of Wisconsin has been appointed to an assistant professorship at the University of Wyoming.

Mr. M. D. Springer of the University of Illinois has been appointed to an assistant professorship at Michigan State College.

Mr. M. L. Stein of the University of California at Los Angeles has accepted a position as mathematician at the Institute for Numerical Analysis, University of California at Los Angeles.

Mr. Marvin Stippes of the University of Washington has been appointed to an assistant professorship at Montana State College.

Dean Ellen C. Stokes of New York State College for Teachers has been made director of summer session.

Dean Elijah Swift of the University of Vermont has retired.

Dr. R. C. Taliaferro of St. Johns College, Annapolis, Maryland, has been made master of the Portsmouth Priory School, Portsmouth, Rhode Island.

Assistant Professor W. B. Temple of the Agricultural and Mechanical College of Texas has been appointed to an associate professorship at the Louisiana Polytechnic Institute.

Dr. C. A. Truesdell has been appointed to an associate professorship at the University of Maryland.

Dr. L. I. Wade of Duke University has been appointed to a professorship at the Louisiana State University.

Professor André Weil has been appointed to a professorship at the University of Chicago.

Professor Mary Evelyn Wells of Vassar College has retired.

Dr. A. L. Whiteman of the Office of Chief of Naval Operations, Washington, D. C. has been appointed to an assistant professorship at the University of Southern California.

Mr. Roger Williams has accepted a position as Assistant Editor in the Chemical Engineering Division of McGraw-Hill Publishing Company.

Dr. James Wolfe has been appointed to an assistant professorship at the University of Utah.

The following promotions are announced:

R. G. Archibald, Queens College, to a professorship.

Miriam C. Ayer, Wellesley College, to an assistant professorship.

Frederick Bagemihl, University of Rochester, to an assistant professorship.

Valentine Bargmann, Princeton University, to an associate professorship.

R. A. Baumont, University of Washington, to an associate professorship.

F. P. Beer, Lehigh University, to an associate professorship.

Alice K. Bell, Fresno State College, to an associate professorship.

P. O. Bell, University of Kansas, to a professorship.

Arthur Bernhart, University of Oklahoma, to an associate professorship.

L. G. Black, Purdue University, to an assistant professorship.

Olga Bocanegra-Saldaña, University of Puerto Rico, to an assistant professorship.

Stanley Bolks, Purdue University, to an assistant professorship.

C. B. Boyer, Brooklyn College, to an associate professorship.

Angeline J. Brandt, Wheaton College, Wheaton, Illinois, to an associate professorship.

Robert Breusch, Amherst College, to an associate professorship.

H. W. Brinkmann, Swarthmore College, to a professorship.

H. D. Brunk, Rice Institute, to an assistant professorship.

J. C. Burke, University of Notre Dame, to an assistant professorship.

J. D. Campbell, Rensselaer Polytechnic Institute, to an associate professorship.

M. I. Chernofsky, City College, New York City, to a lectureship.

L. M. Court, Rutgers University, to an assistant professorship.

H. S. M. Coxeter, University of Toronto, to a professorship.

K. W. Crain, Purdue University, to an assistant professorship.

Margaret R. Davis, Southeastern Louisiana College, to a professorship.

M. W. DeJonge, Purdue University, to an assistant professorship.

R. C. H. Dieckmann, San Jose State College, to an associate professorship.

L. F. Doty, University of Cincinnati, to an assistant professorship.

Howard Eves, Oregon State College, to an associate professorship.

M. Celine Fasenmyer, Mercyhurst College, Erie, Pennsylvania, to an assistant professorship.

J. M. Feld, Queens College, to an assistant professorship.

W. A. Ferguson, University of Illinois, to an assistant professorship.

R. H. Fox, Princeton University to an associate professorship.

J. E. Freund, Alfred University, to an associate professorship.

R. E. Fullerton, University of Wisconsin, to an assistant professorship.

H. A. Garabedian, John Hancock Mutual Life Insurance Company, to a vice presidency.

Landis Gephart, University of Dayton, to an assistant professorship.

P. W. Gilbert, Syracuse University, to an assistant professorship.

B. E. Gillam, Drake University, to a professorship.

M. T. Goodrich, Keene Teachers College, Keene, New Hampshire, to an assistant professorship.

Harold Grad, New York University, to an assistant professorship.

R. E. Graves, University of Minnesota, to an assistant professorship.

J. W. Green, University of California at Los Angeles, to an associate professorship.

A. O. Groth, Equitable Life Insurance Company of Iowa, to an associate actuary.

G. B. Haggerty, Rhode Island State College, to an assistant professorship.

Edwin Halfar, University of Nebraska, to an assistant professorship.

Marshall Hall, Ohio State University, to a professorship.

G. J. Haltiner, United States Naval Postgraduate School, Monterey, California, to an associate professorship of aerology.

R. G. Helsel, Ohio State University, to an associate professorship.

F. F. Helton, Central College, Fayette, Missouri, to a professorship.

Erik Hemmingsen, Syracuse University, to an assistant professorship.

Manuel Herschdorfer, Seton Hall College, South Orange, New Jersey, to a professorship.

P. G. Hoel, University of California at Los Angeles, to a professorship.

Alfred Horn, University of California at Los Angeles, to an assistant professorship.

P. M. Hummel, University of Alabama, to a professorship.

L. C. Hutchinson, Polytechnic Institute of Brooklyn, to an associate professorship.

P. C. Irwin, Equitable Life Insurance Company of Iowa, to vice president and actuary.

S. B. Jackson, University of Maryland, to a professorship.

S. A. Jennings, University of British Columbia, to a professorship.

R. N. Johanson, Boston University, to an associate professorship.

Evan Johnson, Pennsylvania State College, to a professorship.

A. W. Jones, Rensselaer Polytechnic Institute, to an associate professorship.

P. S. Jones, University of Michigan, to an assistant professorship.

A. D. Kantz, Southwestern College, Winfield, Kansas, to an assistant professorship.

D. E. Kibbey, Syracuse University, to an associate professorship.

A. L. King, Dartmouth College, to a professorship of physics.

Fred Kiokemeister, Mount Holyoke College, to an associate professorship.

E. R. Kolchin, Columbia University, to an assistant professorship.

Horace Komm, University of Rochester, to an assistant professorship.

H. L. Krall, Pennsylvania State College, to a professorship.

R. R. Kuebler, Dickinson College, Carlisle, Pennsylvania, to an associate professorship.

Joseph Landin, University of Illinois, to an assistant professorship.

W. G. Leavitt, University of Nebraska, to an assistant professorship.

P. E. Lewis, North Carolina State College, to an associate professorship.

R. C. McCankie, Equitable Life Insurance Company of Iowa, to vice president and actuary.

J. J. McCarthy, Newark College of Arts and Sciences, Rutgers University, to an assistant professorship in physics.

Dora McFarland, University of Oklahoma, to a professorship.

G. W. Mackey, Harvard University, to an associate professorship.

W. D. MacKinnon, Equitable Life Insurance Company of Iowa, to actuary.

W. G. Madow, University of North Carolina, to a professorship.

Dis Maly, Rensselaer Polytechnic Institute, to an associate professorship.

H. B. Mann, Ohio State University, to a professorship.

Murray Mannos, Johns Hopkins University, to an assistant professorship.

Kenneth May, Carleton College, Northfield, Minnesota, to an associate professorship.

L. E. Mehlenbacher, University of Detroit, to a professorship.

B. E. Meserve, University of Illinois, to an assistant professorship.

H. L. Meyer, University of Chicago, to an assistant professorship.

E. J. Mickle, Ohio State University, to an associate professorship.

R. L. Moenter, Midland College, Freemont, Nebraska, to an associate professorship.

E. W. Montroll, University of Pittsburgh, to an associate professorship.

B. C. Moore, Agricultural and Mechanical College of Texas, to an associate professorship.

R. C. Morrow, United States Naval Academy, to an associate professorship.

Frederick Mosteller, Harvard University, to an associate professorship.

B. N. Moyls, University of British Columbia, to an assistant professorship.

M. E. Munroe, University of Illinois, to an assistant professorship.

S. B. Myers, University of Michigan, to a professorship.

Albert Newhouse, University of Houston, to an associate professorship.

J. A. Northcott, Columbia University, to a professorship.

Catherine M. O'Brien, College of Mount Saint Vincent, New York, New York, to a deanship.

P. W. Overman, Purdue University, to an assistant professorship.

S. T. Parker, Kansas State College, to an associate professorship.

Edward Paulson, University of Washington, to an assistant professorship.

A. S. Peters, New York University, to an associate professorship.

J. W. Ponds, West Virginia University, to an assistant professorship.

D. H. Potts, Northwestern University, to an assistant professorship.

W. T. Puckett, University of California at Los Angeles, to an associate professorship.

C. F. Rehberg, New York University, to an associate professorship in electrical engineering.

W. P. Reid, Purdue University, to an assistant professorship.

I. F. Ritter, New York University, to an associate professorship.

S. L. Robinson, City College, New York City, to an assistant professorship.

Edward Rosenthall, McGill University, to an associate professorship.

D. R. Ryan, Gonzaga University, Spokane, Washington to an assistant professorship.

Edward Saibel, Carnegie Institute of Technology, to a professorship in mechanics.

S. S. Saslaw, United States Naval Academy, to an associate professorship.

R. B. Saunders, Oregon State College, to an associate professorship.

I. J. Schoenberg, University of Pennsylvania, to a professorship.

M. E. Shanks, Purdue University, to an associate professorship.

Marlow Sholander, Washington University, to an assistant professorship.

W. H. Simons, University of British Columbia, to an assistant professorship.

Ernst Snapper, University of Southern California, to an associate professorship.

D. W. Starr, Southern Methodist University, to a professorship.

J. C. Stewart, Lawrence College, to an associate professorship.

Alexander Tartler, Drexel Institute of Technology, to a professorship.

William Charles Taylor, University of Tennessee Junior College, to an assistant professorship in basic engineering.

D. L. Thomsen, Haverford College, to an assistant professorship.

R. M. Thrall, University of Michigan, to an associate professorship.

G. R. Trott, University of Mississippi, to a professorship.

Rohn Truel, Brown University, to associate professorship of physics.

J. W. Tukey, Princeton University, to an associate professorship.

A. R. Turquette, University of Illinois, to an associate professorship in philosophy.

R. N. Varney, Washington University, to a professorship in physics.

Bernard Vinograde, Iowa State College of Agriculture and Mechanic Arts, to an associate professorship.

R. J. Walker, Cornell University, to a professorship.

Eleanor B. Walters, Delta State Teachers College, to an associate professorship.

C. T. Wang, New York University, to an associate professorship in aeronautical engineering.

J. A. Ward, University of Georgia to a professorship.

W. G. Warnock, Rensselaer Polytechnic Institute, to a professorship.

D. L. Webb, University of Arizona, to an associate professorship.

G. C. Webber, University of Delaware, to a professorship.

C. T. West, Ohio State University, to an assistant professorship in mechanics.

P. A. White, University of Southern California, to an associate professorship.

John Williamson, Queens College, to a professorship.

R. L. Wilson, University of Tennessee, to an assistant professorship.

Marie A. Wurster, Temple University, to an assistant professorship

J. W. T. Youngs, Indiana University, to a professorship.

The following appointments to instructorships are announced: University of Arkansas: Mr. W. C. Guenther; Boston University: Dr. J. B. Giever; Bowling Green State University: Mr. M. R. Moore; Brooklyn College: Mr. J. B. Secrist; University of Buffalo: Miss Janet E. Abbey; Carnegie Institute of Technology: Mr. E. L. Kaplan; Case Institute of Technology: Mr. Deonisie Trifan; University of Connecticut: Mr. W. J. Widmer; City College, New York City: Mr. J. S. Dwork, Mr. David Singer; Columbia University: Barnard College: Dr. Hing Tong; Cornell University: Mr. M. D. Donsker; Duke University: Dr. T. D. Reynolds; Emory University, Emory University, Georgia: Mr. C. R. Partington; University of Florida: Mr. R. W. Young; Georgetown University: Mr. S. G. Reed; Harvard University: Dr. Oscar Goldman, Dr. R. M. Redheffer; Hobart College, Geneva, New York: Mr. R. L. Beinert; Hunter College: Miss Alice Osterberg; University of Illinois: Dr. M. K. Fort, Mr. N. C. Scholomiti; University of Illinois, Undergraduate Division, Chicago: Mr. W. R. Allen; Illinois Institute of Technology: Mr. R. P. Brady, Mr. H. J. Curtis, Dr. M. H. M. Esser; Iowa State Teachers College: Miss

Dorothy M. DeWitt; Johns Hopkins University: Dr. E. A. Coddington; University of Kansas: Dr. I. N. Hernstein, Dr. G. K. Overholtzer; Los Angeles City College: Mr. J. B. Nelson; University of Maryland: Mr. C. V. G. Gorciu, Mr. H. W. Stephens, Mrs. H. A. Thorpe; Massachusetts Institute of Technology: Mr. J. D. Newburgh, Mr. Gordon Raisbeck; Michigan State College: Mr. Leo Lapidus; University of Michigan: Mr. Frank Harary, Dr. W. R. Scott, Dr. R. G. Stanton; New York University: Mr. F. V. Pohle; Northwestern University: Mr. R. H. Lence; Ohio State University: Mr. R. S. Gardner, Mr. J. L. Moll, Mr. H. C. Parrish; Ohio University, Athens, Ohio: Miss Marion Treon; University of Oregon: Dr. C. E. Bures, Mr. W. L. Shepherd; University of Pennsylvania: Dr. R. D. Anderson, Mr. Hyman Kamel; Polytechnic Institute of Brooklyn: Mr. M. S. Klamkin; Princeton University: Mr. L. L. Rauch; Purdue University: Miss Marianne Bernstein, Mr. L. D. Kovach; Rutgers University: Mr. W. W. Boone; Smith College: Dr. Mary K. Peabody; Southern University: Miss Willie B. Campbell; Spring Hill College, Spring Hill, Alabama: Rev. John Moore; Syracuse University: Mr. Helmut Aulbach, Mr. S. G. Campbell, Dr. Eckford Cohen, Mr. Daniel Resch, Dr. C. J. Titus; Temple University: Mr. Joel Pitcairn; University of Texas: Mr. J. M. Hurt, Mr. J. S. MacNerney; Trenton Junior College, Trenton, New Jersey: Mr. A. H. Ferester; Tulane University of Louisiana: Dr. F. E. Clark; University of Utah: Mr. Floyd E. Haupt; Vassar College: Dr. Mary P. Dolciani; University of Vermont: Mr. A. L. Wilson; Wayne University: Mr. Willard Clatworthy; West Virginia University: Mr. E. H. Crisler, Miss Helen S. Pollock; Williams College: Mr. A. F. Bausch; University of Washington: Dr. R. W. Ball; Yale University: Dr. J. G. Wendel; Yeshiva University: Mr. Daniel Block.

Professor Mae R. Anderson of Concordia College died March 28, 1948.

Professor Emeritus C. E. Comstock of Bradley University died April 3, 1948. He had been a member of the Society for forty-five years.

Mr. Lewis Greenwald of the Evans Signal Laboratory died March 14, 1948, at the age of twenty-nine years.

Assistant Professor T. R. Long of the University of Rochester died July 10, 1948. He had been a member of the Society for twenty-six years.

Professor Emeritus F. R. Sharpe of Cornell University died on May 18, 1948, at the age of seventy-eight years. He had been a member of the Society for forty-three years.

by interchanging the order of summation and by observing that the number of b's satisfying $1+c_1>b/a>1-c_1$ and $b\equiv 0 \pmod{m}$ is less than $1+(2\cdot c_1\cdot a/m)$. The same holds for the b's satisfying $1+c_1>b/a>1-c_1$ and $a+b\equiv 0 \pmod{m}$. ($v(m)$ denotes the number of prime factors of m.) Thus

$$\begin{aligned}\sum{}' \prod_{q\mid a\cdot b\cdot(a+b)}\left(1+\frac{3}{q}\right) &< c_6c_1 \sum_1 a \prod_{q\mid a}\left(1+\frac{3}{q}\right)\\ &< 20c_6c_1 \log n \sum_1 \prod_{q\mid a}\left(1+\frac{3}{q}\right)\\ &< 20c_6c_1 \log n \sum_{m=1}^{\infty} \frac{(20 \log n)3^{v(m)}}{m^2}\\ &< c_7c_1(\log n)^2 < \frac{1}{10c_5'}(\log n)^2\end{aligned}$$

if $c_1<1/10\cdot c_7\cdot c_5'$. Hence finally from (8) the number of solutions of (5) is less than

$$n/10 + n/10 < n/4,$$

which proves Lemma 1.

LEMMA 2. *There exists a constant c_8 so that the number of integers $k\leqq n$ satisfying*

$$(9) \qquad d_{k+1}/d_k > t \quad \text{or} \quad d_{k+1}/d_k < 1/t$$

is less than $c_8\cdot n/t^{1/2}$.

It suffices to prove the lemma for large t. We split the integers k satisfying (9) into two not necessarily disjoint classes. In the first class are the k's for which either

$$d_k \geqq t^{1/2}\cdot\log n \quad \text{or} \quad d_{k+1} \geqq t^{1/2}\cdot \log n.$$

In the second class are the k's for which either

$$d_k \leqq (\log n)/t^{1/2} \quad \text{or} \quad d_{k+1} \leqq (\log n)/t^{1/2}.$$

Clearly if (9) is satisfied then k is in one of these classes.

We obtain from $p_n<2\cdot n\cdot\log n$ that the number of k's of the first class is less than $4\cdot n/t^{1/2}$.

As in the proof of Lemma 1 we obtain from our result proved in a previous paper[3] that the number Z of solutions of $d_u=a$, $u\leqq n$ is less than we have

$$U < c_4 n \log n \prod_q \left(1 - \frac{3}{q}\right), \quad q < n^\alpha, \qquad q \nmid a\cdot b\cdot (a+b).$$

It is well known that[4]

$$\prod_{q<x} \left(1 - \frac{3}{q}\right) < \frac{c}{(\log x)^3} \quad \text{and} \quad \prod_q \left(1 - \frac{q}{q^2}\right) > 0.$$

Thus

$$U < c_5 \frac{n}{(\log n)^2} \prod_q \left(1 + \frac{3}{q}\right), \qquad q \mid a\cdot b\cdot (a+b).$$

Hence finally from (6) and $V \leqq U + n^\alpha$,

$$g(n;\, a,\, b) < c_5' \frac{n}{(\log n)^2} \prod_q \left(1 + \frac{3}{q}\right), \qquad q \mid a\cdot b\cdot (a+b). \tag{7}$$

Now we split the k's satisfying (5) into two classes. In the first class put the k's with $d_k > 20 \log n$ and in the second class the other k's. From $p_n < 2\cdot n\cdot \log n$ we deduce that the number of k's of the first class is less than $n/10$.

The number of the k's of the second class is not greater than

$$\sum{}' g(n;\, a,\, b) < c_5' \frac{n}{(\log n)^2} \sum{}' \prod_q \left(1 + \frac{3}{q}\right), \qquad q \mid a\cdot b\cdot (a+b), \tag{8}$$

where the prime indicates that the summation is extended over those a and b with $a < 20\cdot \log n$, $1 + c_1 > b/a > 1 - c_1$. Now

$$\sum{}' \prod_{q \mid a\cdot b(a+b)} \left(1 + \frac{3}{q}\right) \leqq \sum_1 \left(\prod_{q \mid a} \left(1 + \frac{3}{q}\right) \sum_2 \prod_{q \mid b(a+b)} \left(1 + \frac{3}{q}\right)\right.$$

where in $\sum_1$, $a < 20 \log n$ and in $\sum_2$, $1 + c_1 > b/a > 1 - c_1$. We have

$$\begin{aligned} \sum_2 \prod_{q \mid b\,(a+b)} \left(1 + \frac{3}{q}\right) &< \sum_2 \left(\prod_{q \mid b} \left(1 + \frac{3}{q}\right)^2 + \prod_{q \mid a+b} \left(1 + \frac{3}{q}\right)^2\right) \\ &< \sum_2 \left(\prod_{q \mid b} \left(1 + \frac{15}{q}\right) + \prod_{q \mid a+b} \left(1 + \frac{15}{q}\right)\right) \\ &< \sum_{m<3a} 2\left(1 + \frac{2c_1 a}{m}\right) \frac{15^{V(m)}}{m} < c_6 c_1 a, \end{aligned}$$

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[4] See, for example, Hardy-Wright, p. 349.

$$\limsup d_{k+1}/d_k > 1, \qquad \liminf d_{k+1}/d_k < 1.$$

At present I can not decide whether $d_{k+2} > d_{k+1} > d_k$ has infinitely many solutions. The following question might be of some interest: Let $\epsilon_n = 1$ if $d_{n+1} > d_n$, otherwise $\epsilon_n = 0$. It may be conjectured that $\sum_{n=1}^{\infty} \epsilon_n/2^n$ is irrational. I can not even prove that from a certain point on ϵ_n is not alternatively 1 and 0.

In order to prove Theorem 1 we need two lemmas.

LEMMA 1. *For sufficiently small* $c_1 > 0$ *the number of solutions in* k *of the inequalities*

$$1 + c_1 > d_{k+1}/d_k > 1 - c_1, \qquad k \leqq n, \tag{5}$$

is less than $n/4$.

Denote by $g(n; a, b)$ the number of solutions of the simultaneous equations

$$d_{k+1} = a, \qquad d_k = b, \qquad k \leqq n.$$

Denote by V the number of primes $r < 2 \cdot n \cdot \log n$ for which $r+a$ and $r+a+b$ are also primes. Since $p_n < 2 \cdot n \cdot \log n$, we evidently have

$$g(n; a, b) \leqq V. \tag{6}$$

Now let $c_1 > 0$ be sufficiently small and $q_1, q_2, \cdots$ run through the primes less than n^{c_2}. Then V is not greater than n^{c_2} plus the number U of integers $m < 2 \cdot n \cdot \log n$, which satisfy, for all i,

$$m \not\equiv 0 \pmod{q_i}, \qquad m \not\equiv -a \pmod{q_i}, \qquad m \not\equiv -(a+b) \pmod{q_i}.$$

If $q \nmid a \cdot b \cdot (a+b)$ then these three residues are all different. In a previous paper[3] I stated the following theorem: Let $q_1, q_2, \cdots$ be primes all less than n^{c_2}. Associate with each q_i t distinct residues $r_1^{(i)}, \cdots, r_t^{(i)}$. Then the number of integers $m \leqq n$ for which

$$m \not\equiv r_j^{(i)} \pmod{q_i}, \qquad j = 1, 2, \cdots, t;\ i = 1, 2, \cdots,$$

is less than

$$cn \prod_i \left(1 - \frac{t}{q_i}\right).$$

The proof of this theorem follows easily from Brun's method.[3] Thus

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[3] P. Erdös, Proc. Cambridge Philos. Soc. vol. 33 (1937) p. 8, Lemma 2. A book of Rosser and Harrington on Brun's method will soon appear which will contain a detail proof of this result.

ON THE DIFFERENCE OF CONSECUTIVE PRIMES

P. ERDÖS

The present paper contains some elementary results on the difference of consecutive primes. Theorem 2 has been announced in a previous paper.[1] Also some unsolved problems are stated.

Let $p_1=2$, $p_2=3, \cdots, p_k, \cdots$ be the sequence of consecutive primes. Put $d_k=p_{k+1}-p_k$. We have:

THEOREM 1. *There exist positive real numbers* c_1 *and* c_2, $c_1<1, c_2<1$, *such that for every* n *the number of* k*'s satisfying both*

$$(1) \qquad d_{k+1} > (1+c_1)d_k, \qquad k \leqq n,$$

and the number of l*'s satisfying both*

$$(2) \qquad d_{l+1} < (1-c_1)d_l, \qquad l \leqq n,$$

are each greater than $c_2 n$.

We shall prove Theorem 1 later. From Theorem 1 we easily deduce:

THEOREM 2. *For every* t *and all sufficiently large* n *the number of solutions in* k *and* l *of each of the two sets of inequalities*

$$(3) \qquad \left(\frac{p_{k+1}^t + p_{k-1}^t}{2}\right)^{1/t} > p_k, \ k \leqq n; \qquad \left(\frac{p_{l+1}^t + p_{l-1}^t}{2}\right)^{1/t} < p_l, \ l \leqq n,$$

is greater than $(c_2/2)n$.

Let ϵ be sufficiently small but fixed. It is well known that $p_n<2\cdot n\cdot\log n$. Thus the number of $k\leqq n$, with $p_{k+1}>(1+\epsilon)p_k$, is less than $c \log n$. Hence it follows from Theorem 1 that the number of k's satisfying

$$(4) \qquad p_{k+1} < (1+\epsilon)p_k, \qquad d_k > (1+c_1)d_{k-1}, \qquad k \leqq n,$$

is greater than $(c_2/2)n$. A simple calculation now shows that the primes satisfying (4) also satisfy the first inequality of (3) if $\epsilon=\epsilon(c_1)$ is chosen small enough. The second inequality of (3) is proved in the same way, which proves Theorem 2.

Further, we obtain, as an immediate corollary of Theorem 1, that[2]

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[1] P. Erdös and P. Turán, *Some new questions on the distribution of primes*, Bull. mer. Math. Soc. vol. 54 (1948) pp. 371–378.

This result was also stated in the above paper.

$$Z < c_9 n \log n \prod_{q} \left(1 - \frac{2}{q}\right), \qquad q \nmid a,\ q < n^{c_7}.$$

Thus as in Lemma 1

$$Z < c_{10} \frac{n}{\log n} \prod_{p \mid a} \left(1 + \frac{2}{p}\right).$$

Thus the number of k's of the second class is less than

$$2c_{10} \frac{n}{\log n} \sum_{a < \log n / t^{1/2}} \prod_{p \mid a} \left(1 + \frac{2}{p}\right) < 2c_{10} \frac{n}{\log n} \sum_{d=1}^{\infty} \frac{(\log n) 2^{\nu(d)}}{t^{1/2} d^2}$$

$$< \frac{c_{11} n}{t^{1/2}},$$

which proves Lemma 2, with $c_8 = 2 + c_{11}$.

Now we can prove Theorem 1. It will suffice to prove (1). Suppose that (1) is not true. Then for every $c_1 > 0$ and $\epsilon > 0$ there exists an arbitrarily large n so that the number of solutions of

$$d_{k+1} > (1 + c_1) d_k \tag{10}$$

is less than $\epsilon \cdot n$. Consider the product

$$\frac{d_n}{d_1} = \frac{d_2}{d_1} \cdot \frac{d_3}{d_2} \cdots \frac{d_n}{d_{n-1}}.$$

By Lemma 2 the number of $k \leqq n$ satisfying $d_{k+1}/d_k > 2^{2l}$ is less than $c_8 n / 2^l$. Thus by Lemma 1 and (10) we have for every u

$$d_n/d_1 < 2^{2u\epsilon n} \prod_{l \geqq u} (2^{2l})^{c_8 n / 2^l} \cdot (1 + c_1)^{n/4} (1 - c_1)^{n/2}$$

$$< 2^{2u\epsilon n} \exp \sum_{l \geqq u} \frac{c_8 l n \log 4}{2^l} \cdot (1 - c_1)^{n/4}.$$

If ϵ is sufficiently small there is a suitable choice of u such that $2^{2u\epsilon n} < (1 + c_1)^{n/8}$ and

$$\exp \sum_{l \geqq u} \frac{c_8 l n \log 4}{2^l} < (1 + c_1)^{n/8}.$$

Thus $d_n/d_1 < (1 - c_1^2)^{n/4} < 1/n$ for arbitrarily large n, an evident contradiction. This proves (1) and completes the proof of Theorem 1.

SYRACUSE UNIVERSITY

NOTE ON A PROBLEM IN NUMBER THEORY

HAROLD N. SHAPIRO

The problem which we shall consider originated from a conjecture of S. Ulam. For x, p, integers, p a prime, let $x \equiv a \pmod{p}$ where $-p/2 < a < p/2$; and define $\|x\|_p = |a|$. Then if $T(x)$ is a mapping of the nonzero residues modulo p into themselves, we consider the following "approximate multiplicative relation" modulo p,

$$\|T(xy) - T(x)T(y)\|_p < k \tag{1}$$

where k is a fixed integer. The problem is to ascertain simple conditions under which the only solutions to (1) are given by

$$T(x) \equiv x^a \pmod{p}. \tag{2}$$

Clearly, p must be larger than k in order that this be feasible. Also, if we give to $T(x)$ any arbitrary set of integral values between 0 and $k^{1/2}$ we may obtain mappings satisfying (1) but not (2). This then indicates in a sense that the value domain of $T(x)$ must not be too small in order that (2) follow from (1).

The results obtained in this note are derived essentially from the following very simple lemma.

LEMMA. *If for $T(x)$ a mapping of a semigroup G into a ring R we define*

$$\epsilon(x, y) = T(xy) - T(x)T(y), \tag{3}$$

then for any x, y, z of G,

$$\epsilon(x, y)T(z) + \epsilon(xy, z) = T(x)\epsilon(y, z) + \epsilon(x, yz). \tag{4}$$

PROOF. For any x, y, z of G we obtain from the associativity of multiplication:

$$\begin{aligned} T(xyz) &= T(xy)T(z) + \epsilon(xy, z) \\ &= T(x)T(y)T(z) + \epsilon(x, y)T(z) + \epsilon(xy, z) \end{aligned} \tag{5}$$

and

$$\begin{aligned} T(xyz) &= T(x)T(yz) + \epsilon(x, yz) \\ &= T(x)T(y)T(z) + T(x)\epsilon(y, z) + \epsilon(x, yz). \end{aligned} \tag{6}$$

Comparing (5) and (6) yields (4).

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We note that in case both G and R are commutative, as they will be in our applications of the lemma, we have $\epsilon(x, y) = \epsilon(y, x)$ and we may write (4) as

$$\epsilon(x, y)T(z) + \epsilon(z, xy) = \epsilon(y, z)T(x) + \epsilon(x, yz). \tag{7}$$

To begin with we shall consider the case where $T(x)$ is a mapping of the residues modulo n, which are prime to n, into themselves, where n is not necessarily prime. In this case we have the following theorem.

THEOREM 1. *If* (1) $\|T(xy) - T(x)T(y)\|_n < k$, (2) $k < \min_{p|n} p$, *and* (3) $T(x)$ *takes on more than* $8k^2$ *distinct values, then*

$$T(xy) \equiv T(x)T(y) \pmod{n}. \tag{8}$$

PROOF. Suppose that, for some two integers x_1, x_2, (8) does not hold, so that

$$T(x_1x_2) \equiv T(x_1)T(x_2) + \epsilon(x_1, x_2) \pmod{n}$$

where

$$\|\epsilon(x_1, x_2)\|_n < k \tag{9}$$

and $\epsilon(x_1, x_2) \not\equiv 0 \pmod{n}$.

We now apply the lemma to the case where G is the group of residues mod n which are prime to n, and R all residues. We get for any integer y,

$$\epsilon(x_1, x_2)T(y) + \epsilon(y, x_1x_2) \equiv \epsilon(x_2, y)T(x_1) + \epsilon(x_1, x_2y) \tag{10}$$

where from hypothesis (1) we have $\|\epsilon(y, x_1x_2)\|_n < k$, $\|\epsilon(x_2, y\|_n < k$, and $\|\epsilon(x_1, x_2y)\|_n < k$. From the hypothesis $k < \min_{p|n} p$ so that by (9) we see that $(\epsilon(x_1, x_2), n) = 1$. Thus having fixed x_1, x_2, $\epsilon(x_1, x_2)$ is fixed, and (10) determines the value of $T(y)$ uniquely, modulo n. Clearly then $T(y)$ could take on at most $8k^2$ distinct values, whence the theorem.

For the special case where $n = p^\alpha$, the power of an odd prime we get the following theorem.

THEOREM 2. *If* (1) $\|T(xy) - T(x)T(y)\|_{p^\alpha} < k$, (2) $k < p$, *and* (3) $T(x)$ *takes on more than* $8k^2$ *distinct values,*

$$T(x) \equiv x^a \pmod{p^\alpha}.$$

PROOF. From Theorem 1 we see that $T(xy) \equiv T(x)T(y) \pmod{p^\alpha}$. Then if ρ is a primitive root mod p^α we have

$$T(x) \equiv T(\rho^s) \equiv \{T(\rho)\}^s \equiv \rho^{as} \equiv x^a \pmod{p^\alpha}.$$

COROLLARY. *If* $\|T(xy)-T(x)T(y)\|_p<k$ *and* $T(x)$ *takes on more than* $8k^2$ *distinct values, then* $T(x) \equiv x^a \pmod{p}$.

If we now restrict our attention to the case of a prime modulus we see that the above discussion covers those cases for which N = the number of distinct values in the value domain is such that $N<k^{1/2}$ or $N>8k^2$. The question then arises as to what can be said for $k^{1/2}<N \leq 8k^2$. In this direction we have:

THEOREM 3. *If* $N>8k$, *and* $\|T(xy)-T(x)T(y)\|_p<k$, *then, for* $p>8k^2$, N *divides* $p-1$.

PROOF. (a) We first note that $T(1)=1$. For we have for any x

$$T(x)(1-T(1)) = \epsilon(1, x), \qquad \|\epsilon(1, x)\|_p < k,$$

so that if $T(1)\neq 1$, $T(x)$ could take on at most $2k$ values.

(b) If $N>8k^2$ we have already proved above that $T(x) \equiv x^a \pmod{p}$ and it follows in this case that $N \mid (p-1)$.

(c) It remains only to consider the case $8k<M\leq 8k^2$.
Since $N\leq 8k^2$ there must be a value c such that $T(x)=c$ for more than $(p-1)/N$ different x. Let $z_1, \cdots, z_\Delta$, $\Delta\geq(p-1)/N$, be the numbers such that $T(z_i)=c$, $i=1, \cdots, \Delta$. Then for p sufficiently large (for example, $p>8k^2$) $\Delta>1$, and we have from (7), for $i\neq j$,

$$\epsilon(x, y)T(z_i) + \epsilon(z_i, xy) \equiv \epsilon(x, z_i)T(y) + \epsilon(y, z_ix), \tag{11}$$

$$\epsilon(x, y)T(z_j) + \epsilon(z_j, xy) \equiv \epsilon(x, z_j)T(y) + \epsilon(y, z_jx). \tag{12}$$

Subtracting (11) and (12) gives for *all* x, y, i, j,

$$\begin{aligned}\epsilon(z_i, xy) - \epsilon(z_j, xy) \equiv T(y)\{\epsilon(x, z_i) - \epsilon(x, z_j)\} \\ + \epsilon(y, z_ix) - \epsilon(y, z_jx).\end{aligned} \tag{13}$$

If for *some* x, i, j, $\epsilon(x, z_i)\not\equiv\epsilon(x, z_j)$, (13) implies that $T(y)$ has a value domain of not more than $8k$ values, which contradicts the hypothesis. Hence for all x; $i, j=1, \cdots, \Delta$,

$$\epsilon(x, z_i) \equiv \epsilon(x, z_j). \tag{14}$$

This gives $T(xz_i)-T(x)T(z_i) \equiv T(xz_j)-T(x)T(z_j)$ or

$$T(xz_i) = T(xz_j). \tag{15}$$

Replacing x by xz_i^{-1} in (15) we get for all x, $i, j=1, \cdots, \Delta$,

$$T(x) = T(xz_i^{-1}z_j). \tag{16}$$

From (16) we see that $Z = \{z_1^{-1}z_1, z_1^{-1}z_2, \cdots, z_1^{-1}z_\Delta\}$ are Δ distinct numbers such that

$$T(z_1^{-1}z_i) = 1.$$

Then if $\alpha_1, \cdots, \alpha_N$ are the N distinct values in the value domain and $x_1, \cdots, x_N$ are such that $T(x_i) = \alpha_i$, $i = 1, \cdots, N$, we see from (16) that

$$T(x_i Z) = T(x_i) = \alpha_i, \qquad i = 1, \cdots, N.$$

Hence at least Δ numbers map into each α_i, and we have $p-1 \geqq \Delta N \geqq p-1$. Thus $p-1 = N\Delta$ and $N \mid (p-1)$.

COROLLARY 1. *If $N > 8k$, $p > 8k^2$, and $\|T(xy) - T(x)T(y)\|_p < k$, then the set Z of those numbers which map into 1 is a subgroup of order $(p-1)/N$ of the group of residues prime to p, and all the elements of a given coset map into the same number.*

PROOF. To prove that Z is a group we need only show that it is closed. Both this and the remainder of the corollary is clear from (16) and Theorem 2.

COROLLARY 2. *If $N > 8k$, $p > 8k^2$, $\|T(xy) - T(x)T(y)\|_p < k$, and $p-1$ has no divisors which lie in the interval* $(8k, 8k^2)$, *then* $T(x) \equiv x^a$ (mod p).

COROLLARY 3. *If $\|T(xy) - T(x)T(y)\|_p < k$, $k > 0$, $p > 8k^2$ and, for some primitive root ρ, $T(\rho) = 1$ then $N \leqq 8k$.*

PROOF. If $N > 8k$ then the subgroup Z contains ρ and hence Z is the whole group. But this means $N = 1 > 8k$ which is impossible.

PRINCETON UNIVERSITY

A NOTE ON HILBERT'S NULLSTELLENSATZ

RICHARD BRAUER

In a recent paper, O. Zariski[1] has given a very simple proof of Hilbert's "Nullstellensatz." We give here another proof which while slightly longer is still more elementary.

Let K be an algebraically closed field. We consider a system of conditions

$$\begin{gathered} f_1(x_1, x_2, \cdots, x_n) = 0, \qquad f_2(x_1, x_2, \cdots, x_n) = 0, \\ \cdots, f_r(x_1, x_2, \cdots, x_n) = 0; \\ g(x_1, x_2, \cdots, x_n) \neq 0 \end{gathered} \tag{1}$$

where $f_1, f_2, \cdots, f_r$, and g are polynomials in n indeterminates $x_1, x_2, \cdots, x_n$ with coefficients in K. The theorem states that *if the conditions* (1) *cannot be satisfied by any values x_i of K,*[2] *a suitable power of g belongs to the ideal* $(f_1, f_2, \cdots, f_r)$.[3]

PROOF. Let k be the number of x_j which actually appear in $f_1, f_2, \cdots, f_r$ and let x_i be the x_j of this kind with the smallest subscript. Denote by l the number of f_ρ in which x_i actually appears. Let m be the smallest positive value which occurs as degree in x_i of one of the f_ρ.[4] Now define a partial order for the different systems (1) using a lexicographical arrangement. If (1*) is a second system of the same type as (1) and if k^*, l^*, and m^* have the corresponding significance, we shall say that (1*) is *lower* than (1) if either $k^* < k$, or $k^* = k$ and $l^* < l$, or $k^* = k$, $l^* = l$, and $m^* < m$.

Suppose now that Hilbert's theorem is false. Then there exist systems (1) which are not satisfied by any values x_j in K, and for which no power of g lies in $(f_1, f_2, \cdots, f_r)$. Choose such a system (1) taking it as low as possible. Then for all systems (1*) lower than (1) the theorem will hold.

If k, l, m have the same significance as above, one of the f_ρ, say

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[1] Bull. Amer. Math. Soc. vol. 53 (1947) pp. 362–368.

[2] If we wish to formulate the theorem for arbitrary fields K as it is done in Zariski's paper, we have to consider a system of values $x_1, x_2, \cdots, x_n$ belonging to extension fields of finite degree over K. If no such system satisfies the conditions (1), the same conclusion can be drawn. The same proof can be used.

[3] We do not use anything from the theory of ideals except the notation $(f_1, f_2, \cdots, f_r)$ for the set of all polynomials of the form $P_1f_1 + P_2f_2 + \cdots + P_rf_r$, $P_j \in K[x_1, x_2, \cdots, x_n]$, and facts which are immediate consequences.

[4] The numbers k, l, m do not depend on g.

f_1, has degree m in x_i. Set

$$f_1 = hx_i^m + f_1^* \tag{2}$$

where h is the highest coefficient of f_1 as polynomial in x_i.

Neither of the following systems:

$$f_1 = 0, f_2 = 0, \cdots, f_r = 0, h = 0;\ g \neq 0; \tag{3}$$

$$f_1 = 0, f_2 = 0, \cdots, f_r = 0;\ hg \neq 0 \tag{4}$$

can be satisfied by values x_j of K, since otherwise (1) would be satisfied by the same values. Replace (3) by

$$f_1^* = 0, f_2 = 0, \cdots, f_r = 0, h = 0;\ g \neq 0. \tag{3*}$$

Then (3*) too cannot be satisfied by values x_j in K. Clearly, (3*) is lower than (1). Since Hilbert's theorem then holds for (3*), we have

$$g^s \in (f_1, f_2, \cdots, f_r, h) \tag{5}$$

for a suitable exponent s.

In the discussion of (4), we distinguish two cases.

Case A. $l \geqq 2$. Then x_i appears in some f_ρ with $\rho \geqq 2$, say in f_2. Divide f_2 by f_1 considering both as polynomials in x_i alone. If we multiply by a suitable power h^q of the highest coefficient h of f_1, we can remove the denominators and set

$$h^q f_2 = Qf_1 + R$$

where Q and R are polynomials in all the x_j and where R is of degree smaller than m in x_i. The system.

$$f_1 = 0, R = 0, f_2 = 0, \cdots, f_r = 0;\ hg \neq 0 \tag{4*}$$

cannot be satisfied by any values x_j in K, since (4*) would imply (4). But (4*) is lower than (1) and hence Hilbert's theorem holds for (4*). Then, for a suitable exponent t, $(hg)^t \in (f_1, R, f_2, \cdots, f_r)$.

Replacing R by $h^q f_2 - Qf_1$, we obtain

$$h^t g^t \in (f_1, f_2, \cdots, f_r). \tag{6}$$

It follows from (5) that g^{t+st} belongs to

$$g^t(f_1, f_2, \cdots, f_r, h)^t \subseteq g^t(f_1, f_2, \cdots, f_r, h^t) \subseteq (f_1, f_2, \cdots, f_r, g^t h^t).$$

Then (6) shows that $g^{t+st} \in (f_1, f_2, \cdots, f_r)$, in contradiction to the assumption that no power of g belongs to $(f_1, f_2, \cdots, f_r)$.

Case B. $l = 1$. If we succeed again in establishing (6), we have the same contradiction as in the Case A, and Hilbert's theorem will be proved.

In this case divide g^{m+1} by f_1, considering both as polynomials in x_i alone. We may then set

$$h^q g^{m+1} = Qf_1 + R \tag{7}$$

where q is again a positive integer, where Q and R are polynomials in all the x_j, and where the degree of R in x_i is smaller than m. Consider here the system

$$f_2 = 0, f_3 = 0, \cdots, f_r = 0;\ hR \neq 0.^{5} \tag{4**}$$

We wish to show that (4**) cannot be satisfied by values x_j in K. If this were not so, choose a system of values $x_1^*, x_2^*, \cdots, x_n^*$ of K which satisfy the conditions (4**). Replace here x_i^* by an indeterminate x_i, leaving all the other x_j^* fixed. The conditions $f_2 = 0$, $f_3 = 0, \cdots, f_r = 0$, and $h \neq 0$ are not affected, since x_i does not appear in them. As shown by (2), the equation $f_1 = 0$ is of degree m in x_i and has therefore m roots $x_i^{(\mu)}$ in the algebraically closed field K. If g would not vanish when we set $x_i = x_i^{(\mu)}$, we would thus find a system of values of K which satisfies all the conditions (4) and this is impossible. Hence g must vanish when we set $x_i = x_i^{(\mu)}$ and it follows from (7) that the same holds for R. Moreover, as root of the equation $R = 0$ in x_i, the quantity $x_i^{(\mu)}$ has the same multiplicity as for $f_1 = 0$. Thus the equation $R = 0$ of degree less than m in x_i has m roots $x_i = x_i^{(\mu)}$. Consequently, R must vanish identically in x_i. However, for $x_i = x_i^*$, we had $R \neq 0$, as shown by (4**). Thus the assumption that (4**) can be satisfied by values of K leads to a contradiction.

If $r > 1$, the system (4**) is lower than (1) and we may again apply Hilbert's theorem. This shows that a suitable power $(hR)^v$ belongs to $(f_2, f_3, \cdots, f_r)$. This still holds for $r = 1$, when we interpret $(f_2, f_3, \cdots, f_r)$ as the zero ideal. Indeed, since (4**) cannot be satisfied, hR must vanish for all systems of values x_i of K, and hence identically.[6] Now (7) yields

$$(h^{q+1} g^{m+1})^v = (hQf_1 + hR)^v \in (f_1, f_2, \cdots, f_r).$$

If the integer t satisfies the inequalities $t \geqq (q+1)v$, $t \geqq (m+1)v$, then (6) will hold again. But this is all we had to show and the proof of Hilbert's theorem is complete.

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[5] If $r = 1$, the system (4**) is to consist only of the inequality $hR \neq 0$.

[6] We assume the elementary theorem that if a polynomial in several variables vanishes for all systems of values of the underlying field K and if K is either infinite or contains at least sufficiently many elements, the polynomial vanishes identically.

A NONASSOCIATIVE METHOD FOR ASSOCIATIVE ALGEBRAS

SAUNDERS MacLANE[1]

This note exhibits a nonassociative proof for a strictly associative theorem concerned with the "Galois theory" of associative crossed product algebras. The theorem in question has also been established by somewhat more elaborate associative computations; it is perhaps of interest that the nonassociative proof to be given here appears to be both shorter and more conceptual than the associative proof. Practically no technical facts about nonassociative algebras are required for our proof.

Let $K\supset N\supset P$ be fields such that both K and N are finite, separable, and normal extensions of the base field P. The Galois group of K over P, or briefly $\mathcal{G}(K/P)$, will be designated as G, and similarly,

$$(1)\qquad \mathcal{G}(K/P)=G,\qquad \mathcal{G}(K/N)=S,\qquad \mathcal{G}(N/P)=Q.$$

Then $S\subset G$. Each $\alpha\in G$ is an automorphism α: $k\mapsto\alpha\cdot k$ of K, and induces an automorphism $\alpha'\in Q$ of N/P; this correspondence $\alpha\to\alpha'$ provides the natural isomorphism $G/S\cong Q$. We consider functions $k(\alpha,\beta)$ with arguments α, β in G and nonzero values in the field K. The coboundary δk is a similar function of three arguments in G, defined as

$$(2)\qquad \delta k(\alpha,\beta,\gamma)=[\alpha\cdot k(\beta,\gamma)]k(\alpha,\beta\gamma)[k(\alpha\beta,\gamma)k(\alpha,\beta)]^{-1}.$$

It is convenient to assume that any such function k is "normalized," in the sense that $k(I,\beta)=k(\alpha,I)=1$, where I denotes the identity automorphism. The coboundary δk is then also normalized, for it follows that $\delta k(I,\beta,\gamma)=\delta k(\alpha,I,\gamma)=\delta k(\alpha,\beta,I)=1$.

A factor set f of S in the multiplicative group of K is a (normalized) function $f(\sigma,\tau)\in K$ defined for arguments σ, $\tau\in S$ and satisfying the identity $\delta f(\rho,\sigma,\tau)=1$, for all ρ, σ, τ in S. Each factor set f leads to a crossed product (cf. [1, Chap. V][2]) $A=(K,f)$, which is a simple algebra with center N, and which may be represented in terms of elements $u(\sigma)$, one for each $\sigma\in S$, as the set of all sums $a=\sum_\sigma k(\sigma)u(\sigma)$ with arbitrary coefficients $k(\sigma)\in K$ and with the multiplication table

$$(3)\qquad u(\sigma)u(\tau)=f(\sigma,\tau)u(\sigma\tau),\qquad u(\sigma)k=[\sigma\cdot k]u(\sigma),$$

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

for $\sigma, \tau \in S$, $k \in K$. The condition $\delta f = 1$ on the factor set assures the associativity of this algebra. In this algebra, the automorphisms σ of K are inner automorphisms $\sigma \cdot k = C[u(\sigma)]k$, where C denotes conjugation; that is, where

$$C[b] \cdot a = bab^{-1}, \qquad a \in A,$$

for any regular element b of A. In particular, the elements $u(\sigma)$ are regular.

A factor set f, defined originally for arguments σ, τ in S, can always be extended to a nonzero "normalized" function $h(\alpha, \beta) \in K$, defined for all $\alpha, \beta \in G$, so that

$$(4) \qquad f(\sigma, \tau) = h(\sigma, \tau), \qquad \sigma, \tau \in S.$$

THEOREM. *If the factor set f of a crossed product algebra $A = (K, f)$ with center N has an extension h as in* (4) *such that*

$$(5) \qquad \delta h(\alpha, \beta, \gamma) = t(\alpha', \beta', \gamma'), \qquad \alpha, \beta, \gamma \in G,$$

where t is a nonzero normalized function $t(\lambda, \mu, \nu) \in K$ defined for $\lambda, \mu, \nu \in Q$, then every automorphism $\lambda \in Q$ can be extended to an automorphism $w(\lambda)$ of the crossed product algebra $A = (K, f)$, in such fashion that there are regular elements $b(\lambda, \mu)$ in A for each pair $\lambda, \mu \in Q$ with

$$(6) \qquad w(\lambda)w(\mu) = C[b(\lambda, \mu)]w(\lambda\mu),$$

$$(7) \qquad t(\lambda, \mu, \nu)b(\lambda, \mu)b(\lambda\mu, \nu) = [w(\lambda) \cdot b(\mu, \nu)]b(\lambda, \mu\nu).$$

The hypothesis (5) asserts that δh depends only on its arguments α, β, γ modulo $S \subset G$. The conclusion shows that the automorphisms of the algebra A over P induce all the automorphisms $\lambda \in Q$ of its normal subfield N over P. As this is analogous to a fundamental property of fields normal over P, such an algebra A may be called *Q-normal*. The conclusion (7) asserts that t is a noncommutative coboundary δb, where the function b measures the extent to which w is not a homomorphism. In the terminology of Eilenberg and the author [2], this conclusion asserts that the function t of (5) is a "Teichmüller cocycle" of A.

This theorem is the converse part of the main theorem of §10 in [2]; it is this theorem which serves there to characterize those three-dimensional cocycles (t's with $\delta t = 1$) which may appear as the Teichmüller cocycles of central simple Q-normal algebras over N.

The proof now to be given does not depend on these concepts, as developed in [2]. The proof given there [2, §§9, 11] involves certain long identities in δh; the present proof was obtained by the observation that these identities may be regarded as resulting from the re-association of certain products in a suitable nonassociative algebra R constructed from $\boldsymbol{h}$.

Construct R from symbols $\boldsymbol{u}(\alpha)$, one for each $\alpha \in G$, as the set of all formal sums $\sum_\alpha \boldsymbol{k}(\alpha)\boldsymbol{u}(\alpha)$ with arbitrary coefficients $\boldsymbol{k}(\alpha) \in K$. Addition of such sums (by addition of coefficients) and multiplication of a sum by a scalar in P are defined in the standard way, to make R a vector space of dimension m^2 over P, where m denotes the degree of K over P. Multiplication in R is defined by the rules

$$\boldsymbol{u}(\alpha)\boldsymbol{k} = (\alpha\cdot\boldsymbol{k})\boldsymbol{u}(\alpha), \qquad \boldsymbol{u}(\alpha)\boldsymbol{u}(\beta) = \boldsymbol{h}(\alpha, \beta)\boldsymbol{u}(\alpha\beta), \tag{8}$$

and more generally, for $\boldsymbol{k}, \boldsymbol{k}' \in K$, $\alpha, \beta \in G$, by

$$[\boldsymbol{k}\boldsymbol{u}(\alpha)][\boldsymbol{k}'\boldsymbol{u}(\beta)] = [\boldsymbol{k}(\alpha\cdot\boldsymbol{k}')\boldsymbol{h}(\alpha, \beta)]\boldsymbol{u}(\alpha\beta). \tag{9}$$

The product of two sums of the form $\sum_\alpha \boldsymbol{k}(\alpha)\boldsymbol{u}(\alpha)$ is then defined by (9) and the distributive law. This multiplication satisfies both distributive laws, so that R becomes a nonassociative algebra over P. The element $\boldsymbol{u}(I)$ is an identity element of R. Comparison of (8) with (3) indicates that R contains the given associative crossed product algebra A; it is our aim to show that the desired automorphisms of A appear in R as inner automorphisms, much as the automorphisms of a field K become inner automorphisms in an (associative) crossed product algebra.

The deviation from the associative law in R may be measured by using (9) to calculate a triple product in the two possible associations, as

$$\begin{aligned}
&[\boldsymbol{k}\boldsymbol{u}(\alpha)]\{[\boldsymbol{k}'\boldsymbol{u}(\beta)][\boldsymbol{k}''\boldsymbol{u}(\gamma)]\} \\
&\qquad = \{\boldsymbol{k}(\alpha\cdot\boldsymbol{k}')(\alpha\cdot\beta\cdot\boldsymbol{k}'')[\alpha\cdot\boldsymbol{h}(\beta, \gamma)]\boldsymbol{h}(\alpha, \beta\gamma)\}\boldsymbol{u}(\alpha\beta\gamma), \\
&\{[\boldsymbol{k}\boldsymbol{u}(\alpha)][\boldsymbol{k}'\boldsymbol{u}(\beta)]\}[\boldsymbol{k}''\boldsymbol{u}(\gamma)] \\
&\qquad = \{\boldsymbol{k}(\alpha\cdot\boldsymbol{k}')(\alpha\beta\cdot\boldsymbol{k}'')\boldsymbol{h}(\alpha, \beta)\boldsymbol{h}(\alpha\beta, \gamma)\}\boldsymbol{u}(\alpha\beta\gamma).
\end{aligned}$$

Upon comparing the results, using the definition (2) of $\delta\boldsymbol{h}$, it appears that

$$\begin{aligned}
&[\boldsymbol{k}\boldsymbol{u}(\alpha)]\{[\boldsymbol{k}'\boldsymbol{u}(\beta)][\boldsymbol{k}''\boldsymbol{u}(\gamma)]\} \\
&\qquad = \delta\boldsymbol{h}(\alpha, \beta, \gamma)(\{[\boldsymbol{k}\boldsymbol{u}(\alpha)][\boldsymbol{k}'\boldsymbol{u}(\beta)]\}[\boldsymbol{k}''\boldsymbol{u}(\gamma)]).
\end{aligned} \tag{10}$$

In particular, if $\boldsymbol{h}$ were a factor set ($\delta\boldsymbol{h} = 1$), the algebra would be associative. More generally, the equation (10) asserts that $\delta\boldsymbol{h}$ is the "associator" of the elements of the algebra in question. Such associators are similar to the additive associators used by Zorn [4] in studying alternative algebras and are exactly parallel to the associators used by Eilenberg and the author [3] to interpret cohomology groups of abstract groups by means of loops. More explicitly, since t is normalized, $t(I, \mu, \nu) = t(\lambda, I, \nu) = t(\lambda, \mu, I) = 1$, and hence, by (5) $\delta h(\sigma, \beta, \gamma) = \delta h(\alpha, \sigma, \gamma) = \delta h(\alpha, \beta, \sigma) = 1$ for $\sigma \in S$. Since the $u(\sigma)$ for $\sigma \in S$ give the algebra A, this proves that every triple product in R with one factor in the subalgebra A is associative.

LEMMA 1. *For $r \in R$, $\beta \in G$ there is a unique element s in R such that $u(\beta)r = su(\beta)$. If $r \in A$, then $s \in A$ also.*

The uniqueness of s is in effect a cancellation law asserting that $su(\beta) = s'u(\beta)$ for $s, s' \in R$ implies $s = s'$. Indeed, take $s = \sum_\alpha k(\alpha)u(\alpha)$; then by the definition of the product

$$su(\beta) = \sum_\alpha k(\alpha)h(\alpha, \beta)u(\alpha\beta) = \sum_\gamma k(\gamma\beta^{-1})h(\gamma\beta^{-1}, \beta)u(\gamma).$$

On calculating a similar expression for s' and equating like coefficients, one finds that $s = s'$, as asserted. Since the multiplication is distributive, it suffices to prove the existence of s for the case when $r = ku(\alpha)$; in this case $s = (\beta \cdot k)h(\beta, \alpha)[h(\beta\alpha\beta^{-1}, \beta)]^{-1}u(\beta\alpha\beta^{-1})$ has the required property. In particular, since S is a normal subgroup of G, $\beta\sigma\beta^{-1} \in S$ for $\sigma \in S$, so that $s \in A$ whenever $r \in A$.

In using this lemma it is convenient to write s as $\theta(\beta) \cdot r$, so that

$$u(\beta)r = [\theta(\beta) \cdot r]u(\beta), \qquad r \in R. \tag{11}$$

LEMMA 2. *$\theta(\beta)$ is an automorphism of A,*

PROOF. In virtue of the distributive law in R applied to (11), $\theta(\beta)$ is a linear transformation of the vector space R into itself, while the associative law of R valid when one factor lies in A shows that

$$\theta(\beta) \cdot (ar) = [\theta(\beta) \cdot a][\theta(\beta) \cdot r], \qquad a \in A, r \in R;$$

in particular, $\theta(\beta)$ is an endomorphism of the algebra A.

For elements $\beta, \gamma \in G$ and $a \in A$ one has

$$u(\beta)u(\gamma)a = u(\beta)[\theta(\gamma) \cdot a]u(\gamma) = [\theta(\beta) \cdot \theta(\gamma \cdot a)]h(\beta, \gamma)u(\beta\gamma),$$

and by a different route

$$u(\beta)u(\gamma)a = h(\beta, \gamma)u(\beta\gamma)a = h(\beta, \gamma)[\theta(\beta\gamma) \cdot a]u(\beta\gamma),$$

hence

$$\theta(\beta) \cdot [\theta(\gamma) \cdot a] = C[h(\beta, \gamma)] \cdot [\theta(\beta\gamma) \cdot a].$$

Setting here $\gamma = \beta^{-1}$, and observing that $C[h(\beta, \beta^{-1})]$ and $\theta(I) = I$ are automorphisms of A, we find that the product $\theta(\beta)\theta(\beta^{-1})$ is an automorphism of A. By the same token $\theta(\beta^{-1})\theta(\beta)$ is an automorphism of A. The first of these two results shows that $\theta(\beta)$ must map A onto all of A, and the second implies that $\theta(\beta)$ can map no nonzero element of A into zero; together they mean that $\theta(\beta)$ is an automorphism of A, as asserted in the lemma.

For elements k in $K\subset A$, the definition (8) and (11) show that $\theta(\beta)$ is exactly the automorphism β of K. By a theorem in Galois theory, every automorphism $\lambda\in Q$ can be extended to some automorphism $\beta=v(\lambda)$ of K. Therefore λ can be extended to the automorphism $w(\lambda)=\theta(v(\lambda))$ of A, as asserted in the first conclusion of the theorem.

To obtain the remaining conclusions (6) and (7), choose a fixed extension $v(\lambda)\in G$ of each automorphism $\lambda\in Q$, and in particular choose $v(I)=I$. Thus $v(\lambda)'=\lambda$, and there is an η with

$$v(\lambda)v(\mu)=\eta(\lambda,\mu)v(\lambda\mu),\qquad \eta(\lambda,\mu)\in S.$$

(Actually, η is a factor set of Q in S, obtained by regarding G as an extension of S by $Q\cong G/S$, with representatives v.) There is then a regular element $b(\lambda,\mu)\in A$ for each pair $\lambda,\mu\in Q$ such that

$$u(v(\lambda))u(v(\mu))=b(\lambda,\mu)u(v(\lambda\mu)). \tag{12}$$

Indeed, one may define a regular element b with this property by the equation

$$k(\eta(\lambda,\mu),v(\lambda\mu))b(\lambda,\mu)=k(v(\lambda),v(\mu))u(\eta(\lambda,\mu)),$$

for if each side of this equation is multiplied on the right by $u(v(\lambda\mu))$, and if the multiplication rule (8) and the definition of η are used, equation (12) results. Note also that $b(I,\mu)=b(\lambda,I)=1$.

The rule (12), when interpreted in terms of the automorphisms θ of (11), asserts that

$$\theta(v(\lambda))\theta(v(\mu))=C[b(\lambda,\mu)]\theta(v(\lambda\mu)). \tag{13}$$

On the other hand, a triple product of u's may be computed as

$$[u(v(\lambda))u(v(\mu))]u(v(\nu))=b(\lambda,\mu)b(\lambda\mu,\nu)u(v(\lambda\mu\nu))$$

or, on using the "associator" in (10) and the assumption (7), as

$$\begin{aligned}
&[u(v(\lambda))u(v(\mu)]u(v(\nu))\\
&\qquad=[\delta k(v(\lambda),v(\mu),v(\nu))]^{-1}u(v(\lambda))[u(v(\mu))u(v(\nu))]\\
&\qquad=[\delta k(v(\lambda),v(\mu),v(\nu))]^{-1}[w(\lambda)\cdot b(\mu,\nu)]b(\lambda,\mu\nu)u(v(\lambda\mu\nu)).
\end{aligned}$$

By the hypothesis (5) and the fact that $v(\lambda)'=\lambda$, the first factor here becomes $t(\lambda,\mu,\nu)^{-1}$. Comparison of the coefficients of $u(v(\lambda\mu\nu))$ in these two expressions is possible by the cancellation law of Lemma 1: it proves the desired second conclusion (7).

Note. From the conclusion (6), (7) of the theorem it can be argued that $t(\lambda, \mu, \nu) \in N$. This, however, is a direct consequence of the hypothesis (5), for from (5) it follows that

$$[\xi \cdot t(\alpha', \beta', \gamma')]t(\xi', \alpha'\beta', \gamma')t(\xi', \alpha', \beta') = t(\xi'\alpha', \beta', \gamma')t(\xi', \alpha', \beta'\gamma'),$$

for $\xi, \alpha, \beta, \gamma \in G$, hence, setting $\xi = \sigma \in S$, that $\sigma \cdot t(\alpha', \beta', \gamma') = t(\alpha', \beta', \gamma')$, for all σ. Therefore $t(\alpha', \beta', \gamma') \in N$.

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UNIVERSAL QUATERNARY QUADRATIC FORMS

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Let the positive quaternary quadratic form $F=\sum_{i,j=1}^{4} b_{ij}x_ix_j$, where b_{ij} are integers and $b_{ij}=b_{ji}$, have its hessian equal to H'. Then it can be shown that every positive quaternary quadratic form whose hessian is H' is equivalent to a form F in which $b_{11}\leqq(4H')^{1/4}$, $2|b_{1j}|\leqq b_{11}$, $j=2, 3, 4$, and $b_{11}F-(\sum_{j=1}^{4} b_{1j}x_j)^2$ is a reduced positive ternary quadratic form $T=\sum_{i,j=2}^{4}B_{ij}x_ix_j$ whose hessian is b_{11}^2H'. Further if $b_{11}=2$, we may take $b_{1j}\geqq 0$, $B_{23}\geqq 0$ and $B_{23}B_{34}-B_{33}B_{24}\geqq 0$, and in case $B_{23}=0$ we may choose $B_{34}\geqq 0$. A form satisfying these conditions is a reduced form.

This paper is concerned with the determination of the positive forms $f_0=\sum_{i,j=1}^{4} c_{ij}x_ix_j$, in which $c_{ij}=c_{ji}$, c_{ii} and $2c_{ij}$ are integers, which are universal. For the case in which $c_{ij}=0$ if $i\neq j$ the forms which represent all positive integers have been determined by Ramanujan.[1] L. E. Dickson[2] considered those forms in which $c_{1j}=0$, $j=2, 3, 4$. Since the minimum positive integer which is represented by f_0 is 1, this form is equivalent to a form f whose first coefficient is 1. Let $F=2f=\sum_{i,j=1}^{4} a_{ij}x_ix_j$, where $a_{ij}=a_{ji}$ are integers, $a_{11}=2$ and $a_{ii}\equiv 0 \pmod 2$, be a reduced form whose hessian is H. Then $4f=X^2+T$ where $X=\sum_{j=1}^{4} a_{1j}x_j$ and $T=\sum_{i,j=2}^{4} A_{ij}x_ix_j$ $(A_{ij}=a_{11}a_{ij}-a_{1i}a_{1j})$ is a reduced positive ternary quadratic form whose hessian is $4H$. Since f is universal, $4f$ represents all positive multiples of 4. The minimum positive integer M which is represented by the ternary T is $\leqq 8$ and $\equiv 0$ or $3 \pmod 4$. Hence $M=3, 4, 7$ or 8.

The ternary form[3] $MT=U^2+Q$ where $U=Mx_2+A_{23}x_3+A_{24}x_4$, and $Q=Ax_3^2+2Bx_3x_4+Cx_4^2$ is a reduced binary quadratic form whose hessian is $4MH$, and where $A=MA_{33}-A_{23}^2$, $B=MA_{34}-A_{23}A_{24}$ and $C=MA_{44}-A_{24}^2$, while $2|A_{23}|$ and $2|A_{24}|$ are each $\leqq M$. We may choose $A_{23}\geqq 0$ and $B\geqq 0$, while if $A_{23}=0$, we may take $A_{24}\geqq 0$.

The least positive integer A which is represented by Q if $M=3$ is $\leqq 24$ and $\equiv 0$ or $8 \pmod{24}$. If $M=4$, $A\leqq 48$ and $\equiv 0$ or $12 \pmod{16}$ if $A_{23}=0$, and $A\equiv 12$ or $8 \pmod{16}$ if $A_{23}=2$. For $M=7$, $A\leqq 84$ and $\equiv 0, -4, -8$, or $-16 \pmod{28}$ according as $A_{23}=0, 1, 2$ or 3. Finally

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[1] Proc. Cambridge Philos. Soc. vol. 19 (1916–1919) pp. 11–21.

[2] Amer. J. Math. vol. 49 (1927) pp. 39–56.

[3] Landau, *Handbuch · · · Verteilung der Primzahlen*, 1909, p. 545.

if $M=8$, $A\leqq 160$ and $A\equiv 0$, 8, 16, 20, 24 or 28 (mod 32). It follows that the number of reduced universal forms is finite.

The problem has not been completely solved, since there are a number of forms for which, as yet, only empirical results have been obtained. The methods used are, largely, those used by L. E. Dickson. The results have been put in tabular form and are being sent to Professor Dresden for deposit in the library of the Society. Table I gives those forms, 1417 in number, which have been proved to be universal. Since the definition given for a reduced form does not always yield a unique reduced form for each class, certain of the forms in this table will be equivalent. In table II are listed those forms which satisfy the necessary conditions for universality, but which fail to represent certain integers. These integers are specified in the table.

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SOME ANALOGS OF THE GENERALIZED PRINCIPAL AXIS TRANSFORMATION

N. A. WIEGMANN

It is known that two normal matrices can be diagonalized by the same unitary transformation if and only if they commute; this theorem is ordinarily stated for hermitian matrices. Some generalizations of this theorem are known. According to a theorem due to Eckert and Young,[1] if A and B are two $r \times s$ matrices, there are two unitary matrices U and V such that $UAV = D_1$ and $UBV = D_2$, D_1 and D_2 diagonal matrices with real elements, if and only if AB^{ct} and $B^{ct}A$ are hermitian. It is also known that a set of normal matrices $\{A_i\}$ is reducible to diagonal matrices under the same unitary similarity transformation, UA_iU^{ct}, if and only if $A_iA_j = A_jA_i$ for all i and j. (More generally, it is true that a set of matrices $\{A_i\}$ with elements in the complex field and simple elementary divisors is reducible to diagonal matrices under the same similarity transformation if and only if $A_iA_j = A_jA_i$ for all i and j.) The following will be shown to hold:

THEOREM. *If $\{A_i\}$ is an arbitrary set of nonzero $r \times s$ matrices, there are unitary matrices U and V of orders $r \times r$ and $s \times s$, respectively, such that $UA_iV = D_i$, D_i diagonal and real, if and only if $A_iA_j^{ct} = A_jA_i^{ct}$ and $A_j^{ct}A_i = A_i^{ct}A_j$ for all i and j.*

If two unitary matrices U and V exist such that $UA_iV = D_i$, D_i real for all i, then $D_iD_j^{ct} = D_iD_j^{t} = D_jD_i^{t} = D_jD_i^{ct}$ where the D_i are $r \times s$ diagonal matrices (that is, the only nonzero elements appear in the d_{ii} position). Therefore, $A_iA_j^{ct} = A_jA_i^{ct}$.

Conversely, let the relations $A_j^{ct}A_i = A_i^{ct}A_j$ and $A_iA_j^{ct} = A_jA_i^{ct}$ hold for all i, j. The proof is by induction.

(1) The theorem is true for a set of matrices of dimension $1 \times s$, $A_i = [a_i', a_i'', \cdots, a_i^{(s)}]$. For there exist unitary matrices U and V such that[1] $UA_1V = [d_1', 0, \cdots, 0]$ for d_1' real and greater than 0 since $A_1 \neq 0$. For if $UA_iV = [d_i', d_i'', \cdots, d_i^{(s)}]$, it follows from $A_i^{ct}A_1 = A_1^{ct}A_i$ that $d_i'' = d_i''' = \cdots = d_i^{(s)} = 0$ and since $d_1' \cdot \bar{d}_i' = \bar{d}_1' \cdot d_i'$ and d_1' is real, $\bar{d}_i' = d_i'$. In the same way by means of the second of

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[1] Bull. Amer. Math. Soc. vol. 45 (1939) pp. 118–121. See also J. Williamson, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 920–922.

the given conditions, the theorem is true for a set of matrices of dimension $r \times 1$.

(2) Assume the theorem to be true for a set of matrices of dimension $k \times l$ for $k \leqq r$, $l \leqq s-1$ and for $k \leqq r-1$, $l \leqq s$. The theorem will be shown to hold for the dimension $r \times s$ and the induction will then be complete. Let $\{A_i\}$ be a set of matrices of dimension $r \times s$ for which the given conditions hold. Let U and V be such that

$$UA_1V = \begin{bmatrix} D & 0_2 \\ 0_3 & 0_4 \end{bmatrix}$$

where U and V are unitary, D a nonsingular diagonal matrix with real positive diagonal elements, and the submatrices 0_2, 0_3, and 0_4 are null matrices or non-existent. If

$$UA_iV = \begin{bmatrix} G_i & K_i \\ L_i & H_i \end{bmatrix}$$

it follows from $A_1A_i^{ct} = A_iA_1^{ct}$ that $L_i = 0_3$, and from $A_i^{ct}A_1 = A_1^{ct}A_i$ that $K_i = 0_2$. Also, $DG_i^{ct} = G_iD$ and $G_i^{ct}D = DG_i$ from both given conditions. Therefore, $D^2G_i^{ct} = DG_iD = G_i^{ct}D^2$ so $D^2G_i = G_iD^2$. Since D consists of positive real numbers and since G_i commutes with D^2, $DG_i = G_iD$. Then $DG_i^{ct} = G_iD = DG_i$ and since D is nonsingular, $G_i^{ct} = G_i$ for all i. Then from the given relation $A_iA_j^{ct} = A_jA_i^{ct}$, it follows that $G_iG_j^{ct} = G_jG_i^{ct}$ or $G_iG_j = G_jG_i$; therefore the set of hermitian matrices $\{D, G_i\}$ are all commutative in pairs and, by the generalized principal axis theorem, there exists a unitary matrix U_1 which diagonalizes all of them. Let $U_2 = U_1 \dotplus I$ be a unitary matrix of the same dimension as U and $U_3 = U_1^{ct} \dotplus I$ of the same dimension as V. Then,

$$U_2UA_1VU_3 = \begin{bmatrix} D & 0_2 \\ 0_3 & 0_4 \end{bmatrix}, \quad U_2UA_iVU_3 = \begin{bmatrix} D_i & 0_2 \\ 0_3 & H_i \end{bmatrix}$$

for all i where the H_i are either non-existent or of dimension $k \times l$ where $k < r$ and $l < s$. The theorem follows from the induction hypothesis.

It is to be noted that if the set $\{A_i\}$ are all $n \times n$ hermitian matrices for which $A_iA_j^{ct} = A_jA_i^{ct}$ or $A_iA_j = A_jA_i$ holds, the principal axis transformation for hermitian matrices is obtained and $V = U^{ct}$.

According to another result due to Eckert and Young,[1] if A and B are $r \times s$ matrices over the complex field, a necessary and sufficient condition that there exist two unitary matrices U and V such that $UAV = D_1$ and $UBV = D_2$, D_1 and D_2 diagonal, is that AB^{ct} and $B^{ct}A$ be normal. Since this is a generalization of the earlier result, it would seem reasonable to hope for an extension to a set of matrices $\{A_i\}$. A simple example shows that this is not the case, however, and the following theorem holds:

THEOREM. *A necessary and sufficient condition that a set of* $n\times n$ *matrices* $\{A_i\}$ *be brought into diagonal forms by the same unitary* U, V *equivalence transformation,* $UA_iV=D_i$, *is that the products* $A_iA_j^{ct}$ *and* $A_j^{ct}A_i$ *be normal for all* i, j *and that* $A_k(A_j^{ct}A_i)=(A_iA_j^{ct})A_k$ *for all* i, j *and* k.

If $UA_iV=D_i$ for all i, then the given conditions can be easily verified.

Conversely, let $\{A_i\}$ be a set of matrices for which $A_iA_j^{ct}$ and $A_j^{ct}A_i$ are normal and where $A_k(A_j^{ct}A_i)=(A_iA_j^{ct})A_k$. The proof is by induction on the order n. The theorem is trivially true if $n=1$. Assume it to be true for order $k\leqq n-1$. Now consider a system of order n. There are two possibilities: for all i, j, either $A_iA_j^{ct}$ is a scaler matrix or there is at least one pair i, j such that $A_iA_j^{ct}$ is not a scaler.

(1) If for all i, j, $A_iA_j^{ct}$ is a scaler, $A_iA_j^{ct}=k_{ij}I$ and since $A_iA_j^{ct}$ is similar to $A_j^{ct}A_i$, it is true that:

$$\text{(a)} \qquad A_iA_j^{ct}=k_{ij}I=A_j^{ct}A_i \qquad \text{for all } i, j.$$

There are two possibilities: (a) Either all $A_i=k_iU_i$ where the k_i are real positive scalers and U_i are unitary; then all A_i are normal and from the above, $A_iA_j=A_jA_i$ since $A_j=f(A_j^{ct})$. In this case the principal axis transformation theorem applies for normal matrices so $V=U^{ct}$ and the theorem is true. (b) There is at least one A_i, say A_1, not of the above form. There exist two unitary U, V such that $UA_1V=D_1$ is diagonal with real non-negative elements. Also, D_1 is not scaler for then $A_1=U^{ct}D_1V^{ct}=D_1U^{ct}V^{ct}$; but this contradicts the assumption. Let $UA_jV=A_j'$. Then,

$$\begin{aligned}\text{(a)}: \qquad UA_1VV^{ct}A_j^{ct}U^{ct} &= Uk_{1j}U^{ct} = k_{1j}I = D_1A_j'^{ct};\\ V^{ct}A_j^{ct}U^{ct}UA_1V &= V^{ct}k_{1j}V = k_{1j}I = A_j'^{ct}D_1.\end{aligned}$$

Therefore,

$$D_1A_j'^{ct}=A_j'^{ct}D_1$$

and

$$A_j'D_1=D_1A_j'.$$

Since D_1 is not scaler, the A_j' are direct sums of matrices of order

$k \leqq n-1$. But for these matrices the given conditions hold and the theorem is true.

(2) If for some i, j the products $A_iA_j^{ct}$ (and consequently $A_j^{ct}A_i$) are not scaler, there exist for this A_i and A_j unitary matrices U_{ij} and V_{ij} such that

$$U_{ij}A_iV_{ij} = D_i, \qquad U_{ij}A_jV_{ij} = D_j.$$

For all k, $A_k(A_j^{ct}A_i) = (A_iA_j^{ct})A_k$.

Apply the U_{ij}, V_{ij} and obtain

$$U_{ij}A_k(A_j^{ct}A_i)V_{ij} = U_{ij}(A_iA_j^{ct})A_kV_{ij}$$

so

$$U_{ij}A_kV_{ij}V_{ij}^{ct}A_j^{ct}U_{ij}^{ct}U_{ij}A_iV_{ij} = U_{ij}A_iV_{ij}V_{ij}^{ct}A_j^{ct}U_{ij}^{ct}U_{ij}A_kV_{ij},$$

so

$$(U_{ij}A_kV_{ij})(D_j^{ct}D_i) = (D_iD_j^{ct})(U_{ij}A_kV_{ij}).$$

Therefore, the matrix $U_{ij}A_kV_{ij}$ commutes with the nonscaler diagonal matrix $D_j^{ct}D_i$. Since the unitary transformation may be chosen so that like elements of $D_j^{ct}D_i$ appear together in order, $U_{ij}A_kV_{ij}$ is a direct sum of matrices of order $m<n$ for all k. Since the submatrices satisfy these conditions, the theorem is true by induction.

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POLYNOMIALS IN TOPOLOGICAL FIELDS

IRVING KAPLANSKY

1. **Introduction.** Let F be a real closed field in the sense of Artin-Schreier [1, 8],[1] and $f(x)$ a polynomial with coefficients in F. It is known that $f(x)$ attains a maximum and minimum in any interval $a \leqq x \leqq b$. Recently, Habicht [4] has begun the study of polynomials in several variables over a real closed field and proved the following theorem: if $f(x_1, \cdots, x_n)$ is positive for $-m \leqq x_i \leqq m$, then f has a positive lower bound in this region. An equivalent statement is that f maps the region into a closed subset of F, where we give F its order topology (the open intervals are a base for the open sets). This latter formulation suggests possible extensions of Habicht's theorem to more general topological fields.

In this note we shall examine such extensions. In §3 we obtain quite complete results for polynomials in one variable over fields of "type V" (§2). But simple examples show that for two or more variables the situation is more complicated. We do however obtain a result (Theorem 4) which immediately implies Habicht's theorem and which is valid for a somewhat wider class of topological fields than real closed fields in their order topology. A further result for polynomials in two variables appears in Theorem 3.

2. **Fields of type** V. We begin by recalling some definitions. In a topological ring A we call a set B right bounded if for any neighborhood U of 0 there exists a neighborhood V of 0 such that $BV \subset U$. Left boundedness is analogously defined and a set is bounded if it is both right and left bounded. We denote by A_n the n-dimensional vector space over A (Cartesian product topology), and a subset of A_n is bounded if it is bounded coordinate-wise. Any compact set is bounded, and in many arguments such as the following, bounded sets behave somewhat like compact ones.

LEMMA 1. *Let f, g be functions defined on a subset S of A_n and taking values in A. Suppose f, g are bounded and uniformly continuous on S. Then $f+g$ and fg are bounded and uniformly continuous on S.*

The proof is obvious except perhaps for the uniform continuity of fg, which depends upon iterated use of the identity

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$ab - cd = a(b - d) + (a - c)d.$$

By repeated application of Lemma 1, we obtain the following corollary.

LEMMA 2. *A polynomial in n variables is bounded and uniformly continuous on a bounded subset of A_n.*

We shall say that a topological division ring F is of type V if whenever a set $B \in F$ is bounded away from 0 (that is, disjoint from a neighborhood of 0), then B^{-1} is bounded.[2] Except for the omission of any countability restriction, this is the definition given in [6]. Most of the results of [6] are valid without the countability assumption.[3] In particular [6, Lemma 5]: the completion of a division ring of type V is a division ring of type V.

It is easy to verify that the following constitute examples of rings of type V: (a) discrete division rings, (b) division rings with a valuation, (c) ordered division rings in their order topology. It is to be observed that in (a) all sets are bounded, while in (b) and (c) boundedness coincides with its ordinary meaning. Any non-completable division ring is necessarily not of type V: for example, the rationals in the 6-adic topology.

In the sequel we shall require Theorem 11 of [6]. Restated in the Cartan-Bourbaki language of filters,[4] it reads as follows. (We have simplified by using a fixed polynomial instead of a variable one; only this special case is needed in the present paper.)

LEMMA 3. *Let $\{A_i\}$ be an ultrafilter in a topological field F of type V, and suppose there exists a nonconstant polynomial $f(x)$ with coefficients in F such that $\{f(A_i)\}$ is the base of a Cauchy filter. Then $\{A_i\}$ is also Cauchy.*

We shall also make use of the following lemma.

LEMMA 4. *Let F be a topological division ring of type V, and A, B subsets of F. If A, B are bounded away from 0, so is AB. If B is bounded and AB is bounded away from 0, then A is bounded away from 0.*

PROOF. If A, B are bounded from 0, then A^{-1}, B^{-1} are bounded and so is $B^{-1}A^{-1}=(AB)^{-1}$. Hence AB is bounded from 0. (A set whose inverses are bounded must be bounded from 0; this holds in any topological ring.) Again if AB is bounded from 0 and B is bounded, then $B\cdot(AB)^{-1}\supseteq A^{-1}$ is bounded, whence A is bounded from 0.

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[2] This axiom also occurs in Bourbaki [3, chap. III, p. 57, Ex. 13] where it is labelled KT_a. The only overlapping with [6] is in part (b) of this exercise, which is to show that the completion of a division ring satisfying KT_a is a division ring. I regret having missed this reference till after [6] was published.

[3] This does not apply to the main result (Theorem 9) whose proof used the category theorem for complete metric spaces.

[4] Cf. [3] for the relevant definitions.

3. One variable.

We begin with a result which follows readily from Lemma 3.

THEOREM 1. *Let F be a topological field of type V, and suppose F is algebraically closed in its completion G (any element of G algebraic over F is already in F). Let $f(x)$ be a polynomial with coefficients in F. Then the mapping $x\rightarrow f(x)$ is closed, that is, it sends closed sets into closed sets.*

PROOF. Let B be a closed set in F, and suppose b is in the closure of $f(B)$. Then we can find $\{X_i\}$, a base of a filter approaching b, with each X_i in $f(B)$. We write $X_i=f(A_i)$, with $A_i\subset B$. Then $\{A_i\}$ is the base of a filter; we refine it to an ultrafilter which by Lemma 3 is Cauchy, and hence approaches an element a in G. Then $f(a)=b$, whence by hypothesis $a\in F$, $a\in B$. Hence $b\in f(B)$ and $f(B)$ is closed.

REMARK 1. The hypothesis that F is algebraically closed in G is of course satisfied if F is algebraically closed or if it is complete; the condition is closely related to Ostrowski's relative completeness which says that F is separably algebraically closed in G [7, p. 316]. A real closed field in its order topology is algebraically closed in its completion since, as is well known, the completion is again formally real (and even real closed). The necessity of some such condition is shown by an example like x^2-2 over the rationals.

One can make a partial extension of Theorem 1 to rational functions.

THEOREM 2. *Let F be a topological field of type V algebraically closed in its completion, and S a bounded closed subset of F. Then for any rational function $f(x)$ with coefficients in F, $f(S)$ is closed.*[5] *If $f\neq\infty$ on S, $f(S)$ is also bounded.*

PROOF. Let b be in the closure of $f(S)$. Write $f=g/h$, $k=g-bh$, where g, h, k are polynomials. By Lemma 2, $h(S)$ is bounded, and hence $1/h(S)$ is bounded from 0. We cannot have $k(S)$ bounded from 0 for then by Lemma 4, $k/h=f-b$ would be bounded from 0 on S. Hence by Theorem 1, $k(a)=0$ for $a\in S$. Necessarily $h(a)\neq 0$, for otherwise g and h would have the common factor $x-a$. Thus $f(a)=b$, and $f(S)$ is closed.

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[5] By $f(S)$ we mean the range of f, excluding the points where $f=\infty$. It would also be possible to adjoin a formal point at infinity, making its neighborhoods the complements of bounded sets. The theorem is also true with this interpretation.

If $h\neq 0$ on S, then, by Theorem 1, $h(S)$ is bounded from 0, $1/h(S)$ is bounded, and $f(S)$ is bounded.

REMARK 2. That Theorem 2 fails if S is unbounded is evident from an example like $1/(x^2+1)$ over the reals.

4. Several variables. Let $f(x, y)$ be a polynomial with coefficients in F. We consider the mapping $(x, y)\rightarrow f(x, y)$ of F_2 into F. If F is locally compact, the mapping sends bounded closed (=compact) sets into closed sets. But it would seem that nothing short of local compactness will suffice to get this result.

EXAMPLE 1. Completeness does not suffice. Let F be the field of formal power series in a variable t over a field containing infinitely many elements a_i. We topologize F in the usual way, making t^i approach 0. Then the set $(a_i+t^i, -a_i)$ is bounded and closed, but is mapped by $x+y$ into the nonclosed set $\{t^i\}$.

EXAMPLE 2. It does not help if the completion of F is locally compact. In the field of real algebraic numbers, pick null sequences a_i, b_i such that $a_i+b_i\neq 0$, and $\pi+a_i$, $-\pi+b_i$ are rational. Then $x+y$ maps the closed bounded set $(\pi+a_i, -\pi+b_i)$ into a nonclosed set.

These examples indicate that for $f(S)$ to be bounded away from 0 it does not suffice to have S closed and disjoint from the manifold $Z(f)$ consisting of all points where $f=0$. A stronger condition is that S be disjoint from a uniform neighborhood of $Z(f)$, by which we mean that there exists a neighborhood U of 0 such that S is disjoint from $Z(f)+U$. For two variables over an algebraically closed field this does suffice, and moreover S need not be bounded.

THEOREM 3. *Let F be an algebraically closed field of type V, $f(x, y)$ a polynomial with coefficients in F, and $Z(f)$ the manifold $f=0$. Let S be a set disjoint from a uniform neighborhood of $Z(f)$. Then $f(S)$ is bounded away from* 0.

PROOF. Write $f(x, y)=a(y)x^m+\cdots=b(x)y^n+\cdots$. Let α_i, β_j denote the roots of a, b respectively (either or both of these sets may be void in case a or b is a constant). If the point (β_j, α_i) is in $Z(f)$, then by hypothesis S is disjoint from a neighborhood of it. If (β_j, α_i) is not in $Z(f)$, then there is a neighborhood of it in which f is bounded from 0. In either event we thus reduce the problem to a portion of S, say S_1, disjoint from a neighborhood of the points (β_j, α_i). We may decompose S_1 into a portion S_2 where $y-\alpha_i$ is bounded from 0 and a portion S_3 where $x-\beta_j$ is bounded from 0 (the sets S_2, S_3 may of course overlap). It will suffice to prove that f is bounded from 0 on S_2. By Theorem 1, or by a direct application of Lemma 4, $a(y)$ is bounded from 0 on S_2. Now for any fixed y write

$$(1) \qquad f(x, y) = a(y)(x - z_1) \cdots (x - z_m)$$

where $z_1, \cdots, z_m$ are the roots of $f(x, y) = 0$, so that $(z_i, y) \in Z(f)$. For (x, y) in S it follows from our hypothesis that $x - z_i$ is bounded from 0. By Lemma 4, f is bounded from 0 on S_2.

REMARK 3. I have not determined whether Theorem 3 holds for three or more variables. Algebraic closure however is necessary: take $f = x^2 + (xy-1)^2$ over the reals. Here $Z(f)$ is void, yet $f \to 0$. One may explain this failure by the necessity of considering the whole complex manifold $f = 0$. However if we assume that S is *bounded* and F real closed, then a result can be obtained just by consideration of the real manifold, and we prove it for any number of variables. The proof is independent of the preceding results.

THEOREM 4. *Let F be a topological real closed field such that $F(i)$, in its Cartesian product topology, is of type V.*[6] *Let $f(x_1, \cdots, x_n)$ be a polynomial with coefficients in F, and $Z(f) \subset F_n$ the manifold $f = 0$. Let S be a bounded set in F_n, disjoint from a uniform neighborhood of $Z(f)$. Then $f(S)$ is bounded away from* 0.

PROOF. Let us write x for x_1, y for $(x_2, \cdots, x_n)$ and $f(x, y) = a(y)x^m + \cdots$. We shall first prove the theorem under the assumption that $a(y)$ is bounded from 0 on S. Let $z_1, \cdots, z_m$ denote the roots of $f(x, y) = 0$. Following Habicht's fundamental device, we form the $M = m(m-1)/2$ symmetric functions in the quantities $(z_i - z_j)^2$ $(i < j)$. After multiplication by suitable powers of $a(y)$ these become polynomials $G_i(y)$ $(i = 1, \cdots, M)$. We choose the numbering so that G_1 corresponds to the product, G_M to the sum of the roots. Let Z_j denote the manifold $G_1 = G_2 = \cdots = G_j = 0$, and Z_0 the entire space. We may embed S in a sequence of sets $S = S_0 \subset S_1 \subset \cdots \subset S_M$ such that each S_i contains a uniform neighborhood of S_{i-1}, and we may suppose that S_M is still bounded and uniformly bounded from $Z(f)$, while $a(y)$ is still bounded from 0 on S_M (the latter is possible by Lemma 2). Set $T_i = S_i \cap Z_i$. We shall prove: (1) f is bounded from 0 on T_M, (2) if f is bounded from 0 on T_{i+1}, it is likewise bounded from 0 on T_i. Since $T_0 = S_0 = S$, this will prove the desired result.

(1) On Z_M, all the z_i are equal and hence real. It follows from

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[6] This condition is equivalent to the assertion that $1 + x^2$ is bounded from 0 in F, and the latter statement of course holds for any ordered field in its order topology. Also, by [6, Theorem 12], $F(i)$ will be of type V if F is complete.

hypothesis that on T_M, $x-s_i$ is bounded from 0. By Lemma 4 so is $f(x, y)$, which is given by (1).

(2) We suppose f bounded from 0 on T_{i+1} and select a neighborhood V of 0 small enough so that f is still bounded from 0 on $T_{i+1}+V$. Thus it remains to prove that f is bounded from 0 on T_i with $T_i \cap (T_{i+1}+V)$ deleted; call this diminished set T_i^*. We assert that T_i^* is disjoint from a uniform neighborhood of Z_{i+1}. For by construction T_i^* is disjoint from a uniform neighborhood of the complement of T_{i+1}; and within T_{i+1} we have in the formation of T_i^* deleted all points within V of Z_{i+1}. Now Z_{i+1} is the zero manifold of $G_1^2 + \cdots + G_{i+1}^2$. By an induction on n (the number of variables) we can assert that on T_i^*, $G_1^2 + \cdots + G_{i+1}^2$ is bounded from 0. Since G_1 to G_i vanish on T_i^*, we have that G_i is bounded from 0. Now G_{i+1} is (except for a power of $a(y)$ which does not affect the argument) precisely the product of all nonvanishing terms among $(s_i-s_j)^2$. For this to be bounded from 0, all these nonvanishing s_i-s_j must likewise be bounded from 0 (one needs here the fact that the s's are bounded, an easy consequence of the boundedness of the coefficients of f, together with the fact that the leading coefficient is bounded from 0). In particular the imaginary parts of the complex roots, if there are any, must be bounded from 0, since these occur in conjugate pairs. Now we observe that in (1), $x-s_i$ is bounded from 0 for real s_i by hypothesis, while for complex s_i, the imaginary part of $x-s_i$ is already bounded from 0. Hence f is bounded from 0.

We now treat the general case where it is not assumed that $a(y)$ is bounded from 0. Changing notation we write

$$f(x, y) = a_0(y)x^m + a_1(y)x^{m-1} + \cdots + a_m(y),$$
$$h_i(y) = a_0^2 + a_1^2 + \cdots + a_i^2.$$

We note that $Z(h_m) \subseteq Z(f)$, and so S is disjoint from a uniform neighborhood of $Z(h_m)$. Since h_m is independent of x, we have by induction on n that h_m is bounded from 0 on S. Consider now the following proposition which we shall designate as P_i: if for a set $T \subseteq S$, $h_i(T)$ is known to be bounded from 0, then $f(T)$ is bounded from 0. We know the truth of P_0 from the first part of the proof. On the other hand P_m is the theorem we are trying to prove, for we have seen that $h_m(S)$ is bounded from 0. Hence it will suffice to prove that P_i implies P_{i+1}. So we suppose P_i known and $h_{i+1}(T)$ bounded from 0, and have to prove $f(T)$ bounded from 0. Now there is a neighborhood V of 0 such that $a_{i+1} \in V$ implies that $h_i(T)$ is bounded from 0. We delete from T the portion where $a_{i+1} \in V$, for there we know from P_i that f is bounded from 0, and study the remaining part, say T_1. We write

$$r(x, y) = a_{i+1}x^{m-1-i} + \cdots + a_m, \qquad k(x, y) = h_i + r^2.$$

Evidently $Z(k) \subseteq Z(f)$ and hence S is disjoint from a uniform neighborhood of $Z(k)$. Moreover in T_1 the leading coefficient of k is bounded from 0. By the first part of the proof, $k(T_1)$ is bounded from 0. Now by choosing h_i small enough we can make

$$f - r = a_0x^m + \cdots + a_ix^{m-i}$$

arbitrarily small in S (this is because S is bounded). Hence there exists a neighborhood W of 0 such that $h_i \in W$ implies that k, r, f are all bounded from 0 on T_1. In that portion of T_1 where h_i is not in W we have again by P_i that f is bounded from 0. This shows that f is bounded from 0 throughout T and proves P_{i+1}. This completes the proof of Theorem 4.

THEOREM 5 (HABICHT). *Let F be a real closed field in its order topology, and R the region $a_i \leqq x \leqq b_i$ $(i = 1, \cdots, n)$. Then for any rational function $f(x_1, \cdots, x_n)$, $f(R)$ is closed.*

The deduction of Theorem 5 from Theorem 4 is identical with Habicht's but we give it for completeness. We suppose f is a polynomial; the device used in the proof of Theorem 2 carries out the extension to a rational function. We may suppose $f \neq 0$ on R and are to prove $f(R)$ bounded from 0. By induction on n we have $f(R')$ bounded from 0, where R' is the boundary of R. This implies that R' is disjoint from a uniform neighborhood of $Z(f)$. Then R is likewise disjoint from a uniform neighborhood of $Z(f)$ (at this point we are using very strongly the special properties of an order topology). Quotation of Theorem 4 completes the proof.

REMARK 4. This method of proof will work for more general regions, for example, regions defined inductively by the property that the boundary lies on a finite number of hyperplanes. On the other hand our examples above have shown that we cannot take an arbitrary bounded closed set for R. One might hazard the conjecture that a bounded region defined by a finite number of polynomial inequalities will do.

5. **A correction.** Professor Jacobson has pointed out an error in [6] which I take this opportunity to correct. The proof of Lemma 9 is inconclusive since the possibility that L is commutative is not excluded, and in fact the validity of Lemma 9 as stated appears to be an open question. Lemma 9 should be deleted, and part (a) of the

proof of Theorem 9 should be replaced by the following remark: if D is a central algebraic division algebra over a field F, with the property that all separable subfields of D are finite-dimensional, then D is finite-dimensional. (This is a slight generalization of [5, Theorem 7].)

To prove this we take (Zorn's lemma) a maximal separable subfield K and let L be its centralizer. By hypothesis K is finite-dimensional and hence [2, Theorem 13] the centralizer of L is again K. The argument of [5, Lemma 2] is now applicable and shows that $L=K$. By [2, Theorem 13] again, $[D:K]=[K:F]$ is finite and D is finite-dimensional.

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A NOTE ON THE SINGULAR MANIFOLDS OF A DIFFERENCE POLYNOMIAL

RICHARD M. COHN

1. Introduction. In a previous paper[1] we defined essential singular manifolds of a difference polynomial in one unknown, and gave an example of such a manifold. By an obvious extension of this definition we may say that if A is an algebraically irreducible difference polynomial in unknowns $y_1, \cdots, y_n$, then an essential irreducible manifold of A which annuls a polynomial of lower effective[2] order than A in y_k, $1 \leqq k \leqq n$, or free of y_k is an *essential singular manifold of A relative to y_k*. The remaining essential irreducible manifolds of A we shall call, as in the case of a polynomial in one unknown, *ordinary manifolds relative to y_k*, and the totality of solutions in these manifolds the general solution *of A relative to y_k*,

The analogous situation in the theory of algebraic differential equations[3] suggests that the essential singular manifolds of a difference polynomial relative to one unknown are also essential singular manifolds relative to any other unknown. It is the purpose of this paper to show that this is actually the case. It will follow that we may drop the term "relative" from the concepts we have just defined. *The essential irreducible manifolds of an algebraically irreducible difference polynomial may be divided into two classes, singular manifolds and ordinary manifolds. The singular manifolds are, in the sense defined above, singular relative to each unknown present in the difference polynomial. The ordinary manifolds are ordinary relative to each unknown, and the totality of solutions they contain may be called the general solution of the difference polynomial.*

We make use, as in the theory of algebraic differential equations, of the *separants*[4] of a difference polynomial. Let A be a difference

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[1] *Manifolds of difference polynomials*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 133–172, referred to below as M.D.P.; §21.

[2] The effective order of a difference polynomial in y_k is defined in M.D.P. as the difference between the orders of the highest and lowest transforms of y_k appearing effectively in the polynomial.

[3] J. F. Ritt, *Differential equations from the algebraic standpoint*, Amer. Math. Soc. Colloquium Publications, vol. 14, 1932, p. 24.

[4] The perhaps unexpected fact that the separant plays a rôle in the theory of difference equations was observed by Poisson, *Mémoire sur les solutions particulières des équations différentielles et des équations aux différences*, J. École Polytech. vol. 6 (1806) pp. 60–125. Poisson's "particular solutions" do not necessarily lie in essential singular manifolds.

polynomial in $y_1, \cdots, y_n$. The y_k-separant of A is defined as the formal partial derivative $\partial A/\partial y_{kr_k}$, where r_k is the order of A in y_k. It will be easy to derive the result stated above from the following theorem.

THEOREM. *Let A be a difference polynomial in unknowns $y_1, \cdots, y_n$. An essential irreducible manifold of A which is singular relative to y_k annuls the y_k-separant of A.*

We proceed to prove this theorem.

2. A lemma on algebraic polynomials. We shall need solutions of difference polynomials in the form of formal power series expansions. We shall first state and prove a lemma concerning the existence of such solutions for algebraic[3] polynomials.

LEMMA. *Let P be an algebraic polynomial in unknowns $x_1, \cdots, x_n$, z with coefficients in an algebraic field K. Let P vanish, and its formal partial derivative $\partial P/\partial z$ not vanish, when $x_1, \cdots, x_n$, z are set equal to zero. Then there exists a formal series in positive integral powers of $x_1, \cdots, x_n$ with coefficients in K which annuls P formally when it is substituted for z.*

We write down the equations which result from the formal differentiation of $P=0$, when z is considered as a function of $x_1, \cdots, x_n$.

$$\begin{gathered}\frac{\partial P}{\partial z}\frac{\partial z}{\partial x_1}+\frac{\partial P}{\partial x_1}=0, \qquad \frac{\partial P}{\partial z}\frac{\partial z}{\partial x_2}+\frac{\partial P}{\partial x_2}=0,\\ \cdots\cdots\cdots,\\ \frac{\partial P}{\partial z}\frac{\partial z}{\partial x_n}+\frac{\partial P}{\partial x_n}=0.\end{gathered}\tag{1}$$

When we substitute zero for $x_1, \cdots, x_n$, z in these equations they determine, since $\partial P/\partial z$ does not vanish, values for $\partial z/\partial x_i$, $i=1, \cdots, n$. These values lie in K.

We differentiate formally the equations (1) obtaining a system of equations of the form

$$\begin{aligned}&\frac{\partial P}{\partial z}\frac{\partial^2 z}{\partial x_j\partial x_i}+\frac{\partial^2 P}{\partial x_j\partial z}\frac{\partial z}{\partial x_i}+\frac{\partial^2 P}{\partial x_j\partial x_i}\\ &\qquad+\frac{\partial z}{\partial x_j}\left(\frac{\partial^2 P}{\partial z^2}\frac{\partial z}{\partial x_i}+\frac{\partial^2 P}{\partial z\partial x_i}\right)=0, \qquad 1\leqq i, j\leqq n;\ i\leqq j.\end{aligned}\tag{2}$$

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[3] That is, polynomials in the usual sense, not difference polynomials.

Letting $x_1, \cdots, x_n$, z be zero in these equations, and giving the partial derivatives $\partial z/\partial x_i$ the values obtained in (1), we obtain unique values lying in $\mathcal{K}$ for each $\partial^2 z/\partial x_i^2$ and $\partial^2 z/\partial x_i \partial x_j$, $i<j$.

We obtain further equations by differentiating the equations of (2). We differentiate each equation of (2) once with respect to each x_i and retain one equation from each set of equations which differ only in the order of taking the derivatives. These equations we call (3). We treat the system (3) as (2) was treated forming (4). We continue this process and consider the system λ consisting of the systems (1), (2), (3), $\cdots$.

From the equations (3) we obtain values of the derivatives of z of third order by letting $x_1, \cdots, x_n$, z be zero and assigning to derivatives of first and second order the values already computed. In this way we obtain the derivatives of z of all orders successively from the equations of λ. Evidently these will all be elements of $\mathcal{K}$.

Using the computed values of the partial derivatives of z we form the Taylor's series z' for z in powers of the x_i. We shall show that this series has the properties stated in the lemma.

It is only necessary to prove that z' annuls P formally when substituted for z. This will follow if we can show that, for every m, when z is replaced in P by the polynomial z_m consisting of all terms of z' of degree less than m, the resulting polynomial $P(z_m)$ has no terms of degree less than m in the x_i. But this in turn is equivalent to showing that, if we take the formal partial derivatives of $P(z_m)$ with respect to the x_i, all derivatives of order less than m, as well as $P(z_m)$ itself, vanish when the x_i are set equal to zero. Now the equations expressing this condition are obtained from the equation $P=0$ and those equations of the system λ which involve no derivatives of order exceeding $m-1$ by replacing z by z_m and setting $x_1, \cdots, x_n$ equal to zero. But these equations must be valid, for the partial derivatives of z_m of order less than m are identical, when the x_i are zero, with the partial derivatives of z'; and the Taylor's series z' was so constructed that its derivatives satisfy the equations λ and $P=0$ under the stated condition. Thus the lemma is proved.

3. **Proof of the theorem.** We turn now to the proof of the theorem on separants. We consider an algebraically irreducible difference polynomial A with coefficients in a difference field $\mathcal{J}$. The unknowns in A we shall now denote, for convenience, by the letters $u_1, \cdots, u_q$; y. We consider an irreducible manifold $\mathfrak{M}$ annulling A, but not $\partial A/\partial y_r$, where r is the order of A in y. We shall prove that $\mathfrak{M}$ is contained in or coincides with one of the ordinary manifolds of A relative to y.

We may assume that the order and effective order of A in y are equal. For, if not, we could replace y by a new variable $\bar{y}=y_k$, where k is the difference between the order and effective order. A would thus be transformed into a polynomial of equal order and effective order in $\bar{y}$; and the other conditions of the problem remain unaltered.

We denote by Λ the reflexive prime ideal consisting of all polynomials which vanish for all solutions in $\mathfrak{M}$. Let $y=\alpha$, $u_i=\gamma_i$, be a general point of Λ. Upon making the substitution $y=z+\alpha$, $u_i=v_i+\gamma_i$, A becomes a difference polynomial $\bar{A}$ in the v_i and z, which vanishes when z and the v_i are set equal to zero; while $\partial A/\partial y_r$ is carried into the polynomial $\partial\bar{A}/\partial z_r$ which does not vanish with z and the v_i. It follows from the lemma that $\bar{A}$ can be annulled formally by substituting for z_r a series z_r' in positive integral powers of $z, z_1, \cdots, z_{r-1}$, and certain v_{ij}. The coefficients in z_r' lie in the field $\mathcal{G}$ obtained by adjoining α and the γ_i to $\mathcal{F}$.

We now construct a formal powers series from z_r' by the following procedure. First we replace each coefficient by its transform in $\mathcal{G}$. Next we replace z_i by z_{i+1}, $i=0, 1, \cdots, r-1$, throughout the series. Finally we replace each v_{ij} by $v_{i,j+1}$. The resulting series we shall call z_{r+1}''. We substitute for z_r in the series z_{r+1}'' the series z_r'. We obtain thus a series z_{r+1}' in positive integral powers of $z, z_1, \cdots, z_{r-1}$ and certain v_{ij}, with coefficients in $\mathcal{G}$.

Let the series z_{r+2}'' result from z_{r+1}' by replacing each coefficient by its transform, each v_{ij} by $v_{i,j+1}$, and then replacing each z_i, $0\leqq i\leqq r-1$, by z_{i+1}. From z_{r+2}'' we obtain z_{r+2}' by replacing z_r in the expansion by z_r'. We continue in this way to construct series z_i', $i=r, r+1, r+2, \cdots$, in powers of $z, z_1, \cdots, z_{r-1}$ and the v_{ij}, and series z_i'', $i=r+1, r+2, r+3, \cdots$ in powers of $z_1, z_2, \cdots, z_r$ and the v_{ij}. The coefficients of each z_i' and z_i'' lie in $\mathcal{G}$.

Let $y_i'=z_i+\alpha_i$, $0\leqq i<r$, and $y_i'=z_i'+\alpha_i$, $i\geqq r$. Let $y_i''=z_i+\alpha_i$, $1\leqq i<r+1$, and $y_i''=z_i''+\alpha_i$, $i\geqq r+1$. Finally let $u_{ij}'=u_{ij}''=v_{ij}+\gamma_{ij}$, $1\leqq i\leqq q$, $j\geqq 0$.

We consider any polynomial C in the unknowns $u_1, \cdots, u_q$; y with coefficients in the field $\mathcal{F}$. The result of substituting y_i' for y_i and u_{ij}' for u_{ij} in C and its transform C_1 is the pair of formal power series C' and C_1' with coefficients in $\mathcal{G}$. Let C_1'' be the expansion which results from C_1 upon making the substitution of y_i'' for y_i, $i\geqq 1$, and u_{ij}'' for u_{ij}. Evidently C_1'' can be constructed from C' by replacing z_i by z_{i+1}, $0\leqq i<r$, v_{ij} by $v_{i,j+1}$, and each coefficient of the expansion by its transform. C_1' may now be obtained from C_1'' by replacing z_r by z_r'. We see that, if C' vanishes identically, so does C_1'', and therefore C_1'. It follows that the set of all polynomials

which vanish when the y_i and the u_{ij} are replaced by the y_i' and u_{ij}' forms a difference ideal Ψ. Evidently Ψ is prime and contains A.

Let Σ be the set of all polynomials whose transforms of any order are in Ψ. Evidently Σ is a prime reflexive difference ideal. We shall show that no polynomial of order less than r in y, or free of y, holds Σ. First we consider a polynomial D of order not exceeding $r-1$ in y and show that it is not in Ψ. The substitutions $y_i = y_i'$, $u_{ij} = u_{ij}'$ become, in this special case, a linear transformation with an inverse. Since D is not identically zero, the polynomial into which it is carried is not zero. Thus D is not in Ψ. We now assume that all polynomials of effective order not exceeding $r-1$ and order not exceeding $r-1+k$ are not in Ψ. We shall show that a polynomial D of order $r+k$ and effective order not exceeding $r-1$ in y is not in Ψ. We have shown elsewhere[6] that the resultant R with respect to y_{r+k} of D and A_k is a nonzero polynomial of order at most $r-1+k$ and effective order at most $r-1$. Our assumption shows that R is not annulled by the substitutions in question, so that D cannot be annulled by them. We can now show by induction that no polynomial of effective order at most $r-1$ in y is in Ψ, so that no polynomial of order not exceeding $r-1$ in y is in Σ. It follows that Σ is one of the reflexive prime ideals whose solutions constitute the ordinary manifolds of A.

Some transform of any polynomial of Σ must be annulled by the y_i' and u_{ij}'. Now on substituting these quantities for the y_i and u_{ij} in the transform C_k of a polynomial C a term of zero degree is obtained which is equal to the result of substituting α for y and γ_i for u_i in C_k. Since this term must vanish for every polynomial C of Σ and a suitable k depending on C, C_k and therefore C is annulled by the general point of Λ. Then Σ holds Λ. This proves the theorem.

4. **Proof of the principal result.** We now return to the proof of the possibility of the division of the essential irreducible manifolds of a difference polynomial into singular and ordinary manifolds. We consider an algebraically irreducible difference polynomial A in unknowns $y_1, \cdots, y_n$ with coefficients in a difference field $\mathcal{F}$. Let $\mathfrak{M}$ be an essential irreducible manifold of A, and let Λ be the corresponding reflexive prime ideal. We wish to show that $\mathfrak{M}$ is either an essential singular manifold of A relative to each y_i or an ordinary manifold relative to each y_i. For this purpose it will suffice to assume that $\mathfrak{M}$ is an essential singular manifold relative to y_1, and prove that it is singular relative to y_2.

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[6] M. D. P., §14. The presence of u_{ij}, which are not included in the discussion of M.D.P., does not affect the argument.

If we make this assumption it follows that $\mathfrak{M}$ annuls $\partial A/\partial y_{1r}$, where r is the order of A in y_1. Let s be the order of A in y_2. We form the resultant R of A and $\partial A/\partial y_{1r}$, considered as algebraic polynomials in y_{2s}. Since A is irreducible, and cannot be a factor of $\partial A/\partial y_{1r}$, R is a nonzero polynomial, free of y_{2s}, which is annulled by $\mathfrak{M}$. Since R is of lower effective order than A in y_2, $\mathfrak{M}$ must be an essential singular manifold of A relative to y_2. The proof is now complete.

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DISTINCT REPRESENTATIVES OF SUBSETS

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1. Introduction. Let W be a set of elements $a_i \cdot W=\{a_1, \cdots\}$ and let $U\{S_1, \cdots, S_j, \cdots\}$ be an indexed system of subsets of W. We wish to choose distinct representatives of the subsets. If $a_j = R(S_j)$ designates the representative of the subset S_j, then we require $R(S_j) \in S_j$ for all j and $R(S_j) \neq R(S_k)$ if $j \neq k$. It is to be emphasized that subsets are distinguished only by their indices and distinct subsets may contain the same elements. An obviously necessary condition for the existence of distinct representatives is:

Condition C: *Every k distinct subsets contain between them at least k distinct elements, for every finite k.* P. Hall[1] has shown that if the number of subsets is finite, condition C is also sufficient for the existence of a system of distinct representatives, or SDR as we shall abbreviate. This condition is no longer sufficient if the number of subsets is infinite. As a counter example consider $U\{S_0, S_1, \cdots\}$ where $S_0 = \{a_1, a_2, \cdots\}$, $S_i = \{a_i\}$, $i = 1, 2, \cdots$. Here condition C is easily shown to hold for the subsets, but clearly no representative may be selected for S_0 which is not also a representative of some S_i.

In this paper it is shown that condition C is sufficient if every subset S_j is finite, and also an estimate on the number of systems of distinct representatives is given. This latter result is applied to Latin squares.

THEOREM 1. *Given an indexed system* $U\{S_1, \cdots, S_j, \cdots\}$ *of finite subsets of a set* $W\{a_1, \cdots, a_i, \cdots\}$. *If the subsets satisfy condi-*

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[1] P. Hall, *On representatives of subsets*, J. London Math. Soc. vol. 10 (1935) pp. 26–30.

tion C *for every finite k then there exists a system of distinct representatives $R(S_j)$ of the subsets.*

THEOREM 2. *Under the hypotheses of Theorem* 1 *if the smallest subset contains r distinct elements, then there are at least $r!$ different systems of distinct representatives.*

THEOREM 3. *There are at least $n!(n-1)! \cdots 2!\cdot 1!$ different n by n Latin squares.*

2. **Blocks and critical blocks.** A finite system of subsets $\{S_j\}$ of $U\{S_1, \cdots, S_j\}$ together with the set of elements $\{a_i\}$ of these subsets will be called a block B. If we wish to indicate that there are r subsets and s distinct elements in the block B, we write $B=B_{r,s}$. We shall assume that every subset S_j contains only a finite number of elements. Condition C is equivalent to the condition $s \geqq r$ for every block $B_{r,s}$ and we shall assume this to hold. If $s=r$ we say the block $B_{r,r}$ is a critical block.

There is a natural partial ordering for the blocks, if we define $B \subseteq B'$ whenever every subset S_j of the block B is a subset of the block B'. With respect to this partial ordering it is easily verified that the blocks form a distributive lattice, since they are essentially finite sets of S_j's. It will be desirable to consider the void block as an improper critical block $B_{0,0}$.

LEMMA 1. *The union $B_{k,k} \cup B_{l,l}$ and crosscut $B_{k,k} \cap B_{l,l}$ of critical blocks are again critical blocks.*

PROOF. Given two critical blocks $B_{k,k}$ and $B_{l,l}$, let $B_{k,k} \cap B_{l,l} = B_{r,s}$ and $B_{k,k} \cup B_{l,l} = B_{u,v}$. Let $S_1, \cdots, S_r$ be the subsets and $a_1, \cdots, a_s$ be the elements of $B_{r,s}$. Then $s \geqq r$. In $B_{k,k} \cup B_{l,l} = B_{u,v}$, we have $u=k+l-r$ since there are in $B_{u,v}$ the sets $S_1, \cdots, S_r$ and $k-r$ further sets of $B_{k,k}$ and $l-r$ further sets of $B_{l,l}$. Also in $B_{u,v}$ there are at most $k+l-s$ different elements since at least the elements $a_1, \cdots, a_s$ appear in both $B_{k,k}$ and $B_{l,l}$. Hence $k+l-s \geqq v \geqq u = k+l-r$ and also $s \geqq r$, whence $s=r$, $v=u$ and both $B_{r,s}$ and $B_{u,v}$ are critical blocks.

LEMMA 2. *Given an indexed system of finite subsets $U\{S_1, \cdots, S_j, \cdots\}$ of a set of elements $W\{a_1, \cdots, a_i, \cdots\}$ satisfying condition* C *and a proper critical block $B_{k,k}$ of U. If the system $U'\{S_1', \cdots, S_j', \cdots\}$ consists of the subsets $S_i' = S_i$ for $S_i \in B_{k,k}$, and, for $S_j \notin B_{k,k}$, S_j' contains those elements of S_j which are not elements of $B_{k,k}$, then $U'\{S_1', \cdots, S_j', \cdots\}$ also satisfies condition* C.

PROOF. This lemma can be more loosely phrased by saying that deletion of elements of a critical block $B_{k,k}$ from sets not in $B_{k,k}$ does not alter the validity of condition C. Let $B_{r,s}$ be any block of U and $B'_{r,t}$ the corresponding block of U' (no subsets have been deleted). Also let $B_{r,s} \cap B_{k,k} = B_{l,f}$, $B_{r,s} \cup B_{k,k} = B_{m,n}$. The number of elements of $B_{r,s}$ not in $B_{k,k}$ is $n-k$. As $n \geqq m = r+k-l$, we have $n-k \geqq r-l$. In the deleted block $B_{r,t}$ there are l subsets of $B_{k,k}$ and $r-l$ other subsets and there are $f \geqq l$ elements of $B_{k,k}$ and $n-k$ other elements. Hence $t = f+n-k \geqq l+r-l = r$ and so condition C is also satisfied for the deleted block $B'_{r,t}$, and hence for the deleted system U'.

Since an element of a critical block could not be a representative of any subset not in that block, the deleted system U' will have the same SDR's (if there are any) as the system U.

3. The principal theorems.

PROOF OF THEOREM 1. We divide the proof into two cases, according to the cardinal number α of sets S_j in U: Case 1: α finite, Case 2: α infinite.

Case 1: α finite. Suppose first there is a critical block $B_{k,k}$, $1 \leqq k \leqq \alpha-1$, not the whole system U. From the sets of U not in $B_{k,k}$ delete the elements of $B_{k,k}$. Then U', the deleted system, consists of $B_{k,k}$ and another block $B'_{\alpha-k,s}$ with no sets or elements in common. By Lemma 2, U' satisfies condition C, whence both $B_{k,k}$ and $B'_{\alpha-k,s}$ as systems U' and U'' satisfy condition C. By induction both of these have SDR's and, as they are disjoint, together these SDR's form an SDR for U. Now suppose there is no critical block except possibly the whole system U. This means that for an arbitrary block $B_{r,s}$ with $r \leqq \alpha-1$ we have $s \geqq r+1$. Now take an arbitrary element of S_α as a representative and delete this element in every other subset. In this deletion any block $B_{r,s}$ in $S_1, \cdots, S_{\alpha-1}$ goes into a block $B'_{r,t}$ where $t = s$ or $s-1$ since only one element has been deleted. As $s \geqq r+1$, $t \geqq r$ and so Condition C holds for $U'(S'_1, \cdots, S'_{\alpha-1})$ whence by induction there is an SDR for U' and this together with the representative for S_α gives an SDR for U.

This part of the theorem is equivalent to Philip Hall's result.

Case 2: α infinite. If from every set S_j of U we delete an element a_1 which does not belong to any critical block, then the deleted system still satisfies condition C. For in any block $B_{r,s}$ which contains a_1 we have $s \geqq r+1$, and after deletion the block becomes $B'_{r,s-1}$ with $s-1 \geqq r$. Condition C is a property of finite character in the sense used by Tukey[3] since it applies to finite collections of subsets S_j. If we consider the set T of pairs (a_i, S_j), $a_i \in S_j$, the following property of subsets T^* of T will also be a property of finite character. Property D: "After deleting a_i from S_j for every pair $(a_i, S_j) \in T$, the system $U\{S_j\}$ still satisfies Condition C." Hence, using the third form of Zorn's Lemma[3] (Tukey[2] p. 7), we may conclude that there is a maximal system T^* with property D. Put more simply, there is a maximal amount of deleting of elements a_i from subsets S_j which may be done without destroying the validity of Condition C. Let us suppose this done. Then by the first sentence of this paragraph, every element belongs to a critical block. From Lemma 2, every critical block is disjoint from all sets not in it. Hence we have left only disjoint critical blocks. By Case 1, a critical block, being finite, possesses an SDR. Hence within each critical block deletion may continue until each S_j contains only one element. Hence after maximal deletion there is exactly one element left in each set, and as Condition C is still satisfied, we have left an SDR for U.

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[3] J. W. Tukey, *Convergence and uniformity in topology*, Annals of Mathematics Studies, No. 2, 1940, p. 7.

The reader will note that if initially every element of U belongs to a critical block, the argument goes through without appeal to the axiom of choice.

Proof of Theorem 2. Two systems of distinct representatives are different if they give different representatives for any subset. This theorem gives a lower limit on the number of different SDR's. I assert that in U there is one S_j in which an aribtrary element may be taken as a representative in an SDR. Suppose U contains critical blocks. Choose a minimal critical block $B_{k,k}$. Here an S_j in $B_{k,k}$ has the property that any element of S_j may be a representative. For deleting a chosen element of S_j from the other subsets of $B_{k,k}$ leaves them satisfying condition C, whence the chosen element appears in an SDR for $B_{k,k}$. But in any SDR for U we may replace any SDR for $B_{k,k}$ by any other. On the other hand, if U contains no critical blocks, we may use an arbitrary representative from any one subset S_j without violating condition C by deleting it from the remaining subsets. Hence given U with at least r elements in every S, we may choose fron an appropriate S_j a representative in at least r ways and delete it elsewhere without violating condition C. Now apart from this S_j every other deleted subset contains at least $r-1$ elements. By induction this possesses at least $(r-1)!$ different SDR's. Hence U possesses at least $r(r-1)!=r!$ different SDR's.

4. **Application to Latin squares.** Theorem 2 may be used to improve

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[3] Cf. C. Kuratowski, *L'élimination des nombres transfinis*, Fund. Math. (1922) pp. 76–108, Theorem 42, p. 89.

a result of the author's[4] on Latin squares. Given a Latin rectangle with n columns and $r<n$ rows. The problem of finding an $(r+1)$st row to add to this Latin rectangle is equivalent to finding an SDR for the n sets $C_1 \cdots C_n$ where each C_i consists of the $n-r$ letters not in the ith column. Applying theorem 2, we see that this may be done in at least $(n-1)!$ ways. Hence adding a row at a time we see that the number of distinct n by n Latin squares is at least $n!\,(n-1)! \cdots 2!\,1!$.

This number is surely too small. A better estimate may be obtained by combining the results given here with those of Erdős and Kaplansky.[5] This would increase the first k factors for $k<(\log n)^{3/2-\epsilon}$. The last factors are also too small and it would be interesting to obtain further improvements.

Note added in proof. Another proof of Theorem 1, by C. J. Everett and George Whaples, is to appear in Amer. J. Math. See Bull. Amer. Math. Soc. Abstract 53-5-170.

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[4] Marshall Hall, *An existence theorem for Latin squares*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 387–388.

[5] P. Erdős and I. Kaplansky, *The asymptotic number of Latin rectangles*, Amer. J. Math. vol. 68 (1946) pp. 230–236.

EXISTENCE THEOREMS CONNECTED WITH THE PICARD-VESSIOT THEORY OF HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATIONS

E. R. KOLCHIN

1. **Introduction.** The Picard-Vessiot theory, as recently reformulated by the author,[1] deals with an abstract ordinary differential field $\mathcal{F}$ of characteristic 0 having an algebraically closed field of constants $\mathcal{C}$, and a differential extension field $\mathcal{G}$ over $\mathcal{F}$ with the two properties:

(a) There exists a homogeneous linear differential polynomial $L(y)=y^{(n)}+p_1y^{(n-1)}+\cdots+p_ny$ (each p_i in $\mathcal{F}$) which has a fundamental system of solutions $\eta_1, \cdots, \eta_n$ such that $\mathcal{G}=\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$;[2]

(b) The field of constants of $\mathcal{G}$ is $\mathcal{C}$.

Such a $\mathcal{G}$ is called a *Picard-Vessiot* extension of $\mathcal{F}$. It is to be noted that the extension $\mathcal{G}$ is given, and the existence of the differential polynomial $L(y)$ with the properties (a) and (b) is postulated. It is not immediately apparent, and it would be of interest to know, whether a given $L(y)$, with coefficients p_i in $\mathcal{F}$, always has a fundamental system of solutions $\eta_1, \cdots, \eta_n$ such that $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ is a Picard-Vessiot extension of $\mathcal{F}$ (that is, contains no constant not in $\mathcal{C}$). This question was posed by R. Baer (in his critical note on the then current status of the Picard-Vessiot theory, included among comments by O. Haupt in F. Klein's *Vorlesungen über hypergeometrische Funktionen*, Berlin, 1933), who remarked that the difficulty lay not in proving the existence of a fundamental system of solutions (see PV, §15), but in proving the existence of one which brings in no new constants.

A differential extension field $\mathcal{H}$ of $\mathcal{F}$ may be an extension of $\mathcal{F}$ by integrals, exponentials of integrals, and algebraic functions. If it is, and if the field of constants of $\mathcal{H}$ is still $\mathcal{C}$, then $\mathcal{H}$ is called a *liouvillian* extension of $\mathcal{F}$. The Picard-Vessiot theory provides a group-theoretic answer to the question of when a Picard-Vessiot extension $\mathcal{G}$ of $\mathcal{F}$ is

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[1] *Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations*, Ann. of Math. (2) vol. 49 (1948) pp. 1–42. This paper, referred to below as "PV", contains the necessary background for the present note.

[2] The notation $\mathcal{F}\langle\cdots\rangle$ indicates, as usual, differential field adjunction. Thus $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ is the differential field consisting of all differential rational functions of $\eta_1, \cdots, \eta_n$ with coefficients in $\mathcal{F}$.

(contained in) a liouvillian extension $\mathcal{H}$ (PV, §25). It is natural to broaden this question in two respects: first, by relinquishing the requirement that the field of constants of $\mathcal{H}$ be $\mathcal{C}$ (that is, by demanding only that $\mathcal{H}$ be an extension of $\mathcal{F}$ by integrals, exponentials of integrals, and algebraic functions, without demanding that $\mathcal{H}$ be liouvillian); second, by inquiring whether or not $L(y)$ has at least one solution contained in such an extension $\mathcal{H}$.

It is the purpose of the present note to show how answers to the points raised in the two preceding paragraphs can be obtained (see Theorems 2 and 4, below) as corollaries to a general theorem on algebraic differential equations due to J. F. Ritt. This theorem can be formulated in the following way ($\mathcal{F}$ is an ordinary differential field of characteristic 0, $y_1, \cdots, y_n$ are unknowns, and m is a nonnegative integer less than n).

Ritt's Theorem. *Let* Π *be a prime differential ideal in* $\mathcal{F}\{y_1, \cdots, y_n\}$,[3] *let* J *be a differential polynomial in* $\mathcal{F}\{y_1, \cdots, y_n\}$ *but not in* Π, *let* $\Pi_0 = \Pi \cap \mathcal{F}\{y_1, \cdots, y_m\}$. *Then there exists a differential polynomial* J_0 *in* $\mathcal{F}\{y_1, \cdots, y_m\}$ *but not in* Π_0 *such that every solution of* Π_0 *which is not a solution of* J_0 *can be completed*[4] *into a solution of* Π *which is not a solution of* J.

In Ritt's proof (Trans. Amer. Math. Soc. vol. 48 (1940) pp. 542–552; see especially pp. 543–545) it is assumed that $\mathcal{F}$ consists of functions of a complex variable meromorphic in a given region, but it is not very difficult to modify his proof to obtain a purely algebraic one, valid for abstract $\mathcal{F}$.

2. The existence theorems.

We work with an ordinary differential field $\mathcal{F}$ of characteristic 0 with an algebraically closed field of constants $\mathcal{C}$.

Theorem 1. *Let* Σ *be a proper subset of* $\mathcal{F}\{y_1, \cdots, y_n\}$, *and let* J *be a differential polynomial in* $\mathcal{F}\{y_1, \cdots, y_n\}$ *but not in the perfect differential ideal* $\{\Sigma\}$. *Then* Σ *has a solution* $\eta_1, \cdots, \eta_n$ *for which* $J \neq 0$ *and the field of constants of* $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ *is* $\mathcal{C}$.

Proof. Since $J \notin \{\Sigma\}$, we have $J \notin \Pi$, where Π is one of the prime components of $\{\Sigma\}$ (in the representation of $\{\Sigma\}$ as an intersection of prime differential ideals none of which contains any other). It follows that Π has a solution $\zeta_1, \cdots, \zeta_n$, which is not a solution of J, such that each ζ_i is differentially algebraic over $\mathcal{F}$. (To see this let $y_1, \cdots, y_m$ be a complete set of arbitrary unknowns for Π; by Ritt's theorem any solution of Π_m which is not a solution of a certain differential polynomial $J_0 \notin \Pi_m$ can be completed to a solution of Π which is not a solution of J; but $\Pi_m = (0)$, so that any set $\zeta_1, \cdots, \zeta_m$ of differentially algebraic elements not annulling J_0 can be completed to a solution $\zeta_1, \cdots, \zeta_n$ of Π not annulling J; clearly this solution has the desired property.) We suppose that of all solutions $\zeta_1, \cdots, \zeta_n$ of Π, not annulling J and having the property that each ζ_i is differentially algebraic over $\mathcal{F}$, ours leads to the smallest value of p = degree of transcendency of $\mathcal{F}\langle\zeta_1, \cdots, \zeta_n\rangle$ over $\mathcal{F}$.

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[3] The notation $\mathcal{F}\{\cdots\}$ indicates, as usual, differential ring adjunction. Thus $\mathcal{F}\{y_1, \cdots, y_n\}$ is the differential ring consisting of all differential polynomials in $y_1, \cdots, y_n$ with coefficients in $\mathcal{F}$.

[4] Following Ritt, we say that a solution $\eta_1, \cdots, \eta_m$ of Π_0 can be *completed* into a solution of Π provided there exist elements $\eta_{m+1}, \cdots, \eta_n$ in some extension of $\mathcal{F}\langle\eta_1, \cdots, \eta_m\rangle$ such that $\eta_1, \cdots, \eta_n$ is a solution of Π.

Let γ be a constant in $\mathcal{F}\langle\zeta_1, \cdots, \zeta_n\rangle$, and suppose that γ is transcendental over $\mathcal{C}$. Then (PV, §14, Theorem 2) γ is transcendental over $\mathcal{F}$, too. Introducing a new unknown w, let Γ be the prime differential ideal in $\mathcal{F}\{w, y_1, \cdots, y_n\}$ with generic solution (PV, §11) $\gamma, \zeta_1, \cdots, \zeta_n$. Then $\Gamma_0 = \Gamma \cap \mathcal{F}\{w\}$, which has the generic solution γ, equals $[w']$. By Ritt's theorem there is a $J_0 \in \mathcal{F}\{w\}$, with $J_0 \notin [w']$, such that every solution of $[w']$ (that is, any constant) not annulling J_0 can be completed into a solution of Γ not annulling J. Since there obviously exists a constant $c \in \mathcal{C}$ for which $J_0 \neq 0$, Γ has a solution $c, \eta_1, \cdots, \eta_n$ for which $J \neq 0$. By Gourin's theorem (PV, §13) we have: degree of transcendency of $\mathcal{F}\langle c, \eta_1, \cdots, \eta_n\rangle$ over $\mathcal{F}$ is less than degree of transcendency of $\mathcal{F}\langle\gamma, \zeta_1, \cdots, \zeta_n\rangle$ over $\mathcal{F}$. Therefore $\eta_1, \cdots, \eta_n$ is a solution of Π not annulling J with the property that each η_i is differentially algebraic over $\mathcal{F}$, and we have: degree of transcendency of $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ over $\mathcal{F}$ is less than p. This contradicts the definition of p, shows that γ is algebraic over $\mathcal{C}$, and proves the theorem.

THEOREM 2. *If* $L(y) = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y$, *where each* $p_i \in \mathcal{F}$, *then* $L(y)$ *has a fundamental system of solutions* $\eta_1, \cdots, \eta_n$ *such that the field of constants of* $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ *is* $\mathcal{C}$.

PROOF. The wronskian determinant $W(y_1, \cdots, y_n)$ is not contained in $\{L(y_1), \cdots, L(y_n)\}$, as $W(y_1, \cdots, y_n)$ is of order $n-1$. By Theorem 1, $\{L(y_1), \cdots, L(y_n)\}$ has a solution $\eta_1, \cdots, \eta_n$ such that $W(\eta_1, \cdots, \eta_n) \neq 0$ and the field of constants of $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ is $\mathcal{C}$. Since $W(\eta_1, \cdots, \eta_n) \neq 0$, the elements $\eta_1, \cdots, \eta_n$ are linearly independent over constants, and constitute a fundamental system of solutions of $L(y)$.

REMARK. As observed by the referee, Theorem 2 extends to systems $L_i(y_1, \cdots, y_n) = 0$, $i = 1, \cdots, n$, where $L_i(y_1, \cdots, y_n) = y_i'$

$-\sum_{j=1}^{n} a_{ij}y_j$ (each $a_{ij}\in\mathcal{F}$). It is merely necessary to introduce n^2 unknowns y_{ij} and to note that $D(\cdots, y_{ij}, \cdots) = \det|y_{ij}|$ is of order 0 and therefore not contained in $\{\cdots, L_i(y_{j1}, \cdots, y_{jn}), \cdots\}$. The proof then proceeds as above.

We now consider differential fields which are extensions of $\mathcal{F}$ by integrals, exponentials of integrals, and algebraic functions, that is, extensions of the form $\mathcal{F}\langle\alpha_1, \cdots, \alpha_m\rangle$ where either $\alpha_i' \in \mathcal{F}\langle\alpha_1, \cdots, \alpha_{i-1}\rangle$ or $\alpha_i'/\alpha_i \in \mathcal{F}\langle\alpha_1, \cdots, \alpha_{i-1}\rangle$ or α_i is algebraic over $\mathcal{F}\langle\alpha_1, \cdots, \alpha_{i-1}\rangle$, $i=1, \cdots, m$.

If $\mathcal{K}$ is such an extension of $\mathcal{F}$, and if in addition the field of constants of $\mathcal{K}$ is again $\mathcal{C}$, then $\mathcal{K}$ is said to be a ***liouvillian*** extension of $\mathcal{F}$.

In any case, we introduce (corresponding to PV, §24) ten types of extensions by integrals, exponentials of integrals and algebraic functions, namely, extensions by

(1) integrals, exponentials of integrals, and algebraic functions,
(2) integrals and exponentials of integrals,
(3) exponentials of integrals,
(4) integrals and algebraic functions,
(5) integrals and radicals,
(6) exponentials of integrals,
(7) integrals,
(8) algebraic functions,
(9) radicals,
(10) rational functions.

THEOREM 3. *Let* $\Sigma\subset\mathcal{F}\{y_1, \cdots, y_n\}$, $J\in\mathcal{F}\{y_1, \cdots, y_n\}$, $J\notin\{\Sigma\}$. *If* Σ *has a solution* $\zeta_1, \cdots, \zeta_n$ *for which* $J\neq 0$ *such that* $\mathcal{F}\langle\zeta_1, \cdots, \zeta_n\rangle$ *is contained in an extension of* $\mathcal{F}$ *of one of the types* (1)–(10), *then* Σ *has a solution* $\eta_1, \cdots, \eta_n$ *for which* $J\neq 0$ *such that* $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ *is contained in a liouvillian extension of* $\mathcal{F}$ *of the same type.*

PROOF. Suppose $\mathcal{F}\langle\zeta_1, \cdots, \zeta_n\rangle\subseteq\mathcal{F}\langle\alpha_1, \cdots, \alpha_m\rangle$, where each α_i is, appropriately, either an integral of an element of $\mathcal{F}\langle\alpha_1, \cdots, \alpha_{i-1}\rangle$, an exponential of an integral of such an element, or algebraic over $\mathcal{F}\langle\alpha_1, \cdots, \alpha_{i-1}\rangle$. Then there exist differential polynomials $P_i(u_1, \cdots, u_m)$, $Q(u_1, \cdots, u_m)\in\mathcal{F}\{u_1, \cdots, u_m\}$, where $u_1, \cdots, u_m$ are new unknowns, such that $\zeta_i = P_i(\alpha_1, \cdots, \alpha_m)/Q_i(\alpha_1, \cdots, \alpha_m)$, $i=1, \cdots, m$. There also exist, for $i=1, \cdots, m$, differential polynomials $M_i(u_1, \cdots, u_{i-1})$, $N_i(u_1, \cdots, u_{i-1})\in\mathcal{F}\{u_1, \cdots, u_{i-1}\}$ such that either $\alpha_i' = M_i(\alpha_1, \cdots, \alpha_{i-1})/N_i(\alpha_1, \cdots, \alpha_{i-1})$, or $\alpha_i'/\alpha_i = M_i(\alpha_1, \cdots, \alpha_{i-1})/N_i(\alpha_1, \cdots, \alpha_{i-1})$, or $\alpha_i N_i(\alpha_1, \cdots, \alpha_{i-1})$ is integral and algebraic over $\mathcal{F}\{\alpha_1, \cdots, \alpha_{i-1}\}$. Let Λ be the prime differential ideal in $\mathcal{F}\{y_1, \cdots, y_n, u_1, \cdots, u_m\}$ with generic solution $\zeta_1, \cdots, \zeta_n, \alpha_1, \cdots, \alpha_m$. By Theorem 1, Λ has a solution $\eta_1, \cdots, \eta_n, \beta_1, \cdots, \beta_m$ for which $J(\eta_1, \cdots, \eta_n)Q_1(\beta_1, \cdots, \beta_m) \cdots Q_n(\beta_1, \cdots, \beta_m)N_1 \cdots N_m(\beta_1, \cdots, \beta_{m-1})\beta_1 \cdots \beta_m \neq 0$, such that the field of constants of $\mathcal{F}\langle\eta_1, \cdots, \eta_n, \beta_1, \cdots, \beta_m\rangle$ is $\mathcal{C}$. It is now obvious that $\eta_1, \cdots, \eta_n$ is a solution of Σ not annulling J, that $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle \subseteq \mathcal{F}\langle\beta_1, \cdots, \beta_m\rangle$, and that $\mathcal{F}\langle\beta_1, \cdots, \beta_m\rangle$ is a liouvillian extension of $\mathcal{F}$ of the same type as $\mathcal{F}\langle\alpha_1, \cdots, \alpha_m\rangle$.

A homogeneous linear differential polynomial $L(y) = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y$, with each p_i in $\mathcal{F}$, is *linearly reducible* over $\mathcal{F}$ if there exist two homogeneous linear differential polynomials $M(y)$ and $N(y)$ with coefficients in $\mathcal{F}$ and of positive order, such that $L(y) = M(N(y))$. If $L(y)$ is not linearly reducible over $\mathcal{F}$ it is *linearly irreducible* over $\mathcal{F}$.

THEOREM 4. *Let $L(y) = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y$, with each p_i in $\mathcal{F}$, be linearly irreducible over $\mathcal{F}$. If $L(y)$ has one solution contained in an extension of $\mathcal{F}$ of one of the types* (1)–(10), *then $L(y)$ has a fundamental system of solutions $\eta_1, \cdots, \eta_n$ such that $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ is a liouvillian extension of $\mathcal{F}$ of the same type.*

PROOF. By Theorem 3, $L(y)$ has a solution η contained in a liouvillian extension $\mathcal{K}$ of $\mathcal{F}$ of the required type. By Theorem 2, $L(y)$ has a fundamental system of solutions $\zeta_1, \cdots, \zeta_n$ such that the field of constants of $\mathcal{K}\langle\zeta_1, \cdots, \zeta_n\rangle$ is $\mathcal{C}$. Therefore η is a linear combination over $\mathcal{C}$ of $\zeta_1, \cdots, \zeta_n$, so that $\mathcal{G} = \mathcal{F}\langle\zeta_1, \cdots, \zeta_n\rangle$ is a Picard-Vessiot extension of $\mathcal{F}$ containing η. Letting $\mathfrak{G}$ be the group of all automorphisms of $\mathcal{G}$ over $\mathcal{F}$, we see that, in the linear space over $\mathcal{C}$ spanned by $\zeta_1, \cdots, \zeta_n$, the linear subspace spanned by the set of all elements $\sigma\eta$ $(\sigma \in \mathfrak{G})$ is invariant under $\mathfrak{G}$. Since $L(y)$ is linearly irreducible, the only invariant linear subspaces are the zero space and the whole space (PV, §22, Theorem 1). Therefore there exist n automorphisms $\sigma_1, \cdots, \sigma_n$ in $\mathfrak{G}$ such that $\sigma_1\eta, \cdots, \sigma_n\eta$ are linearly independent over $\mathcal{C}$.

Now, σ_1 can be extended to an isomorphism τ_1 of $\mathcal{K}_1 = \mathcal{K}\langle\zeta_1, \cdots, \zeta_n\rangle$ over $\mathcal{F}$; σ_2 can be extended to an isomorphism τ_2 of $\mathcal{K}_2 = \mathcal{F}\langle\mathcal{K}_1, \tau_1\mathcal{K}_1\rangle$ over $\mathcal{F}$; $\cdots$; σ_n can be extended to an isomorphism τ_n of $\mathcal{K}_n = \mathcal{F}\langle\mathcal{K}_{n-1}, \tau_{n-1}\mathcal{K}_{n-1}\rangle$ over $\mathcal{F}$. Therefore $\sigma_1\eta, \cdots, \sigma_n\eta$ is a fundamental system of solutions of $L(y)$ contained in $\mathcal{K}_0 = \mathcal{F}\langle\tau_1\mathcal{K}, \cdots, \tau_n\mathcal{K}\rangle$. That is, $\sigma_1\eta, \cdots, \sigma_n\eta$ is a solution of $\{L(y_1), \cdots, L(y_n)\}$ for which $W(\sigma_1\eta, \cdots, \sigma_n\eta) \neq 0$, such that $\mathcal{F}\langle\sigma_1\eta, \cdots, \sigma_n\eta\rangle \subseteq \mathcal{K}_0$. Now, $\mathcal{K}_0$ is clearly an extension of $\mathcal{F}$ of the same type that $\mathcal{K}$ is. Therefore, by Theorem 3, $\{L(y_1), \cdots, L(y_n)\}$ has a solution $\eta_1, \cdots, \eta_n$ for which $W(\eta_1, \cdots, \eta_n) \neq 0$, such that

$\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ is contained in a liouvillian extension of $\mathcal{F}$ of the required type. It follows that $\eta_1, \cdots, \eta_n$ is a fundamental system of solutions of $L(y)$, and (see PV, §25) $\mathcal{F}\langle\eta_1, \cdots, \eta_n\rangle$ is itself a liouvillian extension of $\mathcal{F}$ of the required type.

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ON SOME EXAMPLES IN THE THEORY OF POWER SERIES

P. TURÁN

Fabry,[1] Hardy,[2] S. Bernstein[3] and Carleman[4] discovered that for any $\delta > 0$ there exist power series $\sum_0^\infty a_\nu z^\nu$ which are continuous for $|z| \leqq 1$ and for which the series $\sum |a_\nu|^{2-\delta}$ diverges. An elegant example is provided by the series

$$f(z) = \sum_{n=2}^{\infty} \frac{e^{icn \log n}}{n^{1/2} \log^\beta n} z^n, \qquad c \neq 0, \beta > 1, \tag{1}$$

which is continuous for $|z| \leqq 1$ (and even uniformly convergent there);[5] another example for the Carleman singularity, explicitly

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[1] E. Fabry, *Ordre des points singuliers de la série de Taylor*, Acta Math. vol. 36 (1913) pp. 69–194, esp. p. 103.

[2] G. H. Hardy, *A theorem concerning Tayor's series*, Quarterly Journal of Pure and Applied Mathematics vol. 44 (1913) pp. 147–160. In these two papers it was shown that if $\delta > 0$ the series $f_2(z) = \sum_1^\infty e^{i\nu^{1-\delta}} z^\nu / \nu^{1-\delta}$ is uniformly convergent for $|z| < 1$ and in Hardy's paper remark was made (p. 157) upon H. Bohr's problem on constructing a power series which is uniformly but not absolutely convergent for $|z| \leqq 1$.

[3] S. Bernstein, C. R. Acad. Sci. Paris (1914). He gave interesting cosine-polynomials $H(x) = \sum_{\nu=1}^{p-1} b_\nu \cos \nu x$ (p prime $\equiv 1$ mod 4) with the properties $|H(x)| \leqq 1$ and $\sum_{\nu=1}^{p-1} |b_\nu| = (p-1)/p^{1/2}$ which contains the seeds of the Carleman-singularity and were indeed the basis of Carleman's own construction.

[4] T. Carleman, *Über Fourier Koefficienten einer steigen Funktionen*, Acta. Math. vol. 41 (1918) pp. 337–384. Here is asserted explicitly and proved for the first time the existence of a continuous $f_3(x) \sim \sum (a_\nu \cos \nu x + b_\nu \sin \nu x)$ with $\sum_1^\infty (|a_\nu|^{2-\delta} + |b_\nu|^{2-\delta}) = \infty$, δ arbitrarily small: the existence of a continuous power series with the same property seems to be explicitly mentioned at the first time by Sidon; see footnote 9.

[5] See G. H. Hardy and J. E. Littlewood (*Some problems of Diophantine approximation: A remarkable trigonometrical series*, Proc. Nat. Acad. Sci. U.S.A. vol. 2 (1916) pp. 583–586), who considered only the functions $f_1(z) = \sum_{n=2}^\infty (e^{icn \log n}/n^{1/2+\alpha}) z^n$, $c \neq 0$, $0 < \alpha < 1$ (the divergence of the $(2-\delta)$th power of the moduli of the coefficients is not explicitly mentioned there). Series (1) seems to have been discussed for the first time by Zygmund in his book on trigonometric series. He uses there (in a simplified form) an argument due to Hille (*Note on a power series considered by Hardy and Littlewood*, J. London Math. Soc. vol. 4 (1929) pp. 176–182) and based on the application of Van der Corput's estimates to sums of the form $\sum e^{2\pi i \varphi(n)}$.

stated only for trigonometrical series, is given by Landau[6] who used number-theoretical characters and interpolation. Interesting generalizations were given by Gronwall[7] and Sidon[8] in the case of trigonometrical series; the first of these asserts that given any positive $\phi(x)$ with the property $\lim_{x\to 0}\phi(x)=+\infty$ then there is a continuous $f_4(x)\sim\sum_{\nu=0}^{\infty}(a_\nu\cos\nu x+b_\nu\sin\nu x)$ with $\sum_{\nu=0}^{\infty}(a_\nu^2+b_\nu^2)\phi(a_\nu^2+b_\nu^2)=+\infty$, and according to the second to any given $\delta>0$ and sequence of positive integers $n_1<n_2<\cdots$ we can construct a continuous $f_4(x)\sim\sum_0^\infty(a_\nu\cos\nu x+b_\nu\sin\nu x)$ such that $\sum(|a_{n_k}|^{2-\delta}+|b_{n_k}|^{2-\delta})=\infty$. Finally Sidon[9] proved the same theorems for power series.

In what follows I shall give an entirely elementary method of constructing all the required examples except Sidon's generalization (which method works of course in elementary constructions of corresponding examples concerning the theory of Lipschitz condition). This needs no reference either to Van der Corput's important estimations or to special number-theoretical functions and perhaps it is more suitable to lectures. It will be sufficient to show the construction on the Carleman-Sidon phenomenon. Our example will be

$$F(z)=\sum_{n=4}^{\infty}\frac{1}{n^{1+\epsilon/2}}\sum_{\nu=n^2+1}^{(n+1)^2}e^{-i\nu\theta_n}z^\nu=\sum_{\nu=17}^{\infty}C_\nu z^\nu,\qquad \epsilon>0, \tag{2}$$

where the numbers θ_n are defined by

$$\theta_n=(n-2^r)\frac{2\pi}{2^r}\qquad\text{for } n=2^r,(2^r+1),\cdots,(2^{r+1}-1);\quad r=2,3,\cdots; \tag{3}$$

we suppose

$$0<\epsilon<1/10. \tag{4}$$

Then

$$\sum_{\nu=17}^{\infty}|C_\nu|^{2-\epsilon}=\sum_{n=4}^{\infty}\frac{1}{(n^{1+\epsilon/2})^{2-\epsilon}}(2n+1)=+\infty.$$

Hence we have only to show that the series (2) converges uniformly for $|z| = 1$ or, putting $z = e^{i\theta}$, uniformly in θ. For the inner sum we have

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[6] E. Landau, *Bemerkungen zu einer Arbeit von Herren Carleman*, Math. Zeit. vol. 5 (1919) pp. 147–153. The historical data are partly taken from this thorough paper.

[7] T. H. Gronwall, *On the Fourier coefficients of a continuous function*, Bull. Amer. Math. Soc. vol. 27 (1921) pp. 320–321.

[8] S. Sidon, *Ein Satz über trigonometrische Polynome mit Lücken*, J. Reine Angew. Math. vol. 163, pp. 251–252.

[9] S. Sidon, *Bemerkungen über Fourier und Potenzreihen*, Acta Univ. Szeged. vol. 7 (1934–1935) pp. 85–94.

$$\left| \sum_{\nu=n^2+1}^{(n+1)^2} e^{i\nu(\theta-\theta_n)} \right| = \left| \sum_{\nu=0}^{2n} e^{i\nu(\theta-\theta_n)} \right| = \left| \frac{1 - e^{(2n+1)i(\theta-\theta_n)}}{1 - e^{i(\theta-\theta_n)}} \right|$$

$$(5) \qquad < \min\left(2n + 1, \frac{1}{|\sin((\theta - \theta_n)/2)|}\right)$$

$$< 3 \min\left(n, \frac{1}{|\sin((\theta - \theta_n)/2)|}\right).$$

Let us now be given an integer $\omega \geqq 17$; we determine the integer κ by

$$(6) \qquad \kappa^2 < \omega \leqq (\kappa + 1)^2.$$

Then putting

$$R_\omega(z) = \sum_{\nu=\omega}^{\infty} C_\nu z^\nu,$$

we have

$$|R_\omega(e^{i\theta})| \leqq \left| \sum_{\omega<\nu\leqq(\kappa+1)^2} C_\nu z^\nu \right| + \left| \sum_{\nu>(\kappa+1)^2} C_\nu z^\nu \right|$$

$$< \frac{1}{\kappa^{1+\epsilon/2}}(2\kappa + 1) + \left| \sum_{n=\kappa+1}^{\infty} \frac{1}{n^{1+\epsilon/2}} \sum_{\nu=n^2+1}^{(n+1)^2} e^{\nu i(\theta-\theta_n)} \right|$$

$$(7) \qquad < \frac{3}{\kappa^{\epsilon/2}} + \sum_{n=\kappa+1}^{\infty} \frac{1}{n^{1+\epsilon/2}} \left| \sum_{\nu=n^2+1}^{(n+1)^2} e^{\nu i(\theta-\theta_n)} \right|$$

$$< 3\kappa^{-\epsilon/2} + 3 \sum_{n=\kappa+1}^{\infty} n^{-1-\epsilon/2} \min\left(n, \frac{1}{|\sin((\theta - \theta_n)/2)|}\right),$$

using (5). If L is defined by

$$(8) \qquad 2^L \leqq \kappa < 2^{L+1}$$

then from (7) and (8)

$$|R_\omega(e^{i\theta})| < 3\kappa^{-\epsilon/2} + 3 \sum_{n=2^L}^{\infty} n^{-1-\epsilon/2} \min\left(n, \frac{1}{|\sin((\theta - \theta_n)/2)|}\right)$$

$$(9) \qquad < 5\omega^{-\epsilon/4} + 3 \sum_{\nu=L}^{\infty} \sum_{2^\nu \leqq n < 2^{\nu+1}} n^{-1-\epsilon/2} \min\left(n, \frac{1}{|\sin((\theta - \theta_n)/2)|}\right)$$

$$
\begin{aligned}
&< 5\omega^{-\epsilon/6} \\
&\quad + 3\sum_{\nu=L}^{\infty} 2^{-\nu(1+\epsilon/3)} \sum_{2^\nu \leq n < 2^{\nu+1}} \min\left(2^{\nu+1}, \frac{1}{\left|\sin((\theta-\theta_n)/2)\right|}\right) \qquad (9)\\
&= 5\omega^{-\epsilon/6} + 3\sum_{\nu=L}^{\infty} 2^{-\nu(1+\epsilon/3)} S_\nu(\theta).
\end{aligned}
$$

Now we consider $S_\nu(\theta)$. Taking in account the explicit form of the θ_n's we may observe that $S_\nu(\theta)$ is periodical with the period $2\pi/2^\nu$, hence we may suppose $0 \leqq \theta \leqq 2\pi/2^\nu$. Then using the inequality $\sin x \geqq 2/\pi x$ valid for $0 \leqq x \leqq \pi/2$, we obtain

$$
\begin{aligned}
S_\nu(\theta) &\leqq 3\cdot 2^{\nu+1} + \sum_{n=2^\nu+2}^{2^{\nu+1}-1} \min\left(2^{\nu+1}, \frac{1}{\left|\sin((\theta-\theta_n)/2)\right|}\right) \\
&= 3\cdot 2^{\nu+1} + \sum_{m=2}^{2^\nu-1} \min\left(2^{\nu+1}, \frac{1}{\left|\sin 2^{-1}(\theta - 2\pi m/2^\nu)\right|}\right) \\
&\leqq 3\cdot 2^{\nu+1} + 2\sum_{1\leqq m\leqq 2^{\nu-1}} \min\left(2^{\nu+1}, \frac{1}{\sin 2^{-1}(2\pi m/2^\nu - \theta)}\right) \\
&\leqq 3\cdot 2^{\nu+1} + 2\sum_{1\leqq m\leqq 2^{\nu-1}} \min\left(2^{\nu+1}, \frac{1}{\sin((m-1)2\pi/2^{\nu+1})}\right) \\
&= 3\cdot 2^{\nu+1} + 2\sum_{1\leqq m\leqq 2^{\nu-1}} \frac{1}{\sin(m2\pi/2^{\nu+1})} \\
&< 3\cdot 2^{\nu+1} + \frac{1}{2}\sum_{1\leqq m\leqq 2^{\nu-1}} 2^{\nu+1}\frac{1}{m} < 2^{\nu+1}(3+\nu) < 4\nu 2^\nu.
\end{aligned}
$$

Substituting this into (9) we obtain, using (4),

$$
\begin{aligned}
\left|R_\omega(e^{i\theta})\right| &< 5\omega^{-\epsilon/6} + 3\sum_{\nu\geqq L} 2^{-\nu(1+\epsilon/3)} 4\cdot\nu\cdot 2^\nu = 5\omega^{-\epsilon/6} + 12\sum_{\nu\geqq L}\nu 2^{-\nu\epsilon/3} \\
&= 5\omega^{-\epsilon/6} + 12\left[x^L \frac{L-(L-1)x}{(1-x)^2}\right]_{x=2^{-\epsilon/3}} \\
&< 5\omega^{-\epsilon/6} + 24L2^{-L\epsilon/3},
\end{aligned}
$$

and from (8) and (6) for $\omega > \omega_0(\epsilon)$

$$
\begin{aligned}
&< 5\omega^{-\epsilon/6} + 24\left(\frac{2}{\kappa}\right)^{\epsilon/3} \frac{\log \kappa}{\log 2} < 5\omega^{-\epsilon/6} + 24\log\omega\left(\frac{4}{\omega^{1/2}}\right)^{\epsilon/3} \\
&< 30\omega^{-\epsilon/6}.
\end{aligned}
$$

Since ϵ is fixed the series (2) converges for $|z| = 1$ uniformly.

Added in proof (September 8, 1948). The fact that also a continuous power series can possess Carleman singularity has been remarked earlier by O. Szász in his paper *Über Potenzreihen die in Einheitskreise beschränkte Funktionen darstellen*, Math. Zeit. vol. 8 (1920) pp. 222–236; his example is also based on Bernstein's polynomials.[3] As Professor Szász mentioned, the reasoning he and S. Minakshisundaram used in their paper *On absolute convergence of multiple Fourier series*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 36–53, and which is different from mine, furnishes also completely elementary examples in the above sense.

UNIVERSITY OF COPENHAGEN

A GENERALIZATION OF A THEOREM ON LINEAR DIFFERENTIAL EQUATIONS

A. S. AMITSUR

1. Introduction. It is well known that the number of independent solutions of a linear homogeneous differential equation equals its order, and the general solution is a linear combination of these independent solutions. The object of this paper is to prove that in the general case, that is, the derivation is defined in an abstract field (not necessarily commutative), the number of independent solutions does not exceed the order of the equation.

Applying this theorem to the case of inner derivation one obtains a new proof of a theorem due to Artin and Whaples.[1] Another application concerning cyclic fields will be given elsewhere.

2. Abstract linear differential equations. F will denote a field (not necessarily commutative) with an automorphism S: $a \to aS$. A right S-derivation D of F is a mapping of F into a part of itself satisfying:[2]

$$(1) \quad (a+b)D = aD + bD, \qquad (2) \quad (ab)D = (aS)(bD) + (aD)b.$$

If condition (2′) $(ab)D = a(bD) + (aD)(bS)$ holds instead of (2), D will be called a left S-derivation. The constants of the derivation (elements of F whose derivatives are zero) form a subfield C.

Remark. Given a left S-derivation in F, one can define a right S-derivation in a field F^* which is anti-isomorphic to F. Hence any theorem about right derivation can be translated into a theorem on left derivation by suitable changes of "right" into "left."

As usual we denote $a^{(0)} = a$, $a' = aD, \cdots, a^{(n)} = (a^{(n-1)})D$. An equation:

$$(1) \qquad a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_0 x^{(0)} = 0, \qquad (a_n \neq 0,\ a_i \in F)$$

is called a right homogeneous linear differential equation (r.e.) of order n. When the $x^{(i)}$ are written on the left the equation is a left differential equation (l.e.).

Theorem 1. *The elements of F which are solutions of the r.e.* (1) *form a right C-module of dimension at most n.*

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[1] Artin and Whaples, *The theory of simple rings*, Amer. J. Math. vol. 65 (1943) pp. 87–107.

[2] In the present paper operators are written multiplicatively on the right of the element on which they operate.

PROOF. From (1) and (2) follows $(y_1-y_2)^{(\nu)}=y_1^{(\nu)}-y_2^{(\nu)}$ and $(y_1c)^{(\nu)}=y_1^{(\nu)}c$ for any $c\in C$, $\nu\geqq 0$.

Hence with two solutions y_1, y_2 of (1) also y_1-y_2 and $y_1\cdot c$, $c\in C$, are solutions of the r.e. (1). The solutions of (1) form, therefore, a right C-module. Let $y_1, y_2, \cdots, y_{n+1}$ be $n+1$ solutions. The $n+1$ rows of the Wronskian matrix of these solutions

$$W=\begin{pmatrix} y_1^{(0)} & y_2^{(0)} & \cdots & y_{n+1}^{(0)} \\ y_1' & y_2' & \cdots & y_{n+1}' \\ \cdot & \cdot & \cdot & \cdot \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_{n+1}^{(n)} \end{pmatrix}$$

are left dependent because $\sum_{\nu=0}^{n} a_\nu y_\mu^{(\nu)}=0$, $\mu=1, 2, \cdots, n+1$ and $a_n\neq 0$. Hence the columns of W are right dependent.[3] Let r be the right-hand rank of the columns of W. We may assume that the first r columns are the independent, and the $(r+1)$th column is expressed as a linear combination of the columns as follows:

$$(2)\qquad y_{r+1}^{(\nu)}=\sum_{\mu=1}^{r} y_\mu^{(\nu)}b_\mu, \qquad \nu=0, 1, \cdots, n.$$

We shall prove now that (2) holds also for $\nu=n+1$. In fact for any solution y of (1):

$$(3)\qquad y^{(n)}=-\sum_{\nu=0}^{n-1}(a_n^{-1}a_\nu)y^{(\nu)}.$$

By differentiating (3) we have:

$$y^{(n+1)}=-\sum_{\nu=0}^{n-1}[(a_n^{-1}a_\nu)S]y^{(\nu+1)}-\sum_{\nu=0}^{n-1}(a_n^{-1}a_\nu)'y^{(\nu)}=\sum_{\nu=0}^{n}d_\nu y^{(\nu)}.$$

By (2) and (3) we have:

$$\begin{aligned} y_{r+1}^{(n+1)} &= \sum_{\nu=0}^{n} d_\nu y_{r+1}^{(\nu)} = \sum_{\nu=0}^{n}\sum_{\mu=1}^{r} d_\nu y_\mu^{(\nu)} b_\mu \\ &= \sum_{\mu=1}^{r}\left(\sum_{\nu=0}^{n} d_\nu y_\mu^{(\nu)}\right) b_\mu = \sum_{\mu=1}^{r} y_\mu^{(n+1)} b_\mu \end{aligned}$$

which proves (2) for $\nu=n+1$.

By differentiating (2) we obtain:

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[3] See, for example, J. Levitzki, *On the equivalence of the nilpotent elements of a semi-simple ring*, Compositio Math. vol. 5 (1938) Theorem 7, p. 401.

$$y_{r+1}^{(\nu+1)} = \sum_{\mu=1}^{r} (y_\mu^{(\nu)} S) b'_\mu + \sum_{\mu=1}^{r} y_\mu^{(\nu+1)} b_\mu = \sum_{\mu=1}^{r} (y_\mu^{(\nu)} S) b'_\mu + y_{r+1}^{(\nu+1)},$$

hence $0 = \sum_{\mu=1}^{r} (y_\mu^{(\nu)} S) b'_\mu = [\sum_{\mu=1}^{r} y_\mu^{(\nu)} (b'_\mu S^{-1})] S$ and therefore $\sum_{\mu=1}^{r} y_\mu^{(\nu)} (b'_\mu S^{-1}) = 0$; by the independence of the first r columns it follows that $(b'_\mu) S = 0$, and hence $b'_\mu = 0$, $\mu = 1, 2, \cdots, r$; in other words the b_μ are constants. Thus we have proved that the $n+1$ solutions are right dependent over C. q.e.d.

THEOREM 2. *Let $N \leqq F$ be a right C-module of dimension n, then there exists a right differential equation in F of order n whose module of solutions coincides with N.*

PROOF. Let $y_1, y_2, \cdots, y_n$ be a base of N. The "Wronskian matrix"

$$W = \begin{pmatrix} y_1^{(0)} & y_2^{(0)} & \cdots & y_n^{(0)} \\ y_1' & y_2' & \cdots & y_n' \\ \cdot & \cdot & \cdots & \cdot \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{pmatrix}$$

has n columns and $n+1$ rows, hence its rows are left dependent.[3] Therefore $n+1$ elements $a_0, a_1, \cdots, a_n$, not all zero, can be found so that: $\sum_{\nu=0}^{n} a_\nu y_\mu^{(\nu)} = 0$, $\mu = 1, 2, \cdots, n$. As above it follows that the elements of N will be solutions of a r.e. (1). Hence the module of solutions of (1) is of dimension at least n, which by Theorem 1 implies that the order of (1) must be exactly n, and the elements of N constitute the only solutions of (1).

3. **An application.** For any element a of F, the mapping $x \to xa$ of F is an endomorphism of F and will be denoted by a_r. The correspondence $a \to a_r$ is an isomorphism between F and F_r, where F_r is the set of all a_r. F_r is a subfield of the ring of endomorphisms $E(F)$.[4] Similarly the correspondence $a \to a_l$ is an anti-isomorphism between F and F_l, where a_l is the mapping $x \to ax$. For any $a_r \in F_r$ and $b_l \in F_l$ we have $a_r b_l = b_l a_r$.[4]

THEOREM 3. *Let F be a noncommutative field with a centre M. $a \in F$ is algebraic over M of degree n if and only if F is a right or left module of dimension n over F_a, where F_a is the centralizer of a in F (that is, the field of all the elements of F which are commutative with a).*

First we shall prove the following lemma.[5]

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[4] N. Jacobson, *Theory of rings*, Mathematical Surveys, vol. 2, 1943, pp. 15–16.

[5] I am indebted to the referee for the present proof which is simpler than that which was previously given.

Lemma. *An element a of F is algebraic over the centre M if and only if there exists a polynomial $h(\lambda)=\sum_{\nu=0}^{n}h_\nu\lambda^\nu$ $(h_\nu\in F,\ h_n\neq 0)$, which vanishes for all the transforms of a, that is $\sum_{\nu=0}^{n}h_\nu(yay^{-1})^\nu=0$ for each y of F, $y\neq 0$.*

Proof. If there exist polynomials which vanish for all the transforms of a, then let $g(\lambda)=\sum_{\nu=0}^{m}b_\nu\lambda^\nu$ $(b_\nu\in F,\ b_m\neq 0)$ be a polynomial with this property with minimum degree. We may evidently assume that $b_m=1$. Then for every nonzero element z of F:

$$z\left[\sum_{\nu=0}^{m}b_\nu(yay^{-1})^\nu\right]z^{-1}=\sum_{\nu=0}^{m}zb_\nu z^{-1}[(zy)a(zy)^{-1}]^\nu=0.$$

Hence the polynomial $g_z(\lambda)=\sum_{\nu=0}^{m}zb_\nu z^{-1}\lambda^\nu$, where $zb_mz^{-1}=1$ also vanishes for all the transforms of a.

Since all the transforms of a are zeros of the difference polynomial $g(\lambda)-g_z(\lambda)$, whose degree is less than m, it follows by the minimum property of $g(\lambda)$ that all the coefficients of $g(\lambda)-g_z(\lambda)$ are zeros, that is, $b_\nu=zb_\nu z^{-1}$, $\nu=0, 1, \cdots, m$, for every z of F, $z\neq 0$. But this implies that all the coefficients of $g(\lambda)$ are in M, that is, a is algebraic over M.

The converse of the theorem follows from the fact that with a also all its transforms are zeros of the minimum polynomial of a over M.

We turn now to the proof of Theorem 3.

The mapping $x\to xa-ax$ is a right as well as a left derivation of F with $S=E$ the identical automorphism.[6] The field of constants of this inner derivation is the centralizer F_a of F. In the ring of endomorphisms $E(F)$: $D=a_r-a_l$, so $a_r=D+a_l$. a_l is commutative with a_r, hence also with D.

Let F be a right F_a-module of a finite dimension m, then there exists a r.e. of order m (Theorem 2) $z^{(m)}+a_1z^{(m-1)}+\cdots+a_mz^{(0)}=0$ satisfied by all elements of F, so that in the ring of endomorphism,

$$E(F):\qquad h(D)=D^m+D^{m-1}a_{1,l}+\cdots+a_{m,l}=0.$$

Hence

$$h(D)=h(a_r-a_l)=a_r^m+a_r^{m-1}C_{1,l}+a_r^{m-2}C_{2,l}+\cdots+C_{m,l}=0,$$

where

$$C_i=(-1)^i\left[\binom{m}{i}a^i-\binom{m-1}{i-1}a_1a^{i-1}+\cdots+(-1)^ia_i\right].$$

Operating with $h(D)$ on $x\in F$, $x\neq 0$, and multiplying on the right by x^{-1} we get: $[xk(D)]x^{-1} = (xa^m + C_1xa^{m-1} \cdots C_mx)x^{-1} = (xax^{-1})^m + C_1(xax^{-1})^{m-1} + \cdots + C_m = 0$.

---

[6] N. Jacobson, *Theory of rings*, p. 102.

The polynomial $k(\lambda) = \lambda^m + C_1\lambda^{m-1} + \cdots + C_m$ is in F and vanishes for all the transforms of a. Therefore, by the preceding lemma, a is algebraic over M, and its degree is at most m. Our theorem will be completely proved if we show that when a is algebraic over M then F is both a right and left F_a-module of dimension not greater than its degree, as from the preceding part of our proof it follows that $(F:F_a)$ must be equal to the degree of a over the centre M.

Now, let $g(\lambda) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_n$ be the minimum polynomial of a over M. The isomorphism between F and F_r yields: $0 = g(a) \to g_r(a_r) = g_r(D + a_l) = D^n + D^{n-1}b_{1,l} + \cdots + b_{n,l} = 0$ where the $b_{i,l}$ are polynomials in a_l with coefficients in $M_r = M_l$, and therefore belong to F_l. Hence operating with $g_r(D + a_l) = 0$ on $x \in F$ we have $xg_r(D + a_l) = x^{(n)} + b_1x^{n-1} + \cdots + b_nx^{(0)} = 0$ where b_i is the image of $b_{i,l}$ in the anti-isomorphism between F_l and F. Theorem 1 yields, therefore, that F is a right F_a-module of dimension at most n, q.e.d.

Similarly one proves that F is a left F_a-module of dimension at most n. From the proof follows also that if F is a right F_a-module of finite dimension, then it is also a left F_a-module with the same dimension which is equal to the degree of a over the centre M.

HEBREW UNIVERSITY

ON THE LOCATION OF THE ZEROS OF THE DERIVATIVES OF A POLYNOMIAL SYMMETRIC IN THE ORIGIN

J. L. WALSH

If the zeros of a polynomial $p(z)$ when plotted in the z-plane are symmetric in 0: $z=0$, the zeros of the derivative $p'(z)$ of $p(z)$ can profitably be studied by transforming onto the w-plane, with $w=z^2$, and applying known theorems there.[1] It is the purpose of the present note to carry that study somewhat farther than has been previously done, in particular to consider the higher derivatives of $p(z)$.

Under the transformation $w=u+iv=z^2=(x+iy)^2$, an arbitrary line $Au+Bv+C=0$ in the w-plane corresponds to an equilateral hyperbola $A(x^2-y^2)+2Bxy+C=0$ in the z-plane with center 0 or to two perpendicular lines intersecting at 0. A half-plane in the w-plane for which $w=0$ is an interior or exterior point corresponds in the z-plane respectively to the exterior or interior of an equilateral hyperbola whose center is 0; a half-plane for which $w=0$ is a boundary point corresponds to a double sector with vertex $z=0$ and angle $\pi/2$. A point z is considered to be exterior or interior to a hyperbola according as the curve at its nearest point is convex or concave toward z.

We write the given polynomial in the form

$$p(z) = z^l \prod_{j=1}^{q} (z^2 - \alpha_j^2), \qquad \alpha_j \neq 0, \tag{1}$$

and in the w-plane study the polynomials ($w=z^2$)

$$P(w) = P(z^2) = [p(z)]^2, \qquad P'(w) = p(z)\cdot p'(z)/z. \tag{2}$$

Each zero of $P(w)$ corresponds to a zero of $p(z)$ and reciprocally; each zero of $P'(w)$ corresponds to a zero of $p(z)$ or $p'(z)$ and reciprocally except that $z=0$ is a zero of $p'(z)$ unless $z=0$ is a simple zero of $p(z)$.

We have (loc. cit.) by Lucas' Theorem

THEOREM 1. *If the zeros of $p(z)$ are symmetric in 0 and lie in the closed exterior of an equilateral hyperbola with center 0 or in the closed exterior of a double sector with vertex 0 and angle $\pi/2$, then the zeros of $p'(z)$ lie also in that closed exterior.*

If the zeros of $p(z)$ are symmetric in 0 and lie in the closed interior of an equilateral hyperbola with center 0, then the zeros of $p'(z)$ also lie in that closed interior except for a simple zero at 0.

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[1] Walsh, Mathematica vol. 8 (1933) pp. 185–190.

To the polynomial $P(w)$ we now apply the theorem:[3]

Let $P(w)$ be a polynomial in w of degree n with an l-fold ($l>0$) zero at $w=0$ and all the remaining zeros of $P(w)$ in the closed half-plane Π *not containing $w=0$. Then except for a zero at $w=0$ of multiplicity $l-1$, all zeros of $P'(w)$ lie in the closed half-plane obtained by shrinking* Π *toward the origin in the ratio $n:l$.*

The degrees of $p(z)$ and $P(w)$ defined by (1) and (2) are equal, as are the multiplicities of their zeros $z=0$ and $w=0$. Under the transformation $w=z^2$, the line $u=a$ corresponds to the hyperbola $x^2-y^2=a$, and the line $u=la/n$ corresponds to $x^2-y^2=la/n$, so we have

THEOREM 2. *Let $p(z)$ be a polynomial of degree n whose zeros are symmetric in the origin* 0, *let* 0 *be an l-fold zero ($l>0$), and let all the other zeros lie in the closed interior of an equilateral hyperbola H with center* 0. *Then except for an $(l-1)$-fold zero at* 0, *all zeros of $p'(z)$ lie in the closed interior of the hyperbola obtained by shrinking H toward* 0 *in the ratio $(n:l)^{1/2}$.*

We turn now to the higher derivatives of the polynomial $p(z)$. Under the conditions of the first part of Theorem 1, the zeros of every derivative of $p(z)$ not vanishing identically lie in the given closed region.

Suppose, however, the zeros of the polynomial $p(z)$ of degree n lie in the closed interior of an equilateral hyperbola H whose center is 0. The polynomial $p'(z)$ has a simple zero at 0, and by Theorem 1 the remaining zeros of $p'(z)$ lie in the closed interior of H. By Theorem 2, all zeros of $p''(z)$ lie in the closed interior of the hyperbola obtained by shrinking H toward 0 in the ratio $[(n-1):1]^{1/2}$. The higher derivatives of $p(z)$ not vanishing identically have alternately a simple zero at 0 and no zero at 0. Continued application of Theorems 1 and 2, in alternation, then yields further hyperbolas which respectively contain the zeros other than 0 of the kth derivative of $p(z)$. However, except in the cases $k=1$, and $k=2$ with $n\geqq 4$, these hyperbolas are not the most favorable that can be obtained; a zero of $p'(z)$ in Theorem 2 cannot lie on the new hyperbola unless all zeros of $p(z)$ lie on H in two points symmetric in 0. We proceed to prove the following generalization of Theorem 2, the principal result of the present note:

THEOREM 3. *Let $p(z)$ be a polynomial of degree n whose zeros are symmetric in* 0, *let* 0 *be an l-fold zero, and let the remaining zeros lie in the closed interior of a hyperbola H whose center is* 0. *Let $p_0(z)$ be the polynomial of degree n whose zeros are symmetric in* 0, *which has an l-fold*

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[3] Walsh, Trans. Amer. Math. Soc. vol. 22 (1921) pp. 101–116; p. 115.

zero at 0, *and whose remaining zeros lie at the vertices of* H. *Let* z_k, $1 \leqq k \leqq n-2$, *be one of the zeros of* $p_0^{(k)}(z)$ *of smallest positive modulus, and let* H_k *be the equilateral hyperbola with center* 0 *which has* z_k *as a vertex. Then all zeros of* $p^{(k)}(z)$ *other than* 0 *lie in the closed interior of* H_k.

Of course 0 need not be a zero of $p^{(k)}(z)$; whether or not 0 is such a zero depends on l, n, and k. To prove Theorem 3 we need a more powerful result than that used in proving Theorem 2:

Let $Q(w)$ *be a polynomial in* w *of degree* q, *and let the numbers* A_j *be constants. If the locus of the zeros of* $Q(w)$ *is a closed half-plane* Π, *then the locus of the zeros of the polynomial*

$$A_0 w^r Q(w) + A_1 w^{r+1} Q'(w) + \cdots + A_q w^{r+q} Q^{(q)}(w) \tag{3}$$

is also a number of half-planes, identical with the locus of the zeros of the polynomial (3) *with* $Q(w)$ *replaced by* $(w-\alpha)^q$:

$$\begin{aligned} A_0 w^r (w-\alpha)^q + q A_1 w^{r+1} (w-\alpha)^{q-1} + \cdots \\ + q(q-1) \cdot \cdots \cdot 1 A_q w^{r+q} \end{aligned} \tag{4}$$

when the locus of α *is* Π.

The theorem just quoted is essentially a special case of a more general theorem[3] concerning a linear combination of products of derivatives of two polynomials. If polynomial (4) is written as the product of w^{r+q} and a polynomial in $W = (w-\alpha)/w$ whose zeros are $W = W_1, W_2, \cdots, W_q$, it is seen that the common locus of the zeros of (3) and (4) consists of the origin $w=0$ (provided $w=0$ is a zero of (4)) plus a number of half-planes, loci of the points $w = \alpha/(1-W_j)$, $j = 1, 2, \cdots, q$, when the locus of α is Π; if Π does not contain the point $w=0$, a zero $W_j = 1$ is to be ignored; if Π contains the point $w=0$ and if $W_j = 1$ is a zero of (4), the entire plane is to be considered the locus of the zeros of (3) and (4). If Π does not contain 0, and if for real positive α all zeros of (4) lie in the interval $0 \leqq w \leqq \alpha$, then the locus of the zeros of (4) is the origin (if $w=0$ is a zero of (4)) plus a half-plane not containing 0 bounded by a line parallel to the boundary of Π, traced by the zero of (4) nearest to but different from 0 when α traces the boundary of Π.

In Theorem 3 we write

$$\begin{aligned} p(z) &= z^l p_1(z^2), & p_1(0) \neq 0, \\ p'(z) &= l z^{l-1} p_1(z^2) + 2 z^{l+1} p_1'(z^2), \end{aligned}$$

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[3] Walsh, Trans. Amer. Math. Soc. vol. 24 (1922) pp. 163–180; Theorem 9 of that paper applies to two circular regions, here specialized to a single point and a half-plane respectively.

$$p''(z) = l(l-1)z^{l-2}p_1(z^2) + (4l+2)z^l p_1'(z^2) + 4z^{l+2}p_1''(z^2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

Except perhaps for a factor z, these derivatives are precisely of the form (3), with $w=z^2$. If we have $p_0(z)=z^l(z^2-\beta^2)^{(n-l)/2}$, the zeros of $p_0^{(k)}(z)$ lie on the line segment joining β and $-\beta$, and the corresponding zeros in the w-plane lie on the segment joining 0 and β^2; the images in the w-plane of the zeros of $p_1(z)$ lie in a half-plane Π not containing $w=0$; thus by the theorem quoted the images in the w-plane of the zeros of $p^{(k)}(z)$ other than $z=0$ lie in the half-plane whose boundary is parallel to that of Π, which contains Π in its interior, and whose boundary passes through the image of the zero of $p_0^{(k)}(z)$ nearest to but distinct from 0. Theorem 3 follows.

When as in Theorem 1 we study the zeros of a polynomial $p(z)$ that are symmetric in 0, equilateral hyperbolas with center 0 are both (1) the lines of force in the z-plane in the Gaussian field due to two particles symmetric in 0 and (2) the images in the z-plane of the straight lines in the w-plane under the transformation $w=z^2$. Indeed, the lines of force due to particles at $z=\alpha$ and $z=-\alpha$ are loci arg $(z^2-\alpha^2)=$const., arcs of equilateral hyperbolas with center 0; half-lines in the w-plane can be written arg $(w-w_0)=$const., and their images in the z-plane are loci arg $(z^2-w_0)=$const.

A set of points in the z-plane is said to have *m-fold symmetry about* 0 if the set is unchanged by a rotation about 0 through an angle of $2\pi/m$. A polynomial $p(z)$ whose zeros possess m-fold symmetry about 0 is readily studied by means of the transformation $w=z^m$. Under this transformation a straight line in the w-plane not through $w=0$ corresponds to a curve which may be called an *m-hyperbola* with center 0. A closed half-plane in the w-plane containing 0 in its interior corresponds to the closed exterior of an m-hyperbola, and a closed half-plane not containing 0 corresponds to the closed interior of an m-hyperbola. A line through $w=0$ corresponds to m equally spaced lines through $z=0$, and a half-plane in the w-plane bounded by a line through $w=0$ corresponds to m equally spaced sectors in the z-plane, each of angle π/m. The function $[p(w^{1/m})]^m$ is a polynomial in w which is readily studied in the w-plane, and yields results on the zeros of $p'(z)$ in the z-plane. Theorem 3 extends directly to a polynomial $p(z)$ whose zeros possess m-fold symmetry in 0. The m-hyperbolas with center 0 are not merely the images in the z-plane of the straight lines in the w-plane, but also the lines of force in the z-plane due to m particles m-fold symmetric in $z=0$.

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ON POLYNOMIALS AND LAGRANGE'S FORM OF THE GENERAL MEAN-VALUE THEOREM

V. RAMASWAMI

Suppose that in $(a<x<b)$ (hereafter referred to as (a, b)),

(1) $f(x)$ is defined and has derivatives of the first n orders.

Then, from the general mean-value theorem with Lagrange's form of remainder follows the existence of $\theta=\theta(x, h)$, such that

$$f(x+h)=f(x)+\sum_{r=1}^{n-1}\frac{h^r}{r!}f^{(r)}(x)+\frac{h^n}{n!}f^{(n)}(x+\theta h) \tag{2}$$

$$\text{for } a<x<x+h<b.$$

The θ in (2) is sometimes a uniquely determinate function of x and h in the relevant domain $a<x<x+h<b$ (hereafter referred to as R), as, for instance, if $f^{(n+1)}(x)$ exists and is not zero in (a, b). If, further, $f^{(n+1)}(x)$ is continuous in (a, b), it is easily seen that

$$\lim_{h\to+0}\theta(x, h)=\frac{1}{n+1} \qquad \text{in } a<x<b.$$

It is also possible for $\theta(x, h)$ to be an analytic function, for example,

$$\theta(x, h)=h^{-1}\log\left(1+\sum_{r=1}^{\infty}\frac{h^r\Gamma(n+1)}{\Gamma(n+r+1)}\right),$$

which happens when $f(x)=e^x$.

It would, therefore, seem worth while to determine the types of functions that are or are not possible for $\theta(x, h)$. Inquiry in this direction has led to the results of this paper, namely:

THEOREM 1. *If a polynomial* $\theta(x, h)$ *exists such that* (2) *is true with* $\theta(x, h)$ *in place of* θ, *then* $f^{(n+1)}(x)$ *exists in* (a, b) *and either*

(a) $$f^{(n+1)}(x)=0 \qquad \textit{in } (a, b)$$

or

(b) $f^{(n+1)}(x)=a$ *constant* $\neq 0$ *in* (a, b), *and* $\theta(x, h)$ *is uniquely determinate and equal to* $1/(n+1)$ *in* R.

THEOREM 2. *If* (2) *is true with* $\theta(x, h)=c(x)+h^d\phi(x, h)$ *where*

(3) $\phi(x, h)$ *is bounded in* R;

(4) d *is a constant greater than* 1;

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(5) *$\partial\theta/\partial x$, $\partial^2\theta/\partial^2 x$ are continuous in x, and θ is bounded in R;*

(6) *for all sufficiently small h, $1+h(\partial\theta/\partial x)\neq 0$ in R;*

then, also, (a) *and* (b) *of Theorem* 1 *are true.*

It is significant that, if θ is uniquely determined by (2) in R and not equal to $1/(n+1)$, then $\theta=\theta(x, h)$ cannot be equal to a polynomial in R (by Theorem 1) or even to an analytic function (by Theorem 2) satisfying

(7) $\lim_{h\to 0}\partial\theta/\partial h=0$ for every x in (a, b).

[In the following we write $\theta(x, 0)$ for $\lim_{h\to+0}\theta(x, h)$ and $\theta_1(x, 0)$ for $\lim_{h\to+0}(\theta(x, h)-\theta(x, 0))/h$) (which limits obviously exist in the contexts of the two theorems), and θ_{rs} for $(\partial^{r+s}/\partial x^r\partial h^s)\theta$, wherever the latter obviously exists.]

PROOF OF THEOREM 1.

(8) The conditions (5) and (6) are obviously satisfied here and (2) is true by hypothesis.

On account of the consequent boundedness of θ in R, and the continuity of θ in x, follows

(9) $y=x+\theta h$ for every y in (a, b), with any sufficiently small h and a correspondingly chosen x such that (x, h) lies in R.

From (8) and (9) follows

(10) $f^{n+1}(x)$ and $f^{n+2}(x)$ exist and are continuous in (a, b).

Now, from the general mean-value theorem follows

$$\begin{aligned}f(x+h) &= f(x)+\sum_{r=1}^{n}\frac{h^r}{r!}f^{(r)}(x)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x)\\ &\quad+\frac{h^{n+2}}{(n+2)!}f^{(n+2)}(x+\theta_1 h),\quad 0<\theta_1<1,\quad (x, h)\subset R;\end{aligned}\tag{11}$$

and from (2) and the same theorem applied to $f^n(x+\theta h)$ follows

$$\begin{aligned}f(x+h) &= f(x)+\sum_{r=1}^{n}\frac{h^r}{r!}f^{(r)}(x)+\frac{h^{n+1}\theta}{(n)!}f^{(n+1)}(x)\\ &\quad+\frac{h^{n+2}\theta^2}{n!2!}f^{(n+2)}(x+\theta_2\theta h),\quad 0<\theta_2<1,\ (x, h)\subset R.\end{aligned}\tag{12}$$

Subtracting (12) from (11) and making $h\to+0$ after division by h^{n+1}, it follows by (10) that

$$f^{(n+1)}(x)\left[1-(n+1)\theta(x, 0)\right]=0.\tag{13}$$

Using (13) in (11) and (12), and making $h\to+0$ after division of their difference by h^{n+2}, it follows, again by (10),

$$f^{(n+2)}(x)\left[1-\frac{(n+1)(n+2)}{2}\theta^2(x,0)\right] \qquad \text{(14)}$$
$$-f^{(n+1)}(x)(n+1)(n+2)\theta_1(x,0)=0.$$

Now, either

(15a) $f^{(n+1)}(x)=0$ everywhere in (a, b);

or

(15b) $f^{(n+1)}(x)\neq 0$ everywhere in (a, b);

or

(15c) on account of the continuity (by (10)) of $f^{(n+1)}(x)$ there exists a closed interval (a_1, b_1) contained in (a, b) such that $f^{(n+1)}(x)\neq 0$ for $a_1<x<b_1$, and one at least of $f^{(n+1)}(a_1)$ and $f^{(n+1)}(b_1)$ is zero.

If (15c) were possible, then we should have, by (13) and (14),

$$f^{(n+2)}(x)\cdot n/2(n+1)-f^{(n+1)}(x)(n+1)(n+2)\theta_1(x,0)=0$$
$$\text{in } (a_1<x<b_1),$$

and hence $f^{(n+1)}(x)=A\cdot\exp\{\phi(x)\}$ in $a_1<x<b_1$, where $\phi(x)$ is a polynomial and A is a constant, and making $x\to a_1$ or b_1 in this, there would follow that $f^{(n+1)}(x)=0$ in $a_1<x<b_1$, which contradicts (15c). Hence

(16) (15c) is impossible, and $f^{(n+1)}(x)=A\exp\{\phi(x)\}$ in $a<x<b$, where $\phi(x)$ is a polynomial and $A=$ a constant $\neq 0$, if $f^{(n+1)}(x)\neq 0$ for some x in (a, b).

Now differentiating (2) with respect to x and h, as is obviously permissible on account of (10), and subtracting, and dividing by h^{n-1}, it follows that

$$f^{(n)}(x)-f^{(n)}(x+\theta h)=\frac{h}{n}f^{(n+1)}(x+\theta h)[\theta-1+h\theta_{01}-h\theta_{10}] \quad \text{in } R.$$

Differentiating this (possible by (10)) with respect to x and using (16) we get

(17) $\exp\{k(x, h)\}=g(x, h)$ in R, in case (15b), where $k(x, h)=\phi(x)-\phi(x+\theta h)$ and $k(x, h)$ and $g(x, h)$ are polynomials in x and h.

It is now seen by the theory of analytic continuation that (17) is impossible unless $k(x, h)$ is a constant, which again is seen to be zero by keeping x fixed and making $h\to +0$. Hence

$$\phi(x)=\phi(x+\theta h) \quad \text{in } R. \qquad (18)$$

Now from (2) obviously follows

(19) $f(x)$ is a polynomial of degree not greater than n in (a, b) if $\theta(x, h)\equiv 0$. Also, by continuous variation of x and h in R it follows from (18) that

(20) $\phi(x)=$ a constant k in (a, b) if $\theta(x, h)\neq 0$;
and hence, using (16), follows
(21) $f^{(n+1)}(x)=Ae^{k}$ in (a, b) where $A\neq 0$ if $f^{(n+1)}(x)\neq 0$ in (a, b),
Now the theorem follows from (10), (15a), (15b), (16), (19) and (21). since, when $f^{(n+1)}(x)=$ a constant $\neq 0$, $\theta=1/(n+1)$ and is uniquely determined by (2) in R.

Proof of Theorem 2. In this case, the statements (8) to (14) follow as above, and $\theta_1(x, 0)=0$ since $d>1$. Hence (13) and (14) now become

$$f^{(n+1)}(x)[1-(n+1)c(x)]=0 \qquad \text{in } (a, b), \tag{22}$$

$$f^{(n+2)}(x)\left[1-\frac{(n+1)(n+2)}{2}c^2(x)\right]=0 \qquad \text{in } (a, b). \tag{23}$$

Hence either
(24a) $f^{(n+1)}(x)=0$ every where in (a, b),
or
(24b) $f^{(n+1)}(x)=c\neq 0$ for some x in (a, b).
Then, (22) and (23) give
(25) $c(x)=1/(n+1)$ wherever $f^{(n+1)}(x)\neq 0$,
(26) $f^{(n+2)}(x)=0$ wherever $f^{(n+1)}(x)\neq 0$.
The theorem now follows from (10), (24a), (24b), (25) and (26), since, when $f^{(n+1)}(x)=c\neq 0$ in (a, b), θ in (2) is uniquely determined in R.

Note added January 18, 1948. The conclusions (a) and (b) of Theorem 1 are true if (2) holds with $\theta(x, h)$ in place of θ, where

$$\theta(x, h)=\sum_{r=0}^{\infty} h^r\theta_r(x),$$

and $\theta_1(x)$ is a polynomial, $\theta(x, h)$ satisfies (6) and each of the functions $\theta_r(x)$ satisfies (5). The line of proof is briefly as follows:

The arguments up to and including (16) are the same as above, and the equation in (17) is now true with $K(x, h)=\phi(x)-\phi(x+\theta h)$, and $K(x, h)$ and $g(x, h)$ polynomials in h for fixed x. The rest of the argument is the same as before.

The conclusion (20) can also be seen directly as follows: Differentiating (18) with respect to h, we have

$$\phi'(x+\theta h)\left(\theta+h\frac{\partial\theta}{\partial h}\right)=0.$$

Making $h\to 0$ in this and noting that $\theta(x, 0)=1/(n+1)$ in case (15b) we have $\phi'(x)=0$, and hence $\phi(x)=k$, a constant in (a, b).

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PROOF OF A THEOREM OF SAKS AND SIERPINSKI

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The object of this note is to give a direct proof of the following theorem of Saks and Sierpinski:

If $f(x)$ is an arbitrary one-valued real function defined on the closed interval $I=[0, 1]$, there is a $\phi(x)$ of Baire class 2 at most such that for every $\epsilon>0$ the inequality $|f(x)-\phi(x)|<\epsilon$ holds on a set of exterior measure 1.

The proofs in the literature [1, 2][1] depend on the corresponding theorem for measurable functions; namely, if $f(x)$ is measurable, there is a $\phi(x)$ of Baire class 2 at most such that $f(x)=\phi(x)$ almost everywhere.

We first prove a lemma which seems to be new.

LEMMA. *If $f(x)$ is defined on the closed interval $I=[0, 1]$, $\epsilon>0$, $\phi(x)$ continuous, and $|f(x)-\phi(x)|<\epsilon$ on a set of exterior measure greater than $1-\epsilon$, then for every $\eta>0$ there is a continuous $\psi(x)$ such that $|\phi(x)-\psi(x)|<\epsilon$ on a set of measure greater than $1-\epsilon$ and $|f(x)-\psi(x)|<\eta$ on a set of exterior measure greater than $1-\eta$.*

PROOF. $f(x)$ is exteriorly approximately continuous almost everywhere in the sense that for almost every $\xi\in I$ the set of points x for which $f(\xi)-k<f(x)<f(\xi)+k$ has exterior metric density 1 at ξ, for every $k>0$. There is a δ, with $0<\delta<\eta$, such that $|f(x)-\phi(x)|<\epsilon-\delta$ on a set E of exterior measure greater than $1-\epsilon$. Since $f(x)$ is exteriorly approximately continuous almost everywhere, every $\xi\in E-Z$, with Z of measure zero, is in a sequence, $\{I_{\xi n}\}$, $i=1, 2, \cdots$, of closed intervals, whose lengths converge to zero, such that the set of points x for which $|f(\xi)-f(x)<\delta$ has relative exterior measure exceeding $1-\delta/2$ in each $I_{\xi n}$. Moreover, since $\phi(x)$ is continuous, the $I_{\xi n}$ may be chosen so that the saltus of $\phi(x)$ in $I_{\xi n}$ is less than δ for every n. Consider the totality of intervals

$$I=[I_{\xi n}], \quad \xi\in E-Z, n=1, 2, \cdots .$$

By Vitali's covering theorem, since $m_e(E-Z)>1-\epsilon$, there is a finite number $I_{\xi_1 n_1}$, $I_{\xi_2 n_2}$, $\cdots$, $I_{\xi_k n_k}$ of disjoint intervals of I the sum, $\sum_{i=1}^{k} m(I_{\xi_i n_i})$, of whose lengths is $1-\alpha>1-\epsilon$. Let $G=I-\sum_{i=1}^{k} I_{\xi_i n_i}$ and let $G'\subset G$ be the points of G at which $f(x)$ is exteriorly approxi-

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[1] Numbers in brackets refer to the references cited at the end of the paper.

mately continuous. Every $\zeta \in G'$ is in a sequence $\{I_{\zeta m}\}$ of closed intervals contained in G, whose lengths converge to zero, such that for every m, the set of points x for which $|f(\zeta)-f(x)|<\delta$ has relative exterior measure in $I_{\zeta m}$ exceeding $1-\delta/2$. Consider the totality of intervals

$$J = [I_{\zeta m}], \qquad \zeta \in G', m = 1, 2, \cdots .$$

There is then a finite number of disjoint intervals $I_{\zeta_1 m_1}, I_{\zeta_2 m_2}, \cdots, I_{\zeta_K m_K}$ of J such that $\sum_{i=1}^{K} m(I_{\zeta_i m_i})$ exceeds $\alpha-\delta/2$. Now, let

$$\psi(x) = f(\xi_i) \qquad \text{for every } x \in I_{\xi_i n_i},\ i = 1, 2, \cdots, k,$$

$$\psi(x) = f(\zeta_i) \qquad \text{for every } x \in I_{\zeta_i m_i},\ i = 1, 2, \cdots, K,$$

and let $\psi(x)$ be defined elsewhere so as to be continuous. Suppose $x \in I_{\xi_i n_i}$, $1 \leqq i \leqq k$.

$$\begin{aligned} |\psi(x) - \phi(x)| \leqq |\psi(x) - f(\xi_i)| + |f(\xi_i) - \phi(\xi_i)| \\ + |\phi(\xi_i) - \phi(x)| < 0 + (\epsilon - \delta) + \delta = \epsilon. \end{aligned}$$

But $\sum_{i=1}^{k} m(I_{\xi_i n_i}) > 1-\epsilon$. Hence $|\psi(x)-\phi(x)|<\epsilon$ on a set of measure greater than $1-\epsilon$. Moreover, since the relative exterior measure of the set of points for which $|\psi(x)-f(x)|<\delta<\eta$ exceeds $1-\delta/2$ in every $I_{\xi_i n_i}$, $i=1, 2, \cdots, k$, and in every $I_{\zeta_i m_i}$, $i=1, 2, \cdots, K$, and these intervals are disjoint, the exterior measure of this set exceeds

$$\begin{aligned} \left(1-\frac{\delta}{2}\right)\left[\sum_{i=1}^{k} m(I_{\xi_i n_i}) + \sum_{i=1}^{K} m(I_{\zeta_i m_i})\right] \\ \geqq \left(1-\frac{\delta}{2}\right)\left[(1-\alpha) + \left(\alpha - \frac{\delta}{2}\right)\right] \\ = \left(1-\frac{\delta}{2}\right)^2 > 1 - \delta > 1 - \eta. \end{aligned}$$

This completes the proof.

We now prove the theorem of Saks and Sierpinski. Let $f(x)$ be an arbitrary function defined on $I = [0, 1]$. Let $\phi_0(x) \equiv 0$. By the lemma, there is a continuous $\phi_1(x)$ such that $|\phi_1(x)-f(x)|<1/2$ on a set of exterior measure greater than $1/2$. Having defined the continuous functions

$$\phi_1(x), \phi_2(x), \cdots, \phi_{n-1}(x)$$

with $|\phi_{n-1}(x)-f(x)|<1/2^{n-1}$ on a set of exterior measure greater than $1-1/2^{n-1}$ there is, by the lemma, a continuous $\phi_n(x)$ such that $|\phi_n(x)-\phi_{n-1}(x)|<1/2^{n-1}$ on a set of measure greater than $1-1/2^{n-1}$

and $|\phi_n(x)-f(x)|<1/2^n$ on a set of exterior measure greater than $1-1/2^n$. Let

$$\phi(x) = \limsup_{n\to\infty} \phi_n(x).$$

Then $\phi(x)=\lim_{n\to\infty}$ [l.u.b. $(\phi_n(x), \phi_{n+1}(x), \cdots)$], and, since l.u.b. $(\phi_n(x), \phi_{n+1}(x), \cdots)$ is lower semi-continuous, for every n, $\phi(x)$ is the limit of a nonincreasing sequence of lower semi-continuous functions and is, therefore, of Baire class 2 at most. Moreover, the sequence $\{\phi_n(x)\}$ itself converges almost everywhere to $\phi(x)$. Accordingly, for every $\epsilon>0$ and every $\eta>0$, there is an n, such that $|\phi_n(x)-\phi(x)|<\epsilon/2$ on a set of measure greater than $1-\eta/2$ and $|\phi_n(x)-f(x)|<\epsilon/2$ on a set of exterior measure greater than $1-\eta/2$. Hence, the set of points for which $|\phi(x)-f(x)|<\epsilon$ is of exterior measure greater than $1-\eta$ for every $\eta>0$ and is, therefore, of exterior measure one.

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A RATIO TEST

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The d'Alembert ratio test has numerous extensions that are effective in some cases where the d'Alembert test fails; that is, when the limit of the ratio a_n/a_{n-1} is 1. Examples are Raabe's test and Gauss' test. The following test is simpler than any of these, and is easy to prove and to remember.

THEOREM. *A series of positive terms* (1) $\sum a_n$ *converges if* $\lim_{n\to\infty} (a_n/a_{n-1})^n < 1/e$, *and diverges if* $\lim_{n\to\infty} (a_n/a_{n-1})^n > 1/e$.

More generally, (1) *converges if* lim sup $(a_n/a_{n-1})^n < 1/e$. It diverges if $(a_n/a_{n-1})^n \geqq 1/e$ *for all* n *sufficiently large. In particular* (1) *diverges if* lim inf $(a_n/a_{n-1})^n > 1/e$.

COROLLARY. *A series of real or complex terms* $\sum a_n$ *converges absolutely if* lim sup $|a_n/a_{n-1}|^n < 1/e$.

The proof is by the comparison ratio test with $\sum n^{-s}$ as the comparison series. To prove the convergence part of the test, suppose lim sup $(a_n/a_{n-1})^n = e^{-d} < e^{-1}$. Then $d > 1$. Consider the series (2) $\sum b_n = \sum n^{-s}$ with $1 < s < d$. Then we have lim $(b_n/b_{n-1})^n$ $=$ lim $(1-1/n)^{sn} = e^{-s}$. Since $e^{-d} < e^{-s}$, then ultimately $(a_n/a_{n-1})^n$ $< (b_n/b_{n-1})^n$, and therefore $a_n/a_{n-1} < b_n/b_{n-1}$. It follows that (1) converges, since (2) does.

To prove the divergence part of the test, suppose $(a_n/a_{n-1})^n \geqq 1/e$ for n sufficiently large. For the harmonic series (3) $\sum c_n = \sum n^{-1}$, we have $(c_n/c_{n-1})^n = (1-1/n)^n < e^{-1}$ for all n. Hence ultimately $(a_n/a_{n-1})^n$ $> (c_n/c_{n-1})^n$, and therefore $a_n/a_{n-1} > c_n/c_{n-1}$. Since (3) diverges, so does (1). This completes the proof. In the same manner one can prove the following generalization.

THEOREM. *A series of positive terms* (1) $\sum a_n$ *converges if* lim sup $(a_n/a_{n-k})^n < 1/e^k$, *and diverges if* $(a_n/a_{n-k})^n \geqq 1/e^k$ *for all* n *sufficiently large.*

As an application, consider the Dirichlet series (4) $\sum a_n n^{-s}$, where $s = \sigma + i\tau$. Suppose lim $|a_n/a_{n-1}|^n = e^{-d}$. Then it follows from the test that the series (4) converges absolutely at every point of the halfplane $\sigma > 1-d$, and converges absolutely at no point of the halfplane $\sigma < 1-d$.

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THE STRUCTURE OF MINIMAL SETS

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1. Introduction. A minimal set is a topological space X acted on by a topological group T such that the orbit closure of every point in X coincides with X. If G is a relatively dense subgroup of T, the orbit closure under G of a point of X may or may not coincide with X. In this paper the properties of the orbit closures under such a subgroup are studied and several of the possibilities are analyzed. There is displayed an example of a regularly almost periodic point such that not all points in the orbit closure are regularly almost periodic.

2. The decomposition of a minimal set by relatively dense subgroups. Let X be a compact metric space and let T be an additive abelian topological group. Let f be a continuous transformation of the product space $X\times T$ into X. We denote the image of the point $x\times t$ under f by either $f(x, t)$ or $f^t(x)$. We assume that f defines a transformation group in that if $t, s\in T$, $x\in X$, then

$$f^0(x) = x, \tag{1}$$

$$f^{s+t}(x) = f^s(f^t(x)). \tag{2}$$

It is easily shown that for fixed t in T, the transformation defined by $x\to f^t(x)$ is a homeomorphism of X onto X. With f satisfying the stated conditions we say that T is a *transformation group* acting on X.

The subset Y of X is *invariant* under the subset A of T if $f^a(Y)=Y$ for all a in A. If G is a subgroup of T and Y is a subset of X which is invariant under G, then Y is a topological space, G is a topological group, and f defines a continuous transformation of the product $Y\times G$ onto Y such that (1) and (2) are satisfied for all points y in Y and all element pairs g, h in G. Thus G is a transformation group acting on Y.

The *orbit* of x is the set $f(x, T)$. If A is a subset of T, the *orbit of x under A* is the set $f(x, A)$.

The space X is *minimal* under T if for every x in X, $\bar{f}(x, T)=X$, where $\bar{f}(x, T)$ denotes the closure of the set $f(x, T)$. The subset A of T is *relatively dense* in T if there exists a compact subset C of T such that $T=A+C$. The point x of X is *almost periodic* under T if, corresponding to any neighborhood U of x, there exists a set A, relatively dense in T, such that $f(x, A)\subset U$.

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We shall assume throughout that X is minimal under T. It follows from a theorem of Gottschalk [1, Theorem 1, p. 762][1] that each point of X is almost periodic under T.

Let G be a relatively dense subgroup of T. It follows from another theorem of Gottschalk [1, Theorem 4, p. 764] that each point x of X is almost periodic under G. Thus the set $\bar{f}(x, G)$ is invariant and minimal under G. If x and y are arbitrary points of X, the sets $\bar{f}(x, G)$ and $\bar{f}(y, G)$ are either disjoint or identical and G effects a decomposition of X into disjoint closed sets which are minimal under G.

Let X and Y be minimal sets with associated transformation groups T and S, and transformations f and g, respectively. The minimal sets X and Y will be said to be ***equivalent*** if there exists a homeomorphism h of X onto Y and an isomorphism I between T and S such that if $x \in X$, $t \in T$ and $I(t) = s \in S$, then

$$f^t(x) = h^{-1}g^s[h(x)].$$

In this case it will be said that X and Y are ***equivalent by virtue of the homeomorphism h and the isomorphism I***. If the groups T and S happen to be identical, X and Y will be said to be ***equivalent by virtue of the homeomorphism h***.

THEOREM 1. *Let X be a compact metric space which is minimal under the transformation group T and let G be a relatively dense subgroup of T. Then if x and y are arbitrary points of X, there exists an element u of T such that $f^u[\bar{f}(x, G)] = \bar{f}(y, G)$ and the sets $\bar{f}(x, G)$ and $\bar{f}(y, G)$, which are minimal under G, are equivalent by virtue of f^u.*

Since G is a relatively dense subgroup of T, there exists a compact set A in T such that $T = A + G$. Let X_1 be the subset of X consisting of all points $f(z, a)$ such that $z \in \bar{f}(x, G)$ and $a \in A$. If follows from the compactness of $\bar{f}(x, G)$ and A that X_1 is compact. But if t is any element of T, then $t = a + g$, with a in A and g in G, and since $f^g(x) \in \bar{f}(x, G)$, we infer that $f^t(x) = f[f^g(x), a] \in X_1$. Since the orbit of x is everywhere dense in X, X_1 is everywhere dense in X and thus, X_1 being a closed set, $X_1 = X$. In particular, we have shown that there exists a point z in $\bar{f}(x, G)$ and an element u of T such that $f^u(z) = y$.

The transformation f^u is a homeomorphism of X onto X. If g is any element of G, $f^u[f^g(z)] = f^g[f^u(z)] = f^g(y)$ and since the set $f(x, G)$ is everywhere dense in $\bar{f}(x, G)$, while the set $f(y, G)$ is everywhere dense in $\bar{f}(y, G)$, it follows that $f^u[\bar{f}(x, G)] = \bar{f}(y, G)$. If w is any point of $\bar{J}(x, G)$ and g is any element of G, then

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$f^{g}(w) = f^{-b}[f^{g}\{f^{b}(w)\}]$$

and $\bar{J}(x, G)$ and $\bar{J}(y, G)$ are equivalent by virtue of f^{b}.

THEOREM 2. *The decomposition of X by G is continuous.*

Let g denote the mapping of $X \times G$ onto X defined by f. According to a theorem of Gottschalk [1, Theorem 5, p. 765] it is sufficient to prove that g is weakly almost periodic. That is, corresponding to $\epsilon > 0$, there exists a compact set C in G such that if x is any point of X, every translate of C by an element of G contains an element s such that $g(x, s) \in U_\epsilon(x)$. There exists a compact set B in T such that $T = B + G$. Let z be any point of X. Corresponding to $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in \bar{J}(z, G)$ and $b \in B$, then $f^b(U_\delta(y)) \subset U_\epsilon(f^b(y))$. This follows from the compactness of the sets $\bar{J}(z, G)$ and B. Now g is weakly almost periodic on the minimal set $\bar{J}(z, G)$ and hence there exists a compact set C in G such that if $u \in \bar{J}(z, G)$ then every translate $C + g^*$ of C by an element g^* of G contains an element s such that $g(u, s) \in U_\delta(u)$. We show that the set C satisfies the desired condition with respect to every point x in X. For there exists a point u in $\bar{J}(z, G)$ and an element b in B such that $x = f^b(u)$. Let s be an element in $C + g^*$ such that $f^s(u) \in U_\delta(u)$. But then $f^b(f^s(u)) \in U_\epsilon(f^b(u))$ or $g(x, s) \in U_\epsilon(x)$, which is the desired result.

3. Reducibility and regular almost periodicity. If we consider all possible relatively dense subgroups of T and the resulting decompositions of X, several possibilities arise. The minimal set X will be said to be *totally minimal* if there exists a point x of X such that for every relatively dense subgroup G of T, $\bar{J}(x, G) = X$. It is clear that if this property holds for one point of X, it holds for all points of X. It is easy to construct examples of totally minimal sets. If X is a circle of circumference unity and α is a positive rational number, the rotations of X about its center through integral multiples of α define a transformation group acting on X with T the discrete group of integers. It is readily verified that in this case X is a minimal set which is totally minimal. More generally, if X is a connected minimal set for which the group T is discrete, and G is a relatively dense subgroup of T, then, since any compact subset of a discrete group is finite, the number of sets in the decomposition of X by G is finite. Since X is connected and these sets are closed and disjoint, there can be only one such set, namely $\bar{J}(x, G)$, and X is totally minimal. However X can be totally disconnected and yet a totally minimal set. Examples of totally minimal sets for which X is the Cantor discontinuum and T is the group of integers have been constructed by Hedlund [1, Theorem 6.4, p. 619].

If T is the group of real numbers with its natural topology we say that f defines a *continuous flow* in X. It is possible to display examples of compact manifolds which are totally minimal under continuous flows. A continuous flow in the topological space X is said to be *topologically mixing* if corresponding to any two open subsets A and B of X there exists a real number r such that for $|t| > r, f^t(A) \cap B \neq \emptyset$. It is clear that if X is minimal under a continuous flow which is topologically mixing, then X is totally minimal. Continuous flows in compact three-dimensional manifolds which are minimal and topologically mixing have been constructed by Hedlund ([2, p. 250]. The result is not stated specifically, but it is remarked that if the fundamental region, together with its boundary, lies interior to the unit circle, the horocycle flow has the property that every motion is transitive. Thus the horocycle flow is minimal. It is easy to show that this flow is topologically mixing). These considerations suggest a converse problem, namely as to whether, in the case of a set which is minimal under a continuous flow, total minimality implies that the flow is topologically mixing. The authors have not been able to verify or disprove this conjecture.

In contrast to the case in which X is totally minimal it may happen that corresponding to any $\epsilon > 0$ there exists a relatively dense subgroup G of T such that if D is any one of the members of the decomposition of X by G, diam. $D < \epsilon$. Generalizing a definition due to G. T. Whyburn [1, p. 250] we say that in this case the *transformation group T acting on X is regularly almost periodic.* It will also be said in this case that the minimal set X is *completely reducible.*

If the transformation group T is periodic in the sense that there exists a relatively dense subgroup G such that $f(x, G) = x$ for all x in X, then T is regularly almost periodic and the minimal set X is completely reducible. It is not difficult to construct a minimal set X for which T is the group of integers, X is a Cantor discontinuum, and X is completely reducible. In such a case T cannot be periodic in the sense defined.

Again generalizing a definition of Whyburn [1, p. 250] we say that the *point x of X is regularly almost periodic* with respect to T if corresponding to any $\epsilon > 0$ there exists a relatively dense subgroup G of T such that diam. $\bar{f}(x, G) < \epsilon$. In this case we say that the minimal set X is *reducible* at x and if every point of X is regularly almost periodic we say that X is *pointwise reducible.* If the minimal set X is completely reducible, it is clearly pointwise reducible. As to whether the converse of this is true in general seems to be difficult to determine, but it is possible to show that pointwise reducibility implies complete reducibility if the group T is discrete. To that end we prove an elementary lemma.

LEMMA 1. *If T is a discrete, abelian, topological group and G_1 and G_2 are relatively dense subgroups of T, then $G_1 \cap G_2$ is a relatively dense subgroup of T, of G_1 and of G_2.*

The proof of this lemma follows readily from the theorem (Kurosch [1, p. 47]) which states that the intersection of a finite number of subgroups of finite index is a group of finite index and the fact that a relatively dense subgroup of T is of finite index.

THEOREM 3. *If X is a compact minimal set which is pointwise reducible and T is discrete, then X is completely reducible.*

For if $\epsilon > 0$ and x is any point of X, there exists a relatively dense subgroup G of T such that diam. $\bar{f}(x, G) < \epsilon/2$. It follows from Theorem 2 that there exists a neighborhood $N(x)$ of x such that if $y \in N(x)$ then diam. $\bar{f}(y, G) < \epsilon$. A finite number of these neighborhoods cover X and let such a set be $N_1, N_2, \cdots, N_m$ with corresponding groups, $G_1, G_2, \cdots, G_m$. According to Lemma 1, the intersection $G^* = G_1 \cap G_2 \cap \cdots \cap G_m$ of these relatively dense subgroups of T is a relatively dense subgroup of T. If z is any point of X, there exists an integer i such that $z \in N_i$ and then we have diam. $\bar{f}(z, G^*) \leqq$ diam. $\bar{f}(z, G_i) < \epsilon$. It follows that X is completely reducible.

If the transformation group T acting on X is regularly almost periodic it is almost periodic and hence the family T is equi-uniformly continuous (cf. Gottschalk [3, Theorem 2, p. 635]).

THEOREM 4. *If X is compact and minimal under the equi-uniformly continuous transformation group T, and if X is reducible at one point, then X is completely reducible.*

Since the group T is equi-uniformly continuous, corresponding to $\epsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $d[f^t(x), f^t(y)] < \epsilon/2$ for all t in T. If $\bar{x}$ is the point at which X is reducible, there exists a relatively dense subgroup G of T such that diam. $\bar{f}(\bar{x}, G) < \delta$. But then if $y = f^t(x)$ is any point in the orbit of x, it follows that $\bar{f}(y, G) = \bar{f}[f^t(\bar{x}), G] = f^t[\bar{f}(\bar{x}, G)]$ and hence diam. $\bar{f}(y, G) \leqq \epsilon/2$. In view of Theorem 2, we see that if x is any point of X, diam. $\bar{f}(x, G) < \epsilon$. The proof of the theorem is complete.

The compact set X, minimal under the transformation group T, will be said to be 0-*reducible* if corresponding to $\delta > 0$ there exists a relatively dense subgroup G of T and a point x of X such that diam. $\bar{f}(x, G)<\delta$. If X is a 0-reducible minimal set, each point of X will be said to be ***isochronous***.

THEOREM 5. *A necessary and sufficient condition that $y\in X$ be isochronous is that corresponding to $\delta>0$ there exists a relatively dense subgroup G of T and an element t of T such that* diam. $\bar{f}(y, t+G)<\delta$.

Since $\bar{f}(y, t+G)=\bar{f}[f^t(y), G]$, it is clear that the condition is sufficient. To prove the necessity, suppose that y is isochronous. Then corresponding to $\delta>0$ there exists a relatively dense subgroup G of T and a point x of X such that diam. $\bar{f}(x, G)<\delta/2$. It follows from Theorem 2 that there exists a neighborhood $N(x)$ of x such that $z\in N(x)$ implies diam. $\bar{f}(z, G)<\delta$. But the orbit of y is everywhere dense in X and thus there exists a t in T such that $f^t(y)\in N(x)$. We now have diam. $\bar{f}(y, t+G)=$ diam. $\bar{f}[f^t(y), G]<\delta$ and the proof of the theorem is complete.

THEOREM 6. *If the compact set X is minimal under T and 0-reducible, there exists a point of X at which X is reducible.*

We show first that if y is any point of X and $U_\epsilon(y)$ is any ϵ-neighborhood of y, there exists a point z in $U_\epsilon(y)$ and a relatively dense subgroup G of T such that $\bar{f}(z, G)\subset U_{2\epsilon}(y)$. Since f is weakly almost periodic, corresponding to $\epsilon>0$, there exists a compact set A in T such that if v is any point of X, each translate of A contains an element t for which $f(v, t)\in U_{\epsilon/2}(y)$. Since X is minimal, there exists an s in T such that $f^s(v)\in U_{\epsilon/2}(v)$. It follows that the set $A-s$ contains an element $t=a-s$, $a\in A$, such that $f^t[f^s(v)]\in U_{\epsilon/2}(f^s(v))$. But then $f^a(v)\in U_{\epsilon/2}[f^s(v)]\subset U_\epsilon(y)$. Thus $f[U_\epsilon(y), -A]=X$. Since the set A is compact, there exists a $\delta>0$ such that if u and v are any points of X at distance apart less than δ and a is any element of A, then $d[f^a(u), f^a(v)]<\epsilon/2$. Since X is 0-reducible, there exists a point $\bar{x}$ in X and a relatively dense subgroup G of T such that diam. $\bar{f}(\bar{x}, G)<\delta$. Now let a be any element of A such that $f^a(\bar{x})\in U_\epsilon(y)$. But then diam. $f^a[\bar{f}(\bar{x}, G)]=$ diam. $\bar{f}[f^a(\bar{x}), G]<\epsilon$ and if we let $f^a(\bar{x})=z$, we have the first of our desired results.

Now let $\epsilon_1>\epsilon_2>\cdots$ be a sequence of real numbers with $\lim_{n\to\infty}\epsilon_n=0$. Let U_1 be a sphere in X with radius ϵ_1. It follows from the first part of the proof that there exists a point x_1 in U_1 and a relatively dense subgroup G_1 of T such that $\bar{f}(x_1, G_1)\subset U_1$. From Theorem 2 we infer the existence of a sphere V_1 with center x_1 such that for every y in $\bar{V}_1$ it is true that $\bar{f}(y, G_1)\subset U_1$. In particular it follows that $\bar{V}_1\subset U_1$. Now let U_2 be a sphere of radius ϵ_2 in V_1. By a repetition of the argument just given, there exists a point x_2, a sphere V_2 with center x_2, and a relatively dense subgroup G_2 of T such that $\bar{V}_2 \subset U_2 \subset V_1$ and if $z \in \bar{V}_2$ then $\bar{f}(z, G_2) \subset U_2$. Continuing this process we obtain a sequence $V_1 \supset V_2 \supset \cdots$ of spheres such that the radius of V_i is less than ϵ_i, $i = 1, 2, \cdots$, and a sequence of relatively dense subgroups $G_1, G_2, \cdots$ of T such that if x_i is any point of $\bar{V}_i$ then diam. $\bar{f}(x_i, G_i) < \epsilon_i$. Since $\bar{V}_i$ is compact, there must exist a point $x^* \in \bigcap_{i=1}^{\infty} \bar{V}_i$. Now X is reducible at x^* and the proof of the theorem is complete.

COROLLARY 6.1. *If X is compact and minimal under the equi-uniformly continuous transformation group T and if X is 0-reducible, then X is completely reducible.*

If the group T is discrete it is possible to add several results which are not necessarily true in the general case.

THEOREM 7. *If X is a compact set which is minimal under the discrete transformation group T and G is a relatively dense subgroup of T then:*

(a) *If x is regularly almost periodic under T, x, considered as in $\bar{f}(x, G)$ acted on by G, is regularly almost periodic under G;*

(b) *If x is isochronous under T, x, considered as in $\bar{f}(x, G)$ acted on by G, is isochronous under G.*

Statement (a) of the theorem is a simple application of Lemma 1.

To prove (b) suppose that x is isochronous under T. It follows from Theorem 6 that there exists a point y of X which is regularly almost periodic under T. We infer from (a) that y, considered as in $\bar{f}(y, G)$ acted on by G, is regularly almost periodic under G. But according to Theorem 1, the sets $\bar{f}(x, G)$ and $\bar{f}(y, G)$ are equivalent and there must be a point of $\bar{f}(x, G)$ which is regularly almost periodic under G. This implies that x, considered as in $\bar{f}(x, G)$ acted on by G, is isochronous under G.

4. An isochronous point which is not regularly almost periodic. Let S be the space consisting of the symbols a and b and let I denote the ordered set $\cdots, -1, 0, 1, 2, \cdots$ of all integers. Let X be the set of all mappings from I into S. If x and y are points of X, $x(n)$ denoting the image of n in S under the mapping x, $y(n)$ denoting the image of n in S under the mapping y, we define the distance $d(x, y)$ between x and y as follows:

$$d(x, y) = 0 \qquad \text{if} \quad x(n) = y(n) \ \text{for all } n \text{ in } I,$$

$$d(x, y) = 1 \qquad \text{if} \quad x(0) \neq y(0),$$

$$d(x, y) = (k+2)^{-1} \quad \text{if} \quad x(n) = y(n),\ n = 0, \pm 1, \cdots, \pm k,$$
$$\text{and either } x(-k-1) \neq y(-k-1) \quad \text{or} \quad x(k+1) \neq y(k+1).$$

It is easily shown that this metric satisfies the usual metric axioms and that the metric space X thus defined is compact, perfect and totally disconnected (cf. Morse and Hedlund [1, pp. 819–820]). Thus X is a Cantor discontinuum.

Let x be any point of X and let $x(n)$ denote the image of n in S under the mapping x. If to the integer n we let correspond $x(n+1)$ there is defined a mapping of I into S and we denote the point of x thereby defined by y. It is easy to show that the mapping $x \rightarrow y$ is a homeomorphism H of X onto X. This homeomorphism H and its integral powers define a transformation group T acting on X and $T = I$.

We define a sequence of points of X. It will be convenient to term the symbol b the *dual* of a and the symbol a the *dual* of b. If c stands for either a or b, we denote its dual by $\bar{c}$. Now let α_0 be the point of X defined by the mapping

$$\alpha_0(n) = a, \qquad n = 0, \pm 1, \pm 2, \cdots.$$

Proceeding inductively and assuming that $\alpha_0, \alpha_1, \cdots, \alpha_{m-1}$ have been defined, we define α_m by the mapping

$$\alpha_m(n) = \begin{cases} \alpha_{m-1}(n), & n \neq k2^m,\ k = 0, \pm 1, \cdots, \\ \bar{\alpha}_{m-1}(n), & n = k2^m,\ k = 0, \pm 1, \cdots. \end{cases}$$

The point x of X is *periodic* if there exists an ω in I, $\omega \neq 0$, such that $x(n+\omega) = x(n)$ for all n in I. In this case we say that ω is *a period* of x. If x is periodic, there exists a primitive period ω of x in the sense that $\omega \in I$, $\omega > 0$, and any period of x is an integral multiple of ω. We then say that this primitive period is *the period* of x. Clearly x is periodic under T if and only if x is periodic in the sense just defined.

LEMMA 2. *α_m is periodic with the period 2^m.*

We prove this by induction. It follows immediately from its definition that α_0 is periodic with the period 1. Let us suppose that for $0 \leq i \leq m-1$, α_i is periodic with the period 2^i. If $n \neq k2^m$, k integral, $\alpha_m(n) = \alpha_{m-1}(n) = \alpha_{m-1}(n+2^{m-1}) = \alpha_{m-1}(n+2^m) = \alpha_m(n+2^m)$. If $n = k2^m$, k integral, we have

$$\alpha_m(n) = \bar{\alpha}_{m-1}(n) = \bar{\alpha}_{m-1}(n+2^m) = \alpha_m(n+2^m).$$

Thus 2^m is a period of α_m. If 2^m is not the period of α_m and ω is the period of α_m, then 2^m must be a multiple of ω and ω must be a power of 2. It follows in particular that 2^{m-1} is a period of α_m. But

$$\alpha_m(0) = \bar{\alpha}_{m-1}(0) = \bar{\alpha}_{m-1}(2^{m-1}) = \bar{\alpha}_m(2^{m-1}) = \bar{\alpha}_m(0).$$

From this contradiction we infer that 2^{m-1} is not a period of α_m and 2^m must be the period of α_m. The proof of the lemma is complete.

Remark 1. If k and m are integers such that $k>m$, then $\alpha_k(n) = \alpha_m(n)$ provided $n \neq p2^{m+1}$, p integral.

This follows immediately from the definition of α_m.

Remark 2. The sequence of points $\alpha_0, \alpha_2, \alpha_4 \cdots$ converges in X.

For if k and m are integers such that $k>m$,

$$\alpha_{2k}(n) = \alpha_{2m}(n), \qquad n = 0, \pm 1, \cdots, \pm(2^{2m+1}-1).$$

If we define β by the mapping

$$\beta(n) = \alpha_{2m}(n), \qquad |n| < 2^{2m+1} - 1$$

then $\beta(n)$ is uniquely defined and it is clear that $\alpha_{2m} \to \beta$.

Remark 3. If x is periodic with the period ω, there cannot exist integers k and ω_1 such that $0<\omega_1<\omega$ and

$$x(n+\omega_1) = x(n), \qquad n = k+1, k+2, \cdots, k+\omega. \tag{4}$$

For suppose (4) were true. If p is integral, there exists an integer q such that $p+q\omega = k+i$ with $1 \leqq i \leqq \omega$. But now we have

$$x(p) = x(p+q\omega) = x(k+i) = x(k+i+\omega_1) = x(p+\omega_1).$$

Thus x has ω_1 as period, contrary to the hypothesis that the period of x is ω.

LEMMA 3. *β is not periodic.*

For suppose that β has the period ω. Let the positive integer m be so chosen that $2^{2m} > \omega$. According to the definition of β,

$$\beta(n) = \alpha_{2m}(n), \qquad n = 0, \pm 1, \cdots, \pm(2^{2m+1}-1)$$

and if β has period ω we have

$$\begin{gathered} \alpha_{2m}(k+\omega) = \alpha_{2m}(k), \\ k = -2^{2m+1}+1, -2^{2m+1}+2, \cdots, -2^{2m+1}+2^{2m}. \end{gathered} \tag{5}$$

Since α_{2m} has the period 2^{2m}, it follows from Remark 3 that (5) is impossible. We infer that β cannot be periodic.

Remark 4. $\beta(n) = \alpha_{2m}(n)$, $n \neq k2^{2m+1}$, $k = \pm 1, \pm 2, \cdots$.

LEMMA 4. *Given $\epsilon > 0$ there exist integers l and $k > 0$ such that*

$$d(\beta, H^{l+pk}(\beta)) < \epsilon, \qquad p \in I.$$

Let m be a positive integer such that $2^m < 1/\epsilon$, let $l = 2^{2m}$ and let $k = 2^{2m+1}$. Then if $p \in I$ and $|n| < 2^m$, we have

$$\begin{aligned}\beta(l + pk + n) &= \beta(2^{2m} + p2^{2m+1} + n) = \alpha_{2m}(2^{2m} + p2^{2m+1} + n) \\ &= \alpha_{2m}(n) = \beta(n).\end{aligned}$$

This implies

$$d(\beta, H^{l+pk}(\beta)) \leqq 2^{-m} < \epsilon$$

and the lemma is proved.

Since the set pk, $p = 0, \pm 1, \cdots$ is relatively dense in I, it follows from Lemma 4 that β is an almost periodic point. Thus if $\Gamma(\beta)$ denotes the orbit closure of β in X, $\Gamma(\beta)$ is minimal under the transformation group T. From Lemma 4 and Theorem 5 we infer that β is isochronous and thus $\Gamma(\beta)$ is 0-reducible. From Theorem 6 there exists a point γ of $\Gamma(\beta)$ at which $\Gamma(\beta)$ is reducible and γ is regularly almost periodic with respect to T.

But $\Gamma(\beta)$ is not completely reducible. For if this were the case the transformation group T acting on X would be regularly almost periodic and hence equi-uniformly continuous. Since β is almost periodic but not periodic, there exists a sequence of points $\beta_1, \beta_2, \cdots$ of $\Gamma(\beta)$ such that $\beta_i \rightarrow \beta$ and $\beta_i \neq \beta$, $i = 1, 2, \cdots$. Since $\beta_i \neq \beta$, there exists an integer n such that $\beta_i(n) \neq \beta(n)$. This implies that $d[H^n(\beta_i), H^n(\beta)] = 1$. Thus if $\epsilon = 1/2$, there exists no $\delta > 0$ such that $d(\beta_i, \beta) < \delta$ implies $d[H^p(\beta_i), H^p(\beta)] < \epsilon$ for all p in I. It follows that the transformation group T acting on $\Gamma(\beta)$ is not equi-uniformly continuous and $\Gamma(\beta)$ is not completely reducible.

Since T is discrete, it follows from Theorem 3 that $\Gamma(\beta)$ is not pointwise reducible. Since $\Gamma(\beta)$ is a minimal set, the orbit closure $\Gamma(\gamma)$ of γ is identical with $\Gamma(\beta)$. Thus γ is a regularly almost periodic point with the property that not all points in the orbit closure of γ are regularly almost periodic.

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A ZERO-DIMENSIONAL TOPOLOGICAL GROUP WITH A ONE-DIMENSIONAL FACTOR GROUP

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As can be easily shown, if a locally compact topological group is zero-dimensional, all of its factor groups are zero-dimensional. In this note we give an example of a non locally compact zero-dimensional group with a factor group which is topologically isomorphic to the real numbers, hence one-dimensional.[1]

1. **Preliminaries.** Let $\{\lambda\}$ be a set of indices of cardinality c, and for each λ, let R_λ be a topological isomorph of the additive group of rational numbers. We form the weak product R of the R_λ: an element r of R is a collection $r=\{r_\lambda\}$, $r_\lambda \in R_\lambda$, such that for only a finite number of the λ's is $r_\lambda \neq 0_\lambda$. Under the definitions $r+r'=\{r_\lambda+r'_\lambda\}$, $0=\{0_\lambda\}$, R forms a group.

Now for each $r\in R$, we define $\|r\|=\sum_\lambda |r_\lambda|$. Since all but a finite number of the $r_\lambda=0_\lambda$, this sum exists. Clearly $\|r+r'\|\leqq\|r\|+\|r'\|$, and $\|-r\|=\|r\|$, hence, as can be easily shown, $\|r\|$ defines a metric in R under the definition: the distance from r to r' is $\|r-r'\|$.

Lemma 1. *Let* $\{d_\lambda\}$ *be a set of positive real numbers bounded away from zero, that is, there exists* $d>0$ *such that* $d_\lambda \geqq d$ *for all* λ. *Then*

$$U=\left\{r \,\middle|\, \sum_\lambda \left|\frac{r_\lambda}{d_\lambda}\right| < 1\right\}$$

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[1] Cf. Bourbaki, *Topologie générale*, chap. III, p. 21, exercise 12, for an example of a totally disconnected group with a factor group topologically isomorphic to the reals. This example was pointed out to me by I. Kaplansky.

is an open set containing the origin, and

$$U \subset \left\{ r \,\middle|\, \sum_\lambda \left| \frac{r_\lambda}{d_\lambda} \right| \leq 1 \right\}.$$

Proof. We need only prove that the real valued function $f(r) = \sum_\lambda |r_\lambda / d_\lambda|$ is continuous. Given $r = \{r_\lambda\}$ and $\epsilon > 0$, choose $\delta < d\epsilon$. Now, for any $r' = \{r'_\lambda\}$,

$$\left| \sum_\lambda \left| \frac{r'_\lambda}{d_\lambda} \right| - \sum_\lambda \left| \frac{r_\lambda}{d_\lambda} \right| \right| = \left| \sum_\lambda \frac{|r'_\lambda| - |r_\lambda|}{d_\lambda} \right| \leq \sum_\lambda \left| \frac{|r'_\lambda| - |r_\lambda|}{d_\lambda} \right|$$
$$\leq \frac{1}{d} \sum_\lambda \big| |r'_\lambda| - |r_\lambda| \big|$$
$$\leq \frac{1}{d} \sum_\lambda |r'_\lambda - r_\lambda|$$
$$= \frac{1}{d} \| r' - r \|.$$

Thus, if $\|r' - r\| < \delta$, this last expression is less than ϵ, which proves the continuity and hence the lemma.

Lemma 2. *Let* $\{\alpha_\lambda\}$ *be a bounded set of positive irrational numbers linearly independent with respect to the rationals, that is:*

$$(1) \qquad a_1 \alpha_{\lambda(1)} + \cdots + a_k \alpha_{\lambda(k)} = a \qquad (a_1, \cdots, a_k, a \text{ rational})$$

implies

$$a_1 = \cdots = a_k = a = 0.$$

Then if, in Lemma 1, *we take* $d_\lambda = 1/\alpha_\lambda$, *the resulting U has a vacuous boundary.*

Proof. From the second part of Lemma 1, we need only prove that there is no r such that $\sum_\lambda |r_\lambda / d_\lambda| = 1$. Assume there is. Then, since replacing an r_λ by its negative does not change absolute values, we can assume all the r_λ are non-negative. Then we have $\sum_\lambda r_\lambda / d_\lambda = 1$. But each $d_\lambda = 1/\alpha_\lambda$, hence $\sum_\lambda r_\lambda \alpha_\lambda = 1$, which contradicts the hypothesis on the α_λ's.

Using Lemma 2, we now define a special sequence of neighborhoods $\{U_n\}$ $(n = 0, 1, \cdots)$ which form a basis around the origin. We first take the set of real numbers $1/2 \leq c < 1$ and set them in one-one correspondence with the λ's: $\{c_\lambda\}$. (Our purpose in bringing these in will become clear in §2.) We then choose a set of irrational numbers $\{\alpha_\lambda\}$ with the property (1) (the existence of such a set follows from the existence of a Hamel basis for the reals), and such that for each λ, $c_\lambda < \alpha_\lambda < 1$. This last can always be accomplished by multiplying α_λ by a suitable rational. We then have

$$1/2 \leqq c_\lambda < \alpha_\lambda < 1 \qquad \text{for all } \lambda. \tag{2}$$

Now for each n, we define U_n by taking

$$d_\lambda^{(n)} = \frac{1}{2^n}\cdot\frac{1}{\alpha_\lambda}, \tag{3}$$

and letting

$$U_n = \left\{ r \;\middle|\; \sum_\lambda \left| \frac{r_\lambda}{d_\lambda^{(n)}} \right| < 1 \right\}. \tag{4}$$

Since, for $r \in U_n$,

$$\|r\| = \sum_\lambda |r_\lambda| = \sum_\lambda d_\lambda^{(n)} \left| \frac{r_\lambda}{d_\lambda^{(n)}} \right| \leqq \frac{2}{2^n} \sum_\lambda \left| \frac{r_\lambda}{d_\lambda^{(n)}} \right| < \frac{1}{2^{n-1}},$$

the diameter of U_n is less than $1/2^{n-2}$, hence approaches zero as n goes to infinity. Thus $\{U_n\}$ constitutes a basis around the origin. Since, from Lemma 2, the boundary of each U_n is vacuous, it follows that R is zero-dimensional.

2. **The example.** Let R_* be the additive group of real numbers $\{r_*\}$ with distance defined by $\|r_*\| = 1$ for all r_* different from zero. This makes it discrete. Let $G = R_* \times R$ with distance defined as follows: If $g = (r_*, r)$ then $\|g\| = \|r_*\| + \|r\|$. Since R_* is discrete, the U_n's, now considered as subsets of G, form a basis around the origin of G, hence G is zero-dimensional.

We define the subgroup H of G as the set of all $g = (r_*, r)$ such that $r_* + \sum_\lambda c_\lambda r_\lambda = 0$ (cf. (2)).

Lemma 3. *H is a closed subgroup of G.*

Proof. H is a subgroup, for if $g, g' \in H$ then $r_* + \sum_\lambda c_\lambda r_\lambda = 0$ and $r'_* + \sum_\lambda c_\lambda r'_\lambda = 0$, hence $(r_* - r'_*) + \sum_\lambda c_\lambda (r_\lambda - r'_\lambda) = 0$. To prove H is closed, it is sufficient to show that the real-valued function $f(g) = r_* + \sum_\lambda c_\lambda r_\lambda$ is continuous. Given $g = (r_*, r)$ and $\epsilon > 0$, choose $\delta < \min(\epsilon, 1)$. Consider any $g' = (r'_*, r')$ such that $\|g' - g\| < \delta$. We note first that $r'_* = r_*$, for otherwise $\|r'_* - r_*\| = 1$ and hence $\|g' - g\| \geqq 1 \geqq \delta$. Then

$$\begin{aligned}\left|r_*' + \sum_\lambda c_\lambda r_\lambda' - r_* - \sum_\lambda c_\lambda r_\lambda\right| &= \left|\sum_\lambda c_\lambda r_\lambda' - \sum_\lambda c_\lambda r_\lambda\right| \\ &\leqq \sum_\lambda c_\lambda \left| r_\lambda' - r_\lambda \right| \\ &\leqq \sum \left| r_\lambda' - r_\lambda \right| \qquad \text{(from (2))} \\ &= \|g' - g\| < \delta < \epsilon.\end{aligned}$$

This proves the lemma.

LEMMA 4. *G/H is algebraically isomorphic to the real numbers.*

PROOF. Since $R_* \times 0 \subset G$ is algebraically isomorphic to the real numbers, it is sufficient to prove that G/H is algebraically isomorphic to this subgroup.

(i) Every coset of H contains an element of $R_* \times 0$.

For, let $g = (r_*, r)$ be any element of G, where $r = \{r_\lambda\}$. Then if $g' = (r_* + \sum_\lambda c_\lambda r_\lambda, 0)$, $g' - g = (\sum_\lambda c_\lambda r_\lambda, -r) \in H$. Since $g' \in R_* \times 0$, this proves (i).

(ii) Different elements of $R_* \times 0$ lie in different cosets.

For, let $g = (r_*, 0)$, $g' = (r_*', 0)$, with $r_* \neq r_*'$. Then $g' - g = (r_*' - r_*, 0)$, and since $(r_*' - r_*) + 0 \neq 0$, $g' - g \notin H$. This proves (ii), and with it, Lemma 4.

We can thus denote each element of G/H by a unique real number. From the proof of (i) above, we see that the real number is given by the mapping

$$\pi(r_*, r) = r_* + \sum_\lambda c_\lambda r_\lambda. \tag{5}$$

LEMMA 5. $\pi(U_n) = [-1/2^n < x < 1/2^n]$ *for all* $n = 0, 1, \cdots$.

PROOF. Since the argument is the same for all n, it is sufficient to prove this for U_0. From (4) and (3),

$$U_0 = \left\{ g = (0, r) \,\middle|\, \sum_\lambda \left| \alpha_\lambda r_\lambda \right| < 1 \right\}.$$

(i) $\pi(U_0) \subset [-1 < x < 1]$.

For, if $g = (0, r) \in U_0$, then from (5),

$$\begin{aligned}\left|\pi(g)\right| = \left|\sum_\lambda c_\lambda r_\lambda\right| &\leqq \sum_\lambda \left| c_\lambda r_\lambda \right| \\ &< \sum_\lambda \left| \alpha_\lambda r_\lambda \right| \qquad \text{(from (2))} \\ &< 1.\end{aligned}$$

(ii) $[-1<x<1]\subset\pi(U_0)$.

For consider any real number x such that $-1<x<1$. Since the c_λ's run through all the real numbers from 1/2 to 1, there is a $c_{\lambda'}$ such that

$$x = \epsilon c_{\lambda'},$$

where ϵ is one of the values ± 1, $\pm 1/2$. Hence, if g is the element of G whose λ'-coordinate is ϵ and whose remaining coordinates are 0, we have from (5) that

$$\pi(g) = c_{\lambda'}\epsilon = x.$$

Thus x has an inverse in U_0 under π. Since x was any element of $[-1<x<1]$, (ii) is proved. This establishes the lemma.

Since the set $\{U_n\}$ is a basis around the origin of G, the set $\{\pi U_n\}$ is by definition a basis around zero in G/H. Hence, from Lemma 5, G/H has the topology of the real numbers.

INSTITUTE FOR ADVANCED STUDY

TOPOLOGICAL GROUPS AND GENERALIZED MANIFOLDS

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In a recent paper [4],[1] Montgomery showed that in a locally euclidean 3-dimensional group, any 2-dimensional closed subgroup is also locally euclidean. In this note we prove an analogous result for higher dimensions and more general spaces.

THEOREM. *Let G be a locally compact space which is both a topological group and an n-dimensional orientable generalized manifold. Let H be a closed connected $(n-1)$-dimensional subgroup. Then, if H carries a nonbounding $(n-1)$-cycle, H is also an orientable generalized manifold.*

The terminology used in the statement of this theorem, and in what follows, is that of our two previous papers on generalized manifolds [1, 2], and we assume that the reader is familiar with them.

We make, however, one change. We find it convenient to define infinite cycles in the following way: We add to G an ideal point, g^+, taking as neighborhoods of g^+ those open subsets of G whose closures are not compact. Then $G^+ = G \cup g^+$ is compact. Now an infinite cycle of G is defined to be a relative cycle of G^+ mod g^+. That this definition of infinite cycles is equivalent to the one used in [2] follows from Theorem 1.1 of [2].

LEMMA 1. *Given any neighborhood M of the unit element e of G, there is a neighborhood N of E such that for any infinite cycle Γ^k on H, $0 \leq k \leq n-1$, and for any $g \in N$, $\Gamma^k \sim g \cdot \Gamma^k$ on $M \cdot H$.*

PROOF. Let $M_{n-1} \subset M$ have a compact closure. Choose a sequence

$$M_{n-1} \supset N_{n-1} \supset M_{n-2} \supset \cdots \supset M_0 \supset N_0$$

such that N_i is obtained from M_i by the local connectedness of G in dimension i, and such that $M_i \cdot M_i \subset N_{i+1}$. Finally let N be such that $N \cdot N \subset N_0$.

Now let $g \in N$. To show that $\Gamma^k \sim g \cdot \Gamma^k$ on $M \cdot H$, it is sufficient to show that the coordinates of these cycles on the nerve of any covering U of G are homologous on $(M \cdot H)^+$. To this end, given a covering U, choose $U' <^* U$. Let X be the complement of the union of those sets of U' which contain g^+. Then X is a compact set. Let $X_1 = \overline{M}_0 \cdot X$ and $X_i = \overline{M}_{i-1} \cdot X_{i-1}$. Each X_i is a compact set.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

A finite number of translations of N_{n-1} cover X_n, say

$$g_{1,n-1}\cdot N_{n-1},\ g_{2,n-1}\cdot N_{n-1},\ \cdots,\ g_{k,n-1}\cdot N_{n-1}.$$

For each i, let $U_{i,n-1}$ be a refinement of U' such that any $(n-1)$-cycle on $U_{i,n-1}$ in $g_{i,n-1}\cdot N_{n-1}$ has its projection to U' bounding in $g_{i,n-1}\cdot M_{n-1}$. Let U_{n-1} be a common refinement of these coverings.

Next, a finite number of translations of N_{n-2} cover X_{n-1}. From these we obtain a refinement U_{n-2} of U_{n-1} by the procedure above, this time using the local connectedness of G in dimension $n-2$. Proceeding in this fashion for another $n-2$ steps we arrive at a covering U_0.

Let Γ_0^k and $g\cdot\Gamma_0^k$ be the coordinates of Γ^k and $g\cdot\Gamma^k$ on U_0. We assert that $\pi\Gamma_0^k$ and $\pi g\cdot\Gamma_0^k$ are homologous on U on $(M\cdot H)^+$, where π is the projection from U_0 to U. Let Δ be the cartesian product of $|\Gamma_0^k|$ with a unit segment, subdivided simplicially in such a way that all the vertices of Δ are in the base, $\Delta_0 = |\Gamma_0^k|\times 0$, and in the top, $\Delta_1 = |\Gamma_0^k|\times 1$. Let $\bar{\Delta}$ be the closed subcomplex of Δ generated by those simplexes of Γ_0^k which are on X. We define a partial realization τ' of Δ on U_0 by letting $\tau'\sigma = \sigma$ if $\sigma\in\Delta_0$ and $\tau'\sigma = g\cdot\sigma$ if $\sigma\in\Delta_1$.

τ' induces a partial realization $\bar{\tau}'$ of $\bar{\Delta}$ on U_0. In view of the choices of the coverings made above, the usual argument shows that there is a full realization $\bar{\tau}$ of $\bar{\Delta}$ on U', where, if π_0 is the projection from U_0 to U', $\bar{\tau} = \pi_0\bar{\tau}'$ whenever the latter is defined. Also, $\bar{\tau}\bar{\Delta}$ is on $M_{n-1}\cdot H$.

We can now define a full realization of Δ on U in the following fashion. The projection π_0 can be so chosen that a vertex of U_0 not on X is projected into a vertex of U' which contains g^+. Since $U' <^* U$, a projection π of U' to U can be so chosen that any simplex of U_0 which has a vertex not on X is projected by $\pi\pi_0$ into the simplex of U consisting of those vertices of U which contain g^+. Any cycle in this simplex bounds in this simplex, so $\pi\pi_0\tau'(\Delta-\bar{\Delta})$ can be filled in to make a full realization of $\Delta-\bar{\Delta}$ on U, and this together with $\pi\bar{\tau}\bar{\Delta}$ makes a full realization τ of Δ on U. Since $\Gamma_0^k\times 0\sim\Gamma_0^k\times 1$ on Δ, $\tau(\Gamma_0^k\times 0)\sim\tau(\Gamma_0^k\times 1)$ on U. But $\tau(\Gamma_0^k\times 0) = \pi\pi_0\Gamma_0^k$ and $\tau(\Gamma_0^k\times 1) = \pi\pi_0 g\cdot\Gamma_0^k$. Since it is easily seen from the construction that this homology takes place on $(M\cdot H)^+$, the proof is complete.

Clearly the same proof suffices for the following lemma.

LEMMA 2. *Given any neighborhood M of e in G, there is a neighborhood N of e such that for any closed subset X of H and any cycle Γ^k of H* mod *X, $\Gamma^k\sim g\cdot\Gamma^k$ in $M\cdot H$* mod *$M\cdot X$, whenever $g\in N$.*

LEMMA 3. *If D is an open connected subset of G, then any two points* *of D lie on a compact connected subset of D.*

PROOF. Since G is lc^0, any neighborhood O of a point d of D contains a neighborhood W of d with $W \subset D$ such that any point $w \in W$ lies, together with d, on a compact continuum in $O \cap D$. Now let D_1 be the set of all points of D which can be joined to a fixed point $d_1 \in D$ by compact continua. Then, by the above, D_1 is both open and closed in D. Hence, since D is connected, D_1 is all of D.

LEMMA 4. *If O is a neighborhood of e such that $\mathcal{C}(O\cdot H)$ (where $\mathcal{C}$ means closure) is not all of G, then $O\cdot H-H$ has at least two components.*

PROOF. Let g be a point of $G-\mathcal{C}(O\cdot H)$, and let K be a compact connected set which contains both e and g. Let N be a neighborhood of e in O, chosen by Lemma 1. A finite number of translations of N cover K, and from these we may choose a sequence

$$e \in N, N_1, N_2, \cdots, N_k \ni g$$

where $N_i = g_i \cdot N$ and such that $N_i \cap N_{i+1} \neq 0$. Let $\bar{g}_i$ be a point of $N_i \cap N_{i+1}$. Now, $\bar{g}_{i-1} \in g_i \cdot N$, so $g_i^{-1}\cdot \bar{g}_{i-1} \in N$. Hence, by Lemma 1,

$$\Gamma^{n-1} \sim g_i^{-1} \cdot \bar{g}_{i-1} \cdot \Gamma^{n-1}$$

where Γ^{n-1} is a nonbounding cycle on H. Therefore,

$$g_i \cdot \Gamma^{n-1} \sim \bar{g}_{i-1} \cdot \Gamma^{n-1}.$$

Similarly, $\bar{g}_i \in g_i \cdot N$, and

$$g_i \cdot \Gamma^{n-1} \sim \bar{g}_i \cdot \Gamma^{n-1}.$$

Thus, we have

$$\Gamma^{n-1} \sim g \cdot \Gamma^{n-1}.$$

Now $\Gamma^{n-1} - g\cdot\Gamma^{n-1}$ is a cycle of $H \cup (G-O\cdot H)$ and $\Gamma^{n-1}-g\cdot\Gamma^{n-1} \sim 0$ in G. Hence [3, p. 227 (14.2)] there is in G a cycle Γ^n mod $(H \cup (G-O\cdot H))$ such that $F\Gamma^n = \Gamma^{n-1} - g\cdot\Gamma^{n-1}$. Let $\bar{\Gamma}^n$ be the fundamental n-cycle of G, and let $\bar{\Gamma}_1^n$ be the part of $\bar{\Gamma}^n$ on $G-(O\cdot H-H)$. Let $\bar{\Gamma}_2^n = \bar{\Gamma}^n - \bar{\Gamma}_1^n$.

In a neighborhood of any point of $O\cdot H-H$, Γ^n is homologous to some multiple of $\bar{\Gamma}^n$. If we assume that $O\cdot H-H$ is connected, then (cf. [1, p. 569]) this multiple is the same for all points of $O\cdot H-H$, that is, $r\Gamma^n = \bar{\Gamma}_2^n$.

By definition, $\bar{\Gamma}_1^n$ is on $H \cup (G-O\cdot H)$. Since H and $G-O\cdot H$ are closed and disjoint, and since dim $H < n$, $\bar{\Gamma}_1^n$ must be on $G-O\cdot H$, so $F\bar{\Gamma}_1^n$ is also on $G-O\cdot H$.

But, from $0 = F\Gamma^n = F(\Gamma_1^n + \Gamma_2^n)$, we have

$$\begin{aligned} F\Gamma_1^n &= -F(\Gamma_2^n) = -F(r\Gamma^n) \\ &= -r(\Gamma^{n-1} - g\cdot\Gamma^{n-1}). \end{aligned}$$

This is not on $G - O\cdot H$, since Γ^{n-1} is on H. Thus, the assumption that $O\cdot H - H$ has only one component leads to a contradiction.

We now choose a fixed connected neighborhood of e, satisfying the condition of Lemma 4, and denote by J the product of H by this neighborhood. We note that J is a connected generalized n-manifold. It is not a group, but for any two elements of J which are close enough to H, their product in G is in J.

LEMMA 5. *H is the boundary of each domain of $J - H$.*

PROOF. Let D be any component of $J - H$. Since J is lc^0, D is open. Some point $h \in H$ is a limit point of D, or else J would not be connected. Let O be a neighborhood of e. Then $h\cdot O$ contains a point $d \in D$, that is, $h\cdot o = d$. Now $h^{-1}\cdot d = o \in O$. But $h^{-1}\cdot d$ is also in D. For $o \notin H$, and, since H is connected, $H\cdot o$ lies in one component of $J - H$. Since $h\cdot o = d$ is in D, $H\cdot o$ lies in D, and consequently, $e\cdot o = o$ is in D. Therefore e is a limit point of D. Similarly, if $\bar{h}$ is any other point of H, then the neighborhood $\bar{h}\cdot O$ contains $\bar{h}\cdot o$ which is in $H\cdot o$ and therefore in D. Thus, $\bar{h}$ is a limit point of D, which proves the lemma.

LEMMA 6. *$J - H$ has just two components.*

PROOF.[2] By Lemma 2, it is enough to show that H does not have three complementary domains. Suppose there were three, D_0, D_1 and D_2. Let p_1, p_2 be points in D_1 and D_2 respectively, and let Y_1, Y_2 be neighborhoods of p_1, p_2 such that $\bar{Y}_i$ is compact and is in D_i.

$\gamma^0 = p_1 - p_2$ is a compact 0-dimensional cycle in $Y = Y_1 \cup Y_2$. γ^0 not ~ 0 in $J - H$, since p_1 and p_2 are in different components. But for any point $d_0 \in D_0, \gamma^0 \sim 0$ in $J - d_0\cdot H$. For let O be a neighborhood of e not meeting $d_0\cdot H$, which is in D_0, and let O' be chosen so that every compact 0-cycle in O' bounds in O. Choose $d_1 \in O' \cap D_1$ and $d_2 \in O' \cap D_2$. Then $d_1 \sim d_2$ in O. By Lemma 3, $p_1 \sim d_1$ in D_1, $p_2 \sim d_2$ in D_2. Hence, $p_1 \sim p_2$ in $D_1 \cup D_2 \cup O$, which does not meet $d_0\cdot H$.

Now, by Lemma 5.2 of [2], there is a compact cocycle γ_n in Y such that $(\Gamma^n\cdot\gamma_n) \sim \gamma^0$ in $J - H$, where Γ^n is the fundamental n-cycle of J, and such that $\gamma_n \backsim 0$ in $J - d_0\cdot H$, for any $d_0 \in D_0$. Since γ_n is a compact cocycle of $D_1 \cup D_2$, there is an infinite n-cycle Γ^n of $D_1 \cup D_2$

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[2] The main outline of this proof, and to some extent that of Lemma 7, is derived from Wilder [6, 7].

such that $\mathrm{KI}(\Gamma^n \cdot \gamma_n) = 1$. Let $\Gamma^{n-1} = F\Gamma^n$, so that Γ^{n-1} is an infinite cycle of H.

We now choose a neighborhood M of e which does not meet $\bar{Y}$, and a neighborhood N satisfying the conditions of Lemma 1. Let $d_0 \in D_0 \cap N$. Then $\Gamma^{n-1} \sim d_0 \cdot \Gamma^{n-1}$ in $M \cdot H$. Let $\Gamma^n = \{\Gamma^n_\zeta\}$ and let the chains involved in the homology $\Gamma^{n-1} \sim d_0 \cdot \Gamma^{n-1}$ be $\{C^n_\zeta\}$. Then $\{\Gamma'^n_\zeta\} = \{\Gamma^n_\zeta - C^n_\zeta\}$ is such that $F\Gamma'^n_\zeta = d_0 \cdot \Gamma^{n-1}_\zeta$. By construction, $\mathrm{KI}(\Gamma'^n_\zeta \cdot \gamma^\zeta_n) = \mathrm{KI}(\Gamma^n_\zeta \cdot \gamma^\zeta_n)$, since none of the chains C^n_ζ meet $\bar{Y}$.

$\{\Gamma'^n_\zeta\}$ is not necessarily a Čech cycle. But, for each covering U_ζ, let $U_{\rho(\zeta)}$ be an essential refinement (see [3, II 27: 13]) of U_ζ relative to cycles of J^+ mod $(d_0 \cdot H)^+$. Then $\{\Gamma''^n_\zeta\} = \{\pi^\zeta_{\rho(\zeta)} \Gamma'^n_{\rho(\zeta)}\}$ is a Čech cycle mod $(d_0 \cdot H)^+$ and $\mathrm{KI}(\Gamma''^n_\zeta \cdot \gamma^\zeta_n) = \mathrm{KI}(\Gamma'^n_\zeta\ \gamma_n)$ for all ζ.

But now we have reached a contradiction. For $\gamma_n \sim 0$ in $J - d_0 \cdot H$, so its Kronecker index with any infinite n-cycle of $J - d_0 \cdot H$ must be zero. But $\mathrm{KI}(\Gamma''^n \cdot \gamma_n) = \mathrm{KI}(\Gamma^n \cdot \gamma_n) = 1$.

LEMMA 7. *For each point* $h \in H$, $r^k(J-H, h) = 0$ *for* $1 \leqq k \leqq n-1$ *and* $r^0(J-H, h) = 1$.

PROOF. It is sufficient to consider the case $h = e$. Given any neighborhood V of e, choose a neighborhood V_1 such that $\mathcal{C}(\mathcal{C}(V_1 \cdot V_1) \cdot B(V))$ does not meet $\bar{V}_1$, where $B(V)$ is the boundary of V. Next choose a neighborhood V_2 such that if $\gamma^0 \subset V_2$, then $\gamma^0 \sim 0$ in V_1. Let V_3 be such that $\bar{V}_3 \cdot H$ does not contain all of V_2, and, finally, let W be such that if $\gamma^k \subset W$, then $\gamma^k \sim 0$ in V_3. We assert that for $k \geqq 1$, any γ^k in $W - H$ bounds in $V - H$.

For let A and B be the two components of $J-H$ and let $\gamma^k = \gamma^k_a + \gamma^k_b$, where γ^k_a is the part of γ^k in A. Since $k \geqq 1$, γ^k_a is a cycle and it is sufficient to show that $\gamma^k_a \sim 0$ in $V \cap A$. If it does not, let O be an open set in $W \cap A$ such that γ^k is in O and O does not meet H. Then, by Lemma 5.2 of [2], there is a compact cocycle γ_{n-k} in O such that $\Gamma^n \cdot \gamma_{n-k} \sim \gamma^k$ in O, γ_{n-k} not ~ 0 in $V \cap A$, and $\gamma_{n-k} \sim 0$ in V_3. Let Γ^{n-k} be an infinite cycle of $V \cap A$ such that $\mathrm{KI}(\Gamma^{n-k} \cdot \gamma_{n-k}) = 1$.

In order to apply an argument similar to that of the preceding lemma, we choose a point of B in the following fashion. Let c be a point of B in V_2 and not in $\mathcal{C}(V_3 \cdot H)$. By the choice of V_2, there is a continuum K in V_1 which contains both c and e. Let M be a neighborhood of e such that $M \cdot H$ does not meet $\bar{O}$. Choose N by Lemma 1 and so that $N \cdot K$ is in V_1. $N \cdot K - H$ is an open subset of J and hence is locally connected. Consequently, each component of $N \cdot K - H$ is also open. Let C be that component which contains c. Since C is open and $N \cdot K$ is connected, some point h in H must be a limit point of C. h is in $\overline{V}_1$, since $N\cdot K$ is in V_1, and therefore $D=C\cdot h^{-1}$, containing $d=ch^{-1}$, is an open connected subset of B and e is a limit point of D. Also, since c is not in $V_2\cdot H$, neither is d.

From $N\cdot D$ and N itself a simple chain of regions running from e to d can be extracted, each element of the chain being a translation of N.

Returning now to Γ^{n-k}, let Γ^{n-k-1} be the part of $F\Gamma^{n-k}$ on H, so that Γ^{n-k-1} is a cycle of H mod X, where $X=\overline{V}\cap\mathcal{C}(H-V)$. Using the simple chain above, we have $\Gamma^{n-k-1}\sim d\cdot\Gamma^{n-k-1}$ in $M\cdot D$. Let the chains of this homology be $\{C_\zeta^{n-k}\}$. Then $\Gamma_\zeta^{n-k}-C_\zeta^{n-k}$, for each ζ, is, by the choice of V_1, an infinite cycle of V_2. Also, by the choice of M, no C_ζ^{n-k} meets $\overline{O}$, so $\mathrm{KI}((\Gamma_\zeta^{n-k}-C_\zeta^{n-k})\cdot\gamma_{n-k}^\zeta)=\mathrm{KI}(\Gamma_\zeta^{n-k}\cdot\gamma_{n-k}^\zeta)=1$ for each ζ.

Now we can proceed to the same contradiction we reached in the previous lemma, since $\gamma_{n-k}\sim 0$ in V_2 so its Kronecker index with any infinite cycle of V_2 is zero. This disposes of the case $k\geqq 1$.

For $k=0$, let $\bar{\gamma}^0$ be based on a pair of points, one in $W\cap A$ and the other in $W\cap B$. The proof used above applies to show that any γ^0 in $W-H$ is homologous in $V-H$ to a multiple of $\bar{\gamma}^0$.

LEMMA 8. *For each point h of H, $r_k(h)=0$ for $k<n-1$ and $r_{n-1}(h)=1$.*

This is an immediate consequence of Theorem 6.2 of [2] and Lemma 7.

LEMMA 9. *H is lc^{n-1}.*

PROOF. Given a neighborhood V of e, choose V_1 in V such that any γ^{k+1} on $\overline{V}_1$ bounds in V. Choose $W\subset V_1$ by Lemma 7 so that any γ^k in $A\cap W$ bounds in $A\cap V_1$ and similarly for B. We assert that any γ^k in $W\cap H$ bounds on $\overline{V}\cap H$.

To show this it is enough to show that for any neighborhood O of e, $\gamma^k\sim 0$ in $(O\cdot H)\cap\overline{V}$. In turn, to prove this it is sufficient to show that given any such γ^k and O, and given any covering U_1, then there is a refinement U_2 such that $\pi_2^1\gamma_2^k\sim 0$ in $(O\cdot H)\cap\overline{V}$.

By Lemma 1 we can choose a point $a\in A\cap O$ such that $\gamma^k\sim a\cdot\gamma^k$ in $O\cdot(W\cap H)$ and we can choose a similar point b in $B\cap O$. By the choice of W, $a\cdot\gamma^k\sim 0$ in $A\cap V_1$, and similarly for $b\cdot\gamma^k$. Thus, we have families of chains $\{C_{a,\zeta}^{k+1}\}$ and $\{C_{b,\zeta}^{k+1}\}$ in $O\cdot(W\cap H)$, $\{D_{a,\zeta}^{k+1}\}$ in $A\cap V_1$ and $\{D_{b,\zeta}^{k+1}\}$ in $B\cap V_1$ such that

$$FC_{a,\zeta}^{k+1}=a\cdot\gamma_\zeta^k-\gamma_\zeta^k,\qquad FC_{b,\zeta}^{k+1}=b\cdot\gamma_\zeta^k-\gamma_\zeta^k,$$
$$FD_{a,\zeta}^{k+1}=a\cdot\gamma_\zeta^k,\qquad FD_{b,\zeta}^{k+1}=b\cdot\gamma_\zeta^k.$$

Hence, for each ζ, $D_{a,\zeta}^{k+1}-C_{a,\zeta}^{k+1}+C_{b,\zeta}^{k+1}-D_{b,\zeta}^{k+1}$ is a cycle δ_ζ^{k+1} on U_ζ in

V_1. There is a refinement U_2 of U_1 such that $\pi_2^1\delta_2^{k+1}$ is the coordinate of a Čech cycle, δ^{k+1} on $\overline{V}_1$. By the choice of V_1, $\delta^{k+1}\sim 0$ in V, so there is a chain E^{k+2} on U_1 such that

$$FE^{k+2} = \overset{1}{\pi}_2\delta_2^{k+2}.$$

Let $E^{k+2} = E_a^{k+2} + E_b^{k+2}$, where E_b^{k+2} is the part of E^{k+2} on $\overline{B}$. Now

$$FE_a^{k+2} - \overset{1}{\pi}_2 D_{a,2}^{k+1} = -C_{a,2}^{k+1} + C_{b,2}^{k+1} - D_{b,2}^{k+1} - FE_b^{k+2}.$$

The chain on the right-hand side is in $O\cdot B$ while that on the left is on $\overline{A}$. Hence, since $\overline{A}\cap\overline{B} = H$, $E^{k+1} = FE_a^{k+2} - \pi_2^1 D_{a,2}^{k+1}$ is in $O\cdot H$ and, of course, in V. But

$$F(-E^{k+1}) - F(\overset{1}{\pi}_2 D_{a,2}^{k+1}) = \overset{1}{\pi}_2 a\cdot\gamma_a^k.$$

Hence, $\pi_2^1 a\ \gamma_2^k\sim 0$ in $(O\cdot H)\cap V$. But $a\cdot\gamma_2^k\sim\gamma_2^k$ in $O\cdot(W\cap H)$, so $\pi_2^1\gamma_2^k\sim 0$ in $(O\cdot H)\cap V$.

At this point, we have shown, by Lemmas 8 and 9, that H has the local properties of a generalized manifold. To complete the proof it only remains to show that H is orientable, that is, that it carries an $(n-1)$-cycle which is not carried by any proper closed subset of H.

By Lemma 8, there are neighborhoods O_1 and O_2 of e such that there is an $(n-1)$-cycle mod $H-O_1$ which does not bound mod $H-O_2$. By group translation, every point of H has associated with it such a non-bounding relative $(n-1)$-cycle. Now an argument due to Smith [5] shows that we can carry through in the present situation the proof of Theorem 7.1 of [1] to obtain the desired $(n-1)$-cycle.

In conclusion, we point out that by restricting G, we can lighten the hypothesis on H.

THEOREM. *Let G be a locally compact separable metric topological group which is also an orientable n-dimensional generalized manifold. Let H be a closed connected $(n-1)$-dimensional subgroup. Then H is an orientable generalized manifold if any one of the following conditions is satisfied:*

(1) *H separates some open set of G.*

(2) *For some open set O of H, there is a nonbounding $(n-1)$-cycle of H mod $H-O$.*

(3) *G is locally euclidean.*

The Pontrjagin duality theorem for case (3) and Theorem 6.5 of [2] for case (2) show that both (3) and (2) imply (1). Now the proof of Lemma 1 of [4] shows that (1) yields a neighborhood of H which is separated by H, that is, our Lemma 4. Since this is the only place in our proof where the original hypothesis on H is used, the rest of the proof can remain unchanged.

In case (3), if dim $G=3$, we have Montgomery's theorem, for any 2-dimensional generalized manifold is locally euclidean [8].

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PARACOMPACTNESS AND PRODUCT SPACES

A. H. STONE

A topological space is called *paracompact* (see [2])[1] if (i) it is a Hausdorff space (satisfying the T_2 axiom of [1]), and (ii) every open covering of it can be refined by one which is "locally finite" (= neighbourhood-finite; that is, every point of the space has a neighbourhood meeting only a finite number of sets of the refining covering). J. Dieudonné has proved [2, Theorem 4] that every *separable* metric (= metrisable) space is paracompact, and has conjectured that this remains true without separability. We shall show that this is indeed the case. In fact, more is true; paracompactness is identical with the property of "*full normality*" introduced by J. W. Tukey [5, p. 53]. After proving this (Theorems 1 and 2 below) we apply Theorem 1 to obtain a necessary and sufficient condition for the topological product of uncountably many metric spaces to be normal (Theorem 4).

For any open covering $\mathcal{W} = \{W_\alpha\}$ of a topological space, the *star* $(x, \mathcal{W})$ of a point x is defined to be the union of all the sets W_α which contain x. The space is *fully normal* if every open covering $\mathcal{U}$ of it has a "*Δ-refinement*" $\mathcal{W}$—that is, an open covering for which the stars $(x, \mathcal{W})$ form a covering which refines $\mathcal{U}$.

THEOREM 1. *Every fully normal T_1 space is paracompact.*

Let S be such a space, and let $\mathcal{U} = \{U_\alpha\}$ be a given open covering of S. (We must construct a locally finite refinement of $\mathcal{U}$. Note that S is normal [5, p. 49] and thus satisfies the T_2 axiom.)

There exists an open covering $\mathcal{U}^1 = \{U^1\}$ which Δ-refines $\mathcal{U}$, and by induction we obtain open coverings $\mathcal{U}^n = \{U^n\}$ of S such that $\mathcal{U}^{n+1}$ Δ-refines $\mathcal{U}^n$ ($n = 1, 2, \cdots$, to ∞). For brevity we write, for any $X \subset S$,

$$\begin{aligned} (X, n) &= \text{star of } X \text{ in } \mathcal{U}^n \\ &= \text{union of all sets } U^n \text{ meeting } X \end{aligned} \tag{1}$$

(roughly corresponding to the "$1/2^n$-neighbourhood of X" in a metric space), and

$$(X, -n) = S - (S - X, n). \tag{2}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Thus, since the set (X, n) is evidently open, $(X, -n)$ is closed. Further, it is easily seen that

$$(3) \qquad (X, -n) = \{x \mid (x, n) \subset X\},$$

where (x, n), in conformity with (1), denotes the star of x in $\mathcal{U}^n$; and it readily follows that

$$(4) \qquad ((X, -n), n) \subset X.$$

From the definition of Δ-refinement we have

$$(5) \qquad ((X, n+1), n+1) \subset (X, n).$$

The trivial relations $X \subset Y \rightarrow (X, n) \subset (Y, n)$, $m \geqq n \rightarrow (X, m) \subset (X, n)$, $\overline{X} \subset (X, n)$, and $y \in (x, n) \leftrightarrow x \in (y, n)$ will also be useful.

For convenience, suppose the sets U_α of $\mathcal{U}$ are well-ordered.

Now define, for each α,

$$(6) \qquad V_\alpha^1 = (U_\alpha, -1),\ V_\alpha^2 = (V_\alpha^1, 2), \text{ and } V_\alpha^n = (V_\alpha^{n-1}, n) \qquad (n \geqq 2).$$

Thus $V_\alpha^1 \subset V_\alpha^2 \subset \cdots$, and V_α^n is open if $n \geqq 2$; hence, writing $\bigcup_n V_\alpha^n = V_\alpha$, we have that V_α is open. An easy induction (using (4) and (5)) shows that $(V_\alpha^n, n) \subset U_\alpha$; hence

$$(7) \qquad V_\alpha \subset U_\alpha.$$

Further,

$$(8) \qquad \bigcup V_\alpha = S,$$

since, given $x \in S$, we have $(x, 1) \in$ some U_α (for $\mathcal{U}^1$ Δ-refines $\mathcal{U}$), so that, by (3), $x \in V_\alpha^1 \subset V_\alpha$.

We also have

(9) Given $x \in V_\alpha$, there exists $n > 0$ such that $(x, n) \subset V_\alpha$.

For there exists $n \geqq 2$ such that $x \in V_\alpha^{n-1}$, and then $(x, n) \subset V_\alpha^n \subset V_\alpha$.

Next we define, for each $n > 0$, a transfinite sequence of closed sets $H_{n\alpha}$, by setting

$$(10) \qquad H_{n1} = (V_1, -n), \qquad H_{n\alpha} = \left(V_\alpha - \bigcup_{\beta<\alpha} H_{n\beta}, -n\right).$$

Then we have:

(11) If $\alpha \neq \gamma$, no $U^n \in \mathcal{U}^n$ can meet both $H_{n\alpha}$ and $H_{n\gamma}$.

For we can suppose $\gamma < \alpha$. Then if U^n meets $H_{n\alpha}$, let $x \in U^n \cap H_{n\alpha}$; from (3) and (10), $U^n \subset V_\alpha - \bigcup_{\beta<\alpha} H_{n\beta}$, and so is disjoint from $H_{n\gamma}$.

$$\bigcup_{n,\alpha} H_{n\alpha} = S. \tag{12}$$

For, given $x \in S$, (8) shows that there will be a *first* ordinal α for which $x \in V_\alpha$; and from (9) there exists $n > 0$ such that $(x, n) \subset V_\alpha$. We assert $x \in H_{n\alpha}$. For suppose not. Then, from (10) and (3), (x, n) contains a point y not in $V_\alpha - \bigcup_{\beta<\alpha} H_{n\beta}$; and it follows that $y \in H_{n\beta}$ for some $\beta < \alpha$. But then $x \in (H_{n\beta}, n) \subset ((V_\beta, -n), n) \subset V_\beta$ (from (4)); and this contradicts the definition of α.

Write

$$E_{n\alpha} = (H_{n\alpha}, n + 3), \qquad G_{n\alpha} = (H_{n\alpha}, n + 2). \tag{13}$$

Thus $H_{n\alpha} \subset E_{n\alpha} \subset \overline{E_{n\alpha}} \subset G_{n\alpha}$, and, as is easily seen from (11),

(14) If $\gamma \neq \alpha$, no $U^{n+2} \in \mathcal{U}^{n+2}$ can meet both $G_{n\alpha}$ and $G_{n\gamma}$.

Write $F_n = \bigcup_\alpha \overline{E_{n\alpha}}$. Then F_n *is closed.* For suppose $x \in \overline{F}_n$. Then every open neighbourhood $N(x)$ of x meets some $\overline{E_{n\alpha}}$ and so meets some $E_{n\alpha}$; but if $N(x)$ is contained in the neighbourhood $(x, n+2)$ of x, $N(x)$ can meet at most *one* set $E_{n\alpha}$ (from (14)), so that $x \in \overline{E_{n\alpha}} \subset F_n$.

Finally we define

$$W_{1\alpha} = G_{1\alpha}, \qquad W_{n\alpha} = G_{n\alpha} - (F_1 \cup F_2 \cup \cdots \cup F_{n-1}) \qquad (n > 1);$$

thus the sets $W_{n\alpha}$ are open. We shall show that they form the desired refinement.

In the first place, $\bigcup_{n,\alpha} W_{n\alpha} = S$. For, given $x \in S$, we have $x \in$ some $H_{n\alpha}$ (from (12)) $\subset \overline{E_{n\alpha}}$; let m be the smallest integer for which there exists $\overline{E_{m\beta}} \ni x$. Then $x \in G_{m\beta}$, and $x \notin F_1 \cup \cdots \cup F_{m-1}$, so that $x \in W_{m\beta}$.

Next, $W_{n\alpha} \subset G_{n\alpha} \subset (H_{n\alpha}, n) \subset ((V_\alpha, -n), n) \subset V_\alpha \subset U_\alpha$ (using (4) and (7)). Thus the sets $W_{n\alpha}$ form an open covering $\mathcal{W}$ of S which refines $\mathcal{U}$. All that remains to be proved is that $\mathcal{W}$ is "locally finite." Given $x \in S$, we have as before that $x \in$ some $H_{n\alpha}$, so $(x, n+3) \subset E_{n\alpha} \subset F_n$, and so is certainly disjoint from $W_{k\beta}$ if $k > n$. Further, for a given $k \leqq n$, we have $(x, n+3) \subset U^{n+2} \subset U^{k+2}$, so (13) shows that $(x, n+3)$ can meet $W_{k\beta}$ for at most *one* value of β. Thus the neighbourhood $(x, n+3)$ of x meets at most n of the sets $W_{k\beta}$; and the proof is complete.

Remark. The locally finite refinement $\mathcal{W}$ thus constructed has the additional property that it consists of a countable number of families of sets (formed by the sets $W_{n\alpha}$, n fixed), the sets of each family having pairwise disjoint closures.

Corollary 1. *Every metric space is paracompact.*

For a metric space is fully normal [5, p. 53].

Corollary 2. *The topological product of a metric space and a com-*

pact ($=$*bicompact*) *Hausdorff space is paracompact, and therefore normal.*[3]

This follows from Theorems 5 and 1 of [2].

THEOREM 2. *Every paracompact space is fully normal* (*and* T_1).

Let S be a paracompact space, and let $\mathcal{V}=\{U_\alpha\}$ be a given *locally finite* open covering of S. It will evidently suffice to prove that $\mathcal{V}$ has a Δ-refinement.

Open sets X_α exist, for each α, such that $\overline{X}_\alpha\subset U_\alpha$ and $\bigcup X_\alpha=S$. (This follows by an easy transfinite induction argument from the fact that S is normal; cf. [2, Theorems 1 and 6].) By hypothesis, each $x\in S$ has an open neighbourhood $V(x)$ meeting U_α only for a finite set of α's, say for $\alpha\in A(x)$. Let $B(x)$ be the set of those α's$\in A(x)$ for which $x\in U_\alpha$, and let $C(x)$ be the set of α's $\in A(x)$ for which $x\notin\overline{X_\alpha}$; clearly $B(x)\cup C(x)=A(x)$. Define

$$W(x)=V(x)\cap\bigcap\{U_\alpha\mid\alpha\in B(x)\}\cap\bigcap\{(S-\overline{X}_\alpha)\mid\alpha\in C(x)\}.$$

Evidently $W(x)$ is an open set containg x; hence the sets $\{W(x)\mid x\in S\}$ form an open covering $\mathcal{W}$ of S. To verify that $\mathcal{W}$ is a Δ-refinement of $\mathcal{V}$, let $y\in S$ be given. There exists a set $X_\beta\ni y$; we shall show that the star $(y,\mathcal{W})\subset U_\beta$—that is, that if $y\in W(x)$ then $W(x)\subset U_\beta$. For if $y\in W(x)$ then $W(x)$ meets $\overline{X}_\beta$ and so clearly $\beta\in A(x)$ and $\beta\notin C(x)$. Thus $\beta\in B(x)$, which implies $W(x)\subset U_\beta$, by construction.

Now let N denote the space of positive integers—a countable discrete set—and consider the *product* $T=\prod N_\lambda$ ($\lambda\in\Lambda$) of uncountably many copies of N. More precisely, the points of T are the mappings $x=\{\xi_\lambda\}$ of the uncountable set Λ in N (each $\lambda\in\Lambda$ being mapped on the integer $\xi_\lambda\in N$), and a typical basic neighbourhood U of x in T is obtained by choosing a *finite* set $\mathfrak{R}(U)\subset\Lambda$ and defining U to consist of all points $y=\{\eta_\lambda\}$ such that $\eta_\lambda=\xi_\lambda$ for all $\lambda\in\mathfrak{R}(U)$. $\mathfrak{R}(U)$ will be called the "set of coordinates *restricted* in U."

THEOREM 3. *The space T is not normal.*

For each positive integer k, let A^k be the set of all points $x=\{\xi_\lambda\}\in T$ satisfying: for each positive integer n other than k, there is at most one λ for which $\xi_\lambda=n$.

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[3] It can be shown that the topological product of a metric space and a normal countably compact space is normal, though not necessarily paracompact. (A space is "countably compact" if every infinite subset has a limit point in the space; cf. [5, p. 42]. For metric spaces this is equivalent to compactness.)

It is easy to see that the sets A^k are closed and pairwise disjoint. Hence, if T were normal, there would exist disjoint open sets U, V such that $U\supset A^1$, $V\supset A^2$. We shall show that this leads to a contradiction.

We shall define inductively sequences of points $x_n\in A^1$, of integers $0<m(1)<m(2)<\cdots$, and of elements $\lambda_n\in\Lambda$, as follows. Define x_1 to be the point $\{\xi_\lambda\}$ for which $\xi_\lambda=1$ (all $\lambda\in\Lambda$). Evidently $x_1\in A^1\subset U$, so x has a basic neighbourhood $U_1\subset U$. Let $\mathfrak{R}(U_1)$ consist of the $m(1)$ elements λ_k $(1\leqq k\leqq m(1))$. When x_n and $\lambda_1, \lambda_2, \cdots, \lambda_{m(n)}$ have been defined, in such a way that $x_n\in A^1$ and $\lambda_1, \cdots, \lambda_{m(n)}$ are the coordinates restricted in a basic neighbourhood $U_n\subset U$ of x_n, we define x_{n+1} by: $\xi_\lambda=k$ if $\lambda=\lambda_k$ $(1\leqq k\leqq m(n))$, and $\xi_\lambda=1$ otherwise. Clearly $x_{n+1}\in A^1$, so that x_{n+1} has a basic neighbourhood $U_{n+1}\subset U$; and we can always suppose that $\mathfrak{R}(U_{n+1})$ contains $\mathfrak{R}(U_n)$ as a proper subset. Let $\mathfrak{R}(U_{n+1})$ have $m(n+1)$ elements, and enumerate the elements of $\mathfrak{R}(U_{n+1})-\mathfrak{R}(U_n)$ as $\lambda_{m(n)+1}, \cdots, \lambda_{m(n+1)}$. The induction is now complete.

Now define a point $y=\{\eta_\lambda\}$ by: $\eta_\lambda=k$ if $\lambda=\lambda_k$ $(k=1, 2, \cdots,$ to $\infty)$ and $\eta_\lambda=2$ otherwise. Clearly $y\in A^2\subset V$, so y has a basic neighbourhood $V_0\subset V$. Since $\mathfrak{R}(V_0)$ is finite, there exists an n such that $\lambda_k\in\Lambda-\mathfrak{R}(V_0)$ whenever $k>m(n)$. Finally, define $z=\{\zeta_\lambda\}$ by:

$$\begin{aligned}
\zeta_\lambda &= k \quad \text{if} \quad \lambda=\lambda_k \text{ with } k\leqq m(n),\\
\zeta_\lambda &= 1 \quad \text{if} \quad \lambda=\lambda_k \text{ with } m(n)<k\leqq m(n+1), \quad \text{and}\\
\zeta_\lambda &= 2 \quad \text{otherwise.}
\end{aligned}$$

We evidently have $z\in U_{n+1}\cap V_0\subset U\cap V$, giving the desired contradiction.

COROLLARY. *If a product of nonempty T_1 spaces is normal, all but at most a countable number of the factor spaces must be countably compact.*[2]

For otherwise their product would contain a closed subset homeomorphic with T; and a closed subset of a normal space is normal.

THEOREM 4. *The following statements about a product of nonempty metric spaces are equivalent.*

(i) *The product is normal.*

(ii) *The product is fully normal (or paracompact).*

(iii) *At most $\aleph_0$ of the factor spaces are noncompact.*

In fact, (ii)→(i) [2, Theorem 1], (i)→(iii) (Theorem 3, Corollary), and (iii)→(ii) from Theorem 1, Corollary 2, since the compact factors have a compact product[3] and the product of the remaining factors is metrisable.[4]

REMARK. In Theorem 4, the hypothesis that the factor spaces be metric cannot be much weakened. This is shown by an example of R. H. Sorgenfrey (see [4]), in which the product of a paracompact (and thus fully normal) space with itself is not even normal.

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[3] A theorem of Tychonoff; see, for example, [5, p. 75] for a simple proof.

[4] See, for example, [3, p. 88].

TRANSITIVITY AND EQUICONTINUITY[1]

W. H. GOTTSCHALK

Let X be a metric space with metric ρ and let G be a group of homeomorphisms on X. If $x\in X$ and $g\in G$, then xg denotes the image of the point x under the transformation g. If $x\in X$ and $F\subset G$, then xF denotes $\bigcup_{g\in F} xg$. G is said to be *algebraically transitive* provided that $xG=X$ for some $x\in X$ (and therefore for every $x\in X$). G is said to be *topologically transitive* provided that $(xG)^*=X$ for some $x\in X$, where the star denotes the closure operator. G is said to be *equicontinuous* provided that to each $\epsilon>0$ there corresponds $\delta>0$ such that $x, y\in X$ with $\rho(x, y)<\delta$ implies $\rho(xg, yg)<\epsilon$ $(g\in G)$.

With respect to the following lemma compare [4].[2]

LEMMA. *If X is a complete separable metric space and also a multiplicative group, if the center of X is dense in X and if the function xy*

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[2] Numbers in brackets refer to the bibliography at the end of the paper.

(x, $y\in X$) is continuous in x for each fixed y, then the function xy is continuous simultaneously in x and y.

PROOF. We first show that xy is continuous in y for each fixed x. Let $x_0\in X$ and let $\{x_n\}$ be a sequence of points in the center of X such that $x_n\to x_0$. Now $y\in X$ implies $yx_n=x_ny\to x_0y$. Hence the sequence $\{yx_n\}$ of continuous functions of y converges pointwise to the function x_0y of y. Thus x_0y is of Baire class 1 and has at least one point of continuity. (See [3, pp. 184, 189].) By right translation it follows that x_0y is continuous in y for all values of y.

Since xy is continuous in x and y separately, xy is of Baire class 1 in (x, y). (See [3, p. 180].) Hence xy has at least one point (x, y) of continuity and by left and right translations we see that xy is continuous in (x, y) for all values of (x, y).

THEOREM. *If X is a compact metric space, if G is a topologically transitive abelian group of homeomorphisms on X and if H is the group of all homeomorphisms on X which commute with every element of G, then the following statements are pairwise equivalent:* (1) *H is algebraically transitive*; (2) *H is equicontinuous*; (3) *G is equicontinuous.*

PROOF. Assume H is algebraically transitive. There exists $e\in X$ such that $(eG)^*=X$. If $x\in X$, $h\in H$ and $eh=e$, then there exists a sequence $\{g_n\}$ in G such that $eg_n\to x$ whence $eg_n=ehg_n=eg_nh\to xh$ and $xh=x$. We conclude that for $x\in X$ there exists exactly one element of H, denoted h_x, such that $eh_x=x$. Define a product in X as follows: If x, $y\in X$, then $xy=eh_xh_y$. It is readily verified that X is a group such that every element of eG commutes with every element of X, eG is dense in X and the function xy is continuous in x for each fixed y. By the lemma, xy is continuous on $X\times X$. Since $X\times X$ is compact, xy is uniformly continuous on $X\times X$. It follows that to each $\epsilon>0$ there corresponds $\delta>0$ such that x, $y\in X$ with $\rho(x, y)<\delta$ implies $\rho(xz, yz)<\epsilon$ ($z\in X$). This statement is just the statement of the equicontinuity of H. Thus we have shown that (1) implies (2).

Obviously (2) implies (3).

Assume G is equicontinuous. Let C denote the space of all continuous transformations of X into X supplied with the usual metric. Since C is complete and G is totally bounded (see [1]), G^* in C is compact. Choose $e\in X$ so that $(eG)^*=X$. Let $x\in X$ and let $\{g_n\}$ be a sequence in G for which $eg_n\to x$. Select a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ so that $g_{n_i}\to h\in G^*$ and $g_{n_i}^{-1}\to h_0\in G^*$. Now $y\in X$ implies $y=(yg_{n_i})g_{n_i}^{-1}\to(yh)h_0$ and $y=yhh_0$. Similarly $y=yh_0h$ ($y\in X$). Hence h is a homeomorphism of X onto X. Since also $hg=gh$ ($g\in G$), we conclude that

$h \in H$. Furthermore $eh = x$. Thus $eH = X$ and H is algebraically transitive. The proof is completed.

The following corollary solves a problem proposed by Hedlund [2, bottom p. 617].

COROLLARY. *Let a flow be defined on a compact metric space X so that X is a minimal orbit-closure. Then the flow is equicontinuous if and only if for every pair of points of X there exists an orbit-preserving homeomorphism on X transforming one of these points into the other.*

The above corollary permits a rephrasing of a conjecture of G. D. Birkhoff [5, problems 2 and 3], namely: If a continuous flow on an n-dimensional manifold is pointwise almost periodic, then the flow is almost periodic (equicontinuous) on each orbit-closure. (See [1] for terms used.)

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CORRECTION TO MY PAPER "NOTE ON AFFINELY CONNECTED MANIFOLDS"

SHIING-SHEN CHERN

In my recent paper [1][1] *Note on affinely connected manifolds* I gave a proof of a theorem on affinely connected manifolds. As was pointed out by H. Whitney, the fact that I made use of in the proof, that the space of all real matrices (a_i^j), $|a_i^j|>0$, is simply connected, is erroneous. But the theorem I intended to prove is true. I present in the following lines a revised proof, in which are clarified at the same time certain ambiguities of that note.

We begin by considering the space $M(n)$ of all real matrices (a_i^j), $i, j=1, \cdots, n$ with $\Delta=|a_i^j|>0$. Let $R(n)$ denote the group space of the proper orthogonal group in n variables. There is a natural way to imbed $R(n)$ in $M(n)$ and it is well known that $R(n)$ is a deformation retract of $M(n)$ [2]. In particular, it follows that $R(n)$ and $M(n)$ have the same homotopy type and hence isomorphic homotopy groups. Thus the fundamental group of $M(n)$ is free cyclic if $n=2$ and is cyclic of order two if $n\geqq 3$.

We denote by $\psi: R(n)\rightarrow M(n)$ the identity mapping and by $f: M(n)\rightarrow R(n)$ the deformation such that under f every point of $R(n)$ remains fixed. Let ψ and f denote at the same time the induced homomorphisms of the (singular) chains and ψ^* and f^* the corresponding dual homomorphisms of the cochains. Since f is a deformation, we have, for every one-dimensional cycle Z of $M(n)$, $Z\sim\psi f(Z)$. It follows that

$$\int_Z \frac{d\Delta}{\Delta}=\int_{\psi f(Z)} \frac{d\Delta}{\Delta}=\int_{f(Z)} \psi^*\left(\frac{d\Delta}{\Delta}\right)=0,$$

since $\Delta=1$ in $R(n)$. In other words, in $M(n)$ the integral of $d\Delta/\Delta$ over any one-dimensional cycle is zero.

It is possible to express the differential form $d\Delta/\Delta$ in terms of a_i^j. In fact, let b_j^k be defined by $a_i^j b_j^k=b_i^j a_j^k=\delta_i^k$. Then it is easy to verify that

$$d\Delta/\Delta=da_i^j b_j^i.$$

These remarks on the group manifold being made, let us return to the affinely connected manifold M. Let $\mathfrak{F}$ be the vector bundle of all ordered sets of n linearly independent contravariant vectors through

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[1] Numbers in brackets refer to the references cited at the end of the paper.

a point of M, and let π be the projection of $\mathfrak{F}$ onto M, mapping such a set of vectors into the common origin. Then each fibre of $\mathfrak{F}$ is homeomorphic to $M(n)$. Denoting by π^* the dual homomorphism of π, it follows that a cocycle Φ of M has an inverse image $\pi^*\Phi$ in $\mathfrak{F}$. This is in particular true of our differential form $P = R_{kl}dx^k dx^l$.

As local coordinates in $\mathfrak{F}$ we take the local coordinates x^i of M and the components a_j^i of the n contravariant vectors, which we call $\mathfrak{a}_j$. The components of the absolute differentials $D\mathfrak{a}_j$, referred to the vectors $\mathfrak{a}_k$ themselves, are intrinsic differential forms in $\mathfrak{F}$, independent of the choice of the local coordinate system. If b_i^j are defined as above, they are found to be

$$\omega_j^k = (da_j^i + a_j^l \Gamma_{lm}^i dx^m) b_i^k.$$

Contracting the two indices, we get

$$\omega_i^i = d\Delta/\Delta + \Gamma_{ik}^i dx^k.$$

From the definition of the affine curvature tensor we easily derive the relation

$$d(\Gamma_{ik}^i dx^k) = P.$$

It follows that, in the vector bundle $\mathfrak{F}$,

$$d\omega_i^i = d(d\Delta/\Delta + \Gamma_{ik}^i dx^k) = \pi^* P.$$

Suppose we now integrate the differential form P over a two-dimensional simplicial cycle, on whose one-dimensional skeleton a continuous field of n independent contravariant vectors is defined. The field can be extended continuously over the cycle, with one possible singularity at an interior point of each simplex. Choosing an arbitrarily small neighborhood about each singularity, the integral of P over the cycle will differ as little as we please from a finite sum of integrals $\int d\Delta/\Delta$ over one-dimensional cycles in the space of n independent contravariant vectors, that is, in the space $M(n)$. But such an integral of $d\Delta/\Delta$ is zero, according to our remarks on $M(n)$. It follows that the integral of P over the cycle is zero, as was to be proved.

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RECURSION AND DOUBLE RECURSION

RAPHAEL M. ROBINSON

1. Introduction. We shall apply the results of PRF[1] to construct by double recursion two functions which are not themselves primitive recursive, but which are related in interesting ways to the class of primitive recursive functions. In a sense, this note is a revised version of a paper by Rózsa Péter,[2] much simplified by the use of PRF.

Let Sx denote the successor of x. We shall say that a function $G_n x$ of two variables n and x is defined by a double recursion from certain given functions, if

(1) $G_0 x$ is a given function of x.

(2) $G_{Sn}0$ is obtained by substitution from $G_n z$ (considered as a function of z) and from given functions.

(3) $G_{Sn}Sx$ is obtained by substitution from the number $G_{Sn}x$, from $G_n z$ (considered as a function of z), and from given functions.

It is clear that if the given functions are primitive recursive, then $G_n x$ is a primitive recursive function of x for each fixed n. However, as we shall see, $G_n x$ need not be a primitive recursive function of n and x.

In §2, we shall show that the double recursion

$$G_0 x = Sx, \qquad G_{Sn}0 = G_n 1, \qquad G_{Sn}Sx = G_n G_{Sn} x$$

defines a function $G_n x$ which *majorises* all primitive recursive functions of one variable in the following sense: *If Fx is a primitive recursive function of x, then there exists a number n such that*

$$Fx < G_n x$$

for all x. It is also shown that $G_n x$ is an increasing function of n, so that

$$Fx < G_x x$$

for all sufficiently large x. It follows that $G_x x$ is not a primitive recursive function of x, and hence that $G_n x$ is not a primitive recursive

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[1] R. M. Robinson, *Primitive recursive functions*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 925–942.

[2] R. Péter, *Konstruktion nichtrekursiver Funktionen*, Math. Ann. vol. 111 (1935) pp. 42–60.

function of n and x. The above example is essentially the same as that given by Péter, which was a simplification of one previously given by Ackermann.[3]

In §3, we shall determine two primitive recursive functions Ax and $B(x, y)$, such that the double recursion

$$G_0x = Ax, \qquad G_{Sn}0 = 0, \qquad G_{Sn}Sx = G_nB(x, G_{Sn}x)$$

defines a function G_nx which *generates* all primitive recursive functions of one variable in the following sense: A primitive recursive function $H(n, x)$ can be found, such that *if Fx is a primitive recursive function of x, then there exists a number n such that*

$$Fx = G_nH(n, x).$$

It follows that $G_xH(x, x)$ is not a primitive recursive function of x, and hence that G_nx is not a primitive recursive function of n and x. The above double recursion is of a much simpler form than the one given by Péter for a similar purpose. (In a later paper,[4] she showed how all double recursions can be reduced to a standard form, which is however still not as simple as the above.) Also, the functions Ax and $B(x, y)$ which we use are comparatively simple; they can be obtained by substitution from constant and identity functions, and

$$x + y, \quad x \dot{-} y, \quad x^2, \quad [x^{1/2}], \quad [x/2], \quad [x/3].$$

Here $x \dot{-} y = x - y$ if $x \geqq y$ and $x \dot{-} y = 0$ otherwise. The function $H(n, x)$ which we use is a certain quartic polynomial in n and x.

Both of these results may be derived from PRF, §7, Theorem 3, which states that *all primitive recursive functions of one variable can be obtained by starting with the two functions S and E, and repeatedly using any of the formulas*

$$Fx = Ax + Bx, \qquad Fx = BAx, \qquad Fx = B^x0$$

to construct a new function F from known functions A and B. A second form of the result has E replaced by Q, and $Ax+Bx$ by $|Ax-Bx|$. Here $Ex = x - [x^{1/2}]^2$ is the excess of x over a square, and $Qx = 0^{Ex}$ is the characteristic function of squares.

2. **The majorizing function.** Let the function G_nx be defined by the double recursion

$$G_0x = Sx, \qquad G_{Sn}0 = G_n1, \qquad G_{Sn}Sx = G_nG_{Sn}x.$$

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[3] W. Ackermann, *Zum Hilbertschen Aufbau der reelen Zahlen*, Math. Ann. vol. 99 (1928) pp. 118–133.

[4] R. Péter, *Über die mehrfache Rekursion*, Math. Ann. vol. 113 (1936) pp. 489–527.

It is clear that G_nx is never zero. We shall show that

$$G_nSx > G_nx,$$

so that for any n, G_nx is a strictly increasing function of x, hence $G_nx > x$, that is, $G_nx \geqq Sx$. In the first place, this is true for $n=0$. Now assume for any value of n and prove for Sn. We have indeed

$$G_{Sn}Sx = G_nG_{Sn}x > G_{Sn}x.$$

We shall show next that

$$G_{Sn}x \geqq G_nSx.$$

This may be shown for a fixed n by induction in x. For $x=0$ we have equality by definition. Now assume the inequality for some value of x and prove for Sx. By the inductive hypothesis and the inequality $G_nSx \geqq SSx$, we have

$$G_{Sn}Sx = G_nG_{Sn}x \geqq G_nG_nSx \geqq G_nSSx,$$

as was to be shown. In particular, we have

$$G_{Sn}x > G_nx,$$

so that G_nx is a strictly increasing function of n for a fixed x.

The arguments used up to this point are the same as those given by Péter, although we have modified her function slightly. We shall now show that to every primitive recursive function Fx there exists a number n such that

$$Fx < G_nx.$$

We shall use the result quoted in §1 from PRF in the second form.

We must show that the conclusion holds for Sx and Qx, and that if it holds for Ax and Bx, then it also holds for $|Ax - Bx|$, BAx, and B^x0. We have in the first place

$$Qx \leqq Sx = G_0x < G_1x.$$

Now suppose that

$$Ax < G_kx, \quad Bx < G_lx.$$

If we set $n = \max(k, l)$, then

$$Ax < G_nx, \quad Bx < G_nx.$$

Hence

$$|Ax - Bx| < G_nx, \quad B^x0 < G_n^xG_n1 = G_{Sn}x,$$

and

$$BAx < G_nG_nx < G_nG_{Bn}x = G_{Bn}Sx \leqq G_{BBn}x.$$

3. The generating function. We shall make use of pairing functions $J(u, v)$, Kx, Lx, that is, functions which satisfy

$$KJ(u, v) = u, \qquad LJ(u, v) = v.$$

Such functions establish a one-to-one correspondence between all pairs of numbers and some numbers. We shall want to use functions $J(u, v)$, Kx, Lx, which are primitive recursive, and the conditions $Kx \leqq x$, $Lx \leqq x$ will be needed. We shall also suppose, as in PRF, §4, that $J(0, 0) = 0$, and that if $LSx > 0$, then $KSx = Kx$ and $LSx = SLx$; the interpretation of these conditions is discussed there. Suitable functions are

$$J(u, v) = ((u + v)^2 + u)^2 + v, \qquad Kx = E[x^{1/2}], \qquad Lx = Ex.$$

Since for such pairing functions, $Kx = u$ and $Lx = v$ have infinitely many solutions for x when u and v are given, we can extend the correspondence from one between pairs of numbers and numbers to one between triples of numbers and numbers, by finding two primitive recursive functions $J(u, v, w)$ and Mx, with $J(u, v, 0) = J(u, v)$, and such that

$$KJ(u, v, w) = u, \qquad LJ(u, v, w) = v, \qquad MJ(u, v, w) = w.$$

Since $J(0, 0, 0) = 0$, we see that $K0 = 0$, $L0 = 0$, $M0 = 0$. Suitable functions extending the $J(u, v)$, Kx, Lx given above are

$$J(u, v, w) = ((u + v + w)^2 + u)^2 + v, \qquad Mx = [x^{1/4}] \dot{-} (Kx + Lx).$$

We now define F_nx by the formulas

$$F_0x = Ex, \qquad F_1x = Sx,$$

$$F_{3u+2}x = F_{Ku}x + F_{Lu}x, \qquad F_{3u+3}x = F_{Lu}F_{Ku}x, \qquad F_{3u+4}x = F_u^x0.$$

According to the theorem of PRF quoted in §1, if Fx is a primitive recursive function, then there exists a number n such that $Fx = F_nx$.

We shall now define a function Gx (which is not primitive recursive) such that $G(2v) = F_{Kv}Lv + Mv$. The definition of Gx is completed by supposing that for $v > 0$

$$G(2v - 1) = \begin{cases} F_{Kv}Lv & \text{if} \quad Kv = 3u + 2 \quad \text{or} \quad 3u + 3, \\ F_{Kv}PLv & \text{if} \quad Kv = 3u + 4,\ Lv > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here $Py = y \dot{-} 1$ is the predecessor of y.

The definition of $G(2v)$ was so chosen that

$$F_n x = G(2J(n, x)).$$

Thus Gx may be considered as generating all primitive recursive functions. The essential numbers $F_{Kv}Lv$ were put in alternate places and modified by adding Mv, and then such values were interpolated for $G(2v-1)$ that Gx would satisfy a functional equation

$$GSx = GB(x, Gx),$$

where $B(x, y)$ is a primitive recursive function. Indeed, we shall define $B(x, y)$ so that the function determined from suitable initial values by the formula $G_{Sn}Sx = G_nB(x, G_{Sn}x)$ will approximate to Gx in such a way that

$$F_n x = G_n(2J(n, x)).$$

Thus all primitive recursive functions will be generated by a double recursion.

The definition of $B(x, y)$ is the following: For every $v > 0$, let

$$B(2v-2, y) = \begin{cases} 2J(Ku, Lv, 0) & \text{if } Kv = 3u + 2 \text{ or } 3u + 3, \\ 2J(0, 0, y \dot{-} MPv) & \text{if } Kv = 3u + 4, Lv > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B(2v-1, y) = \begin{cases} 2v & \text{if } Kv = 0 \text{ or } 1, \\ 2J(Lu, Lv, y + Mv) & \text{if } Kv = 3u + 2, \\ 2J(Lu, y, Mv) & \text{if } Kv = 3u + 3, \\ 2J(0, 0, Mv) & \text{if } Kv = 3u + 4, Lv = 0, \\ 2J(u, y, Mv) & \text{if } Kv = 3u + 4, Lv > 0. \end{cases}$$

We proceed to verify that $GSx = GB(x, Gx)$ in all cases.

$x = 2v-2$, $Kv = 3u+2$ or $3u+3$:

$$GB(x, Gx) = F_{Kv}Lv = GSx.$$

$x = 2v-2$, $Kv = 3u+4$, $Lv > 0$:

$$GB(x, Gx) = F_0 0 + (Gx \dot{-} MPv) = F_{KPv}LPv = F_{Kv}PLv = GSx.$$

$x = 2v-2$, other cases:

$$GB(x, Gx) = G0 = 0 = GSx.$$

$x=2v-1$, $Kv=0$ or 1:

$$GB(x, Gx) = G(2v) = GSx.$$

$x=2v-1$, $Kv=3u+2$:

$$\begin{aligned} GB(x, Gx) &= F_{Lu}Lv + (Gx + Mv) = F_{Lu}Lv + F_{Ku}Lv + Mv \\ &= F_{3u+2}Lv + Mv = GSx. \end{aligned}$$

$x=2v-1$, $Kv=3u+3$:

$$GB(x, Gx) = F_{Lu}Gx + Mv = F_{Lu}F_{Ku}Lv + Mv = F_{3u+3}Lv + Mv = GSx.$$

$x=2v-1$, $Kv=3u+4$, $Lv=0$:

$$GB(x, Gx) = F_0 0 + Mv = Mv = F_{3u+4}0 + Mv = GSx.$$

$x=2v-1$, $Kv=3u+4$, $Lv>0$:

$$\begin{aligned} GB(x, Gx) &= F_uGx + Mv = F_uF_{3u+4}PLv + Mv = F_uF_u^{PLv}0 + Mv \\ &= F_u^{Lv}0 + Mv = F_{3u+4}Lv + Mv = GSx. \end{aligned}$$

Let the function Ax be defined by

$$A(2v) = \begin{cases} ELv + Mv & \text{if } Kv = 0, \\ SLv + Mv & \text{if } Kv = 1, \end{cases}$$

and $Ax=0$ otherwise. Then the double recursion $G_0x=Ax$, $G_{Sn}0=0$, $G_{Sn}Sx=G_nB(x, G_{Sn}x)$ defines a function G_nx, which we shall show approaches Gx as n increases in such a way that

$$G_n(2v) = G(2v) \qquad \text{if} \quad Kv \leqq Sn,$$

and also for $v>0$

$$G_n(2v-1) = G(2v-1) \qquad \text{if} \quad Kv \leqq Sn.$$

We shall prove these two equalities by induction in n. If $n=0$, we must have $Kv\leqq 1$, and hence

$$G_0(2v) = A(2v) = F_{Kv}Lv + Mv = G(2v),$$

and for $v>0$

$$G_0(2v-1) = 0 = G(2v-1).$$

Now assume the result for some value of n and prove for Sn. In the first place, $G_{Sn}0=0=G0$. If $v>0$ but $Kv\leqq 1$, we have

$$G_{Sn}(2v) = G_n(2v) = G(2v),$$

$$G_{Sn}(2v-1) = G_n0 = 0 = G(2v-1).$$

It remains to consider the case $2 \leqq Kv \leqq SSn$. Now if $Kv = 3u+2$, $3u+3$, or $3u+4$, we see that for any y, $B(2v-2, y)$ and $B(2v-1, y)$ are both of the form $2w$ with $Kw \leqq u < Kv$, hence $Kw \leqq Sn$. It follows that

$$G_n B(2v - 2, y) = GB(2v - 2, y), \qquad G_n B(2v - 1, y) = GB(2v - 1, y).$$

Hence

$$G_{Sn} Sx = G_n B(x, G_{Sn} x) = GB(x, G_{Sn} x) \quad \text{for } x = 2v - 2 \text{ or } 2v - 1.$$

We shall now prove by induction in v that if $2 \leqq Kv \leqq SSn$ then

$$G_{Sn}(2v - 1) = G(2v - 1), \qquad G_{Sn}(2v) = G(2v),$$

which will complete the proof. For a given value of v, we shall derive both of these, assuming that

$$G_{Sn}(2v - 2) = G(2v - 2)$$

provided that $2 \leqq KPv \leqq SSn$. We have indeed

$$\begin{aligned} G_{Sn}(2v - 1) &= GB(2v - 2, G_{Sn}(2v - 2)) \\ &= GB(2v - 2, G(2v - 2)) = G(2v - 1), \end{aligned}$$

since $B(2v-2, y)$ depends on y only if Kv has the form $3u+4$ and $Lv > 0$, in which case $KPv = Kv$, so that the inductive hypothesis may be used. Finally,

$$G_{Sn}(2v) = GB(2v - 1, G_{Sn}(2v - 1)) = GB(2v - 1, G(2v - 1)) = G(2v).$$

Let

$$H(n, x) = 2J(n, x).$$

If Fx is any primitive recursive function of x, then there exists an n such that $Fx = F_n x$. From what we have proved, we see that

$$Fx = F_n x = G(2J(n, x)) = G_n(2J(n, x)) = G_n H(n, x),$$

thus establishing the result stated in the introduction.

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ON THE RELIABILITY OF THE MEMBRANE THEORY OF SHELLS OF REVOLUTION

C. TRUESDELL[1]

1. Introduction. The reliability of the membrane theory of shells is a somewhat controversial subject.[2] In my previous study of shells of revolution [MT] I have attempted to clarify the nature of the membrane theory as an approximate theory of elastic shells by deducing its differential equations as consequences of the three-dimensional infinitesimal theory of elasticity and of certain further assumptions, by discussing the type of boundary conditions to be used in problems concerning closed shells, and by proving the existence and uniqueness of solutions of the differential equations satisfying these boundary conditions. I developed also apparatus for quickly and efficiently finding the general solutions of the differential equations of the membrane theory for any given specific shell of revolution. Using this apparatus, in the present paper I shall show that:

1. In an open shell, or in a closed shell with a flat, sphere-like apex, the stress-resultants computed from the equations of the membrane theory will not exhibit a critical response to slight perturbations in the meridian curve, provided the curvature of the meridian curve is not changed very much.
2. In a closed shell with a pointed apex, a very slight change in the meridian curve in a very small region near the apex may entail very large changes in the stress resultants at all points of the shell, as computed from the membrane theory.

Both these results presuppose that the boundary condition at the apex is the "ring limit condition" stated at the end of §2. For a discussion of other possible boundary conditions, see §7.

These results and my previous treatment of the membrane theory show that in open shells or in closed shells with a flat, sphere-like apex, the stress resultants computed from the membrane theory

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[1] I wish to thank Dr. Neményi for patient and helpful advice and discussion, and Mr. M. S. Raff and Miss Charlotte Brudno for the calculation of various examples.

[2] The inadequacy of the membrane theory in non-uniform problems for cones was noticed by Neményi, *Beitrage zur Berechnung der Schalen unter unsymmetrischer und unstetiger Belastung*, Bygningsstatiske Meddelelser (Denmark) 1936. See also the example in C. Truesdell, *The membrane theory of shells of revolution*, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 96–166, see pp. 117–118. This latter paper will be denoted henceforth by the letters MT.

with the ring limit condition at the apex may be expected to be good approximations to the correct stress resultants, provided the support of the shell is consistent with a membrane state of stress,[3] but they cast doubt upon the reliability of the membrane theory in problems concerned with pointed shells.

2. **Fundamental apparatus for the subsequent discussion.** Let the meridian curve of the shell be $r=f(z)$, where the z-axis is the axis of revolution. Let N_ϕ and N_θ be the membrane stress resultants at a point in the directions of the meridian and the parallel curve respectively, and let $N_{\theta\phi}$ be the shear resultant. Let X, Y, Z be the components of load per unit area in the directions of the parallel curve, meridian curve, and inward normal respectively. Let subscript n's denote coefficients in complex Fourier series in the azimuth angle θ. Then, as Neményi and I have shown,[4] the quantities $N_{\phi n}$, $N_{\theta n}$, and $N_{\theta\phi n}$ may be derived from the formulas

$$(1)\qquad \begin{gathered} N_{\phi n} = \frac{(1+f'^2)^{1/2}}{f^2}\mathfrak{U}_n, \qquad N_{\theta n} = \frac{f''}{(1+f'^2)^{1/2}}\mathfrak{U}_n - f(1+f'^2)^{1/2}Z_n, \\ in\, N_{\theta\phi n} = \frac{d}{dz}\left(\frac{\mathfrak{U}_n}{f}\right) + f(f'Z_n - Y_n), \end{gathered}$$

where the stress functions $\mathfrak{U}_n(z)$ satisfy the differential equation

$$(2)\qquad \frac{d^2\mathfrak{U}_n}{dz^2} + (n^2-1)\frac{f''(z)}{f(z)}\mathfrak{U}_n = g_n(z),$$

where

$$(3)\qquad \begin{aligned} g_n(z) = {} & -f^2f'\frac{dZ_n}{dz} + [(n^2-3)ff'^2 + n^2f - f^2f'']Z_n \\ & + f^2\frac{dY_n}{dz} + 3ff'Y_n + inf\,(1+f'^2)^{1/2}X_n. \end{aligned}$$

Terms in which $n=0$ or $n=\pm 1$ are more conveniently treated with the aid of special explicit formulas [MT, pp. 128–129, 130–131] which avoid using the differential equation (2); in this investigation we shall limit our analysis to the terms in which $|n|\geq 2$, and our conclusions are therefore restricted to problems involving nonuniform load or support.

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[3] The normal stress only is prescribed at supports, it being supposed the support be such that the concomitant shearing stress is counterbalanced.

[4] P. Neményi and C. Truesdell, *A stress function for the membrane theory of shells of revolution*, Proc. Nat. Acad. Sci. U.S.A. vol. 29 (1943) pp. 159–162; [MT, pp. 126–127].

Let the base of the shell be at $s=0$ and the apex at $s=s_0$. The most common boundary value problems involve given loads and a given mode of support. Then $N_{\phi s}(0)$, and hence $\mathfrak{U}_s(0)$, will be a prescribed quantity. If the shell has an open apex, then also $N_{\phi s}(s_0)$, and hence $\mathfrak{U}_s(s_0)$, will be a prescribed quantity. In the case of a shell with a closed apex, Flügge[5] has suggested that a proper boundary condition may be obtained by writing the equations of equilibrium of an annular section of the shell subtending a colatitude angle $\Delta\phi$, letting $\Delta\phi$ approach zero, and then letting the annular section approach the apex. This boundary condition we shall call the "ring limit condition." Let the apex of the shell be representable in the form

$$f(s) = (s - s_0)^{\mu} g(s), \qquad \mu = 1/2 \text{ or } 1. \tag{4}$$

Then the ring limit condition may be shown [MT, pp. 131–137] to take the form $\mathfrak{U}_n(s_0)=0$, and solutions satisfying this condition will always exist. To follow the arguments of §5 it is essential to realize that the boundary conditions to be imposed on the solution of (2) must involve both the points $s=0$ and $s=s_0$, rather than a single point.

3. **Preliminary observations.** Suppose we have two shells of revolution of nearly the same meridian curve loaded with the same load system and supported in the same way. From the differential equation (2) it is apparent that the difference between the two stress resultant systems will depend essentially on the difference between the two different ratios f''/f. In an open shell, or in portions of a closed shell which are far distant from its apex, we may study the effect of changing curvature very much while changing the shell radius very slightly. With this end in mind in §4 we shall show actually that an arbitrarily large change in the stress resultant distribution of any shell can be produced by introducing a sufficiently large change in the curvature of the meridian curve, at the same time keeping the shell radius arbitrarily close to its original value. This result is presented as of interest in itself, and is not offered as evidence of unreliability of the theory. Supposing, however, that both the change in curvature and the change in shell radius are kept small, so that the two meridian curves would seem hardly distinguishable, from the differential equation (2) we are led to expect singularities in the "complete" stress functions [MT, p. 128] $\mathfrak{U}_{ns}$ at the apex of a closed dome, where f''/f usually [MT, p. 132] becomes infinite. In §5 we

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[5] W. Flügge, *Statik und Dynamik der Schalen*, Berlin, 1934, p. 40.

shall show that if the two apexes are representable in the form (4) with exponent 1/2 (flat-topped shell), a reasonably good fit of the meridian curves will insure an excellent fit of the two values of f''/f at the apex, but that if the exponent is 1 (pointed shell) the two meridian curves may appear to fit very closely with each other and yet the two values of f''/f at the apex may be of entirely different orders of magnitude. We shall show also that a large difference in the two values of f''/f in a very small region near the apex necessarily produces a large difference in the two stress resultant systems *throughout* the shells, casting doubt on the validity of the results of the membrane theory with the ring limit condition when it is applied to problems concerning pointed shells supported or loaded nonuniformly.

Before beginning the detailed analysis, however, let us mention the effect of three other types of special points in the meridian curve. First, a point of inflection in the meridian curve, while it produces an inflection in the complete stress functions $\mathfrak{U}_{n0}$, does not appear either from the equations (1), (2), and (3) or from two special cases treated in detail [MT, pp. 161–163] to cause any sort of singularity or noticeable variation in the stress-resultant system. A point where the tangent to the meridian curve is vertical, $f'=0$, seems to be equally neutral in its effect on the stress-resultant system. A point of horizontal tangency, where $f'=\infty$, will in general produce infinite stress-resultants according to the formulas (1), (2), and (3) unless $f=0$ at the same point. There appears to be no reason to doubt the validity of the membrane theory for shells whose meridian curves possess singularities of these types. The singularity at an apex, however, requires special analysis.

4. The influence of curvature differences. Preparatory to discussing the critical response of the membrane stress resultants to small changes in the curvature of the meridian curve, let us introduce a general superposition principle in the membrane theory of shells of revolution.

Suppose we have two shells whose meridian curves are $r={}^1f(s)$ and $r={}^2f(s)$. Let us superpose these two shells and their loads and obtain a third shell, $r={}^1f(s)+{}^2f(s)$, loaded with the sum of the two original loadings. We shall compare the resulting membrane stress resultants with the sum of the two original stress resultants. We use left superscripts c, 1, and 2 to distinguish quantities associated with the combined, first, and second shells respectively. In particular, ${}^cf(s)={}^1f(s)+{}^2f(s)$. We define the excess stress functions

$$\mathfrak{X}_n = {}^0\mathfrak{U}_n - {}^1\mathfrak{U}_n - {}^2\mathfrak{U}_n. \tag{5}$$

Then from (2) and a straightforward calculation it can be shown that

$$\mathfrak{X}_n'' + (n^2 - 1)\,({}^0f''/{}^0f)\,\mathfrak{X}_n = k_n, \tag{6}$$

where

$$k_n \equiv (n^2 - 1)\left(\frac{{}^1f''\,{}^2f - {}^1f\,{}^2f''}{{}^0f}\right)\left(\frac{{}^1\mathfrak{U}_n}{{}^1f} - \frac{{}^2\mathfrak{U}_n}{{}^2f}\right) + {}^0g_n - {}^1g_n - {}^2g_n. \tag{7}$$

The stress resultant distribution derivable from the excess stress functions $\mathfrak{X}_n$ may be pictured as resulting from a fictitious load on the combined shell. The fictitious load is of two types: The first part of formula (7) represents the geometric change alone, and is independent of the original load systems, while the second part represents the different geometric resolution of the original load systems. In fact

$$k_n = -n^2\,{}^0f(1 + {}^0f'^2)^{1/2}P_n + {}^0g_n - {}^1g_n - {}^2g_n, \tag{8}$$

where

$$P_n \equiv -\frac{n^2 - 1}{n^2}\cdot\frac{{}^1f''\,{}^2f - {}^1f\,{}^2f''}{({}^0f)^2(1 + {}^0f'^2)^{1/2}}\cdot\left(\frac{{}^1\mathfrak{U}_n}{{}^1f} - \frac{{}^2\mathfrak{U}_n}{{}^2f}\right). \tag{9}$$

P_n is a fictitious load distribution in the direction perpendicular to the axis of revolution.

Suppose, for example, we have a shell with meridian $r = {}^1f(z)$ subject to axially symmetric loading but nonuniformly supported: ${}^1g_n \equiv 0$. Let us superpose on it a small waviness, still keeping the apex of the shell closed:

$${}^2f \equiv \epsilon \sin k(z - z_0), \qquad {}^2g_n \equiv 0. \tag{10}$$

Then ${}^0g_n \equiv 0$ and [MT, p. 143]

$${}^2\mathfrak{U}_n = K_n \operatorname{sh}\,(n^2 - 1)^{1/2}k(z - z_0); \tag{11}$$

hence by formula (9),

$$\begin{aligned} P_n = \frac{n^2 - 1}{n^2} &\cdot \frac{({}^1f'' + k^2\,{}^1f)\epsilon \sin k(z - z_0)}{[{}^1f + \epsilon k \sin k(z - z_0)]^2(1 + [{}^1f' + \epsilon k \cos k(z - z_0)]^2)^{1/2}} \\ &\cdot\left(\frac{K_n \operatorname{sh}\,(n^2 - 1)^{1/2}k(z - z_0)}{\epsilon \sin k(z - z_0)} - \frac{{}^1\mathfrak{U}_n}{{}^1f}\right). \end{aligned} \tag{12}$$

$^1\mathfrak{U}_n(s)$ and $^1f(s)$ are fixed functions, so that it is possible to choose k sufficiently larger than their maximum values[6] so that

$$P_n \approx \frac{n^2-1}{n^2} \tag{13}$$
$$\cdot \frac{k^2K_n\,{}^1f\,\mathrm{sh}\,(n^2-1)^{1/2}k(s-s_0)}{[{}^1f+\epsilon\sin k(s-s_0)]^2(1+[{}^1f+\epsilon k\cos k(s-s_0)]^2)^{1/2}}\,.$$

We may now choose ϵ sufficiently small so that ϵk is much smaller than the other magnitudes in formula (13), so that

$$P_n \approx \frac{n^2-1}{n^2}\cdot\frac{k^2K_n\,\mathrm{sh}\,(n^2-1)^{1/2}k(s-s_0)}{({}^1f)^2(1+{}^1f'^2)^{1/2}}\,. \tag{14}$$

Since initially we chose k as large as we pleased it follows that P_n may be made arbitrarily large even when ϵ is arbitrarily small, and that hence the membrane stress resultants derivable from the excess stress function will become arbitrarily large. This result is physically obvious, but has not been proved mathematically until now, so far as I know.

5. Small perturbations of the meridian curve. In §4 we set up apparatus by which we could discover large differences in stress resultant distributions due to a generally poor approximation of one shell by another, but those formulas are not convenient either for showing the absence of large differences when the approximation is close or for demonstrating large differences due to poor fit in the neighborhood of the apex alone. We now outline a method of estimating the difference in the stress resultant systems of two shells, approximately alike and loaded and supported in the same way.

We use the prefix Δ to indicate the difference between a quantity associated with the second shell with meridian curve $r=f(s)+\Delta f(s)$ and the corresponding quantity associated with the first one with meridian curve $r=f(s)$. Suppose that the two shells are of the same height s_0, and that if the apex is closed, each has the same exponent μ in the form (4). The same load distribution in the same geometric resolution is applied to each shell:

$$\Delta X = 0, \qquad \Delta Y = 0, \qquad \Delta Z = 0, \tag{15}$$

and the boundary conditions are the same for each:

$$\Delta N_{\phi n}(0) = 0, \quad \begin{cases} \Delta N_{\phi n}(s_0) = 0 & \text{for an open shell,} \\ \Delta \mathfrak{U}_n(s_0) = 0 & \text{for a closed shell.} \end{cases} \tag{16}$$

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[6] That finite maxima exist for these quantities follows from [MT, pp. 98, 132].

Then $\Delta\mathfrak{U}_n(s)$ will be that solution of the differential equation

$$(17)\qquad (\Delta\mathfrak{U}_n)'' + (n^2 - 1)\left[\frac{f''}{f} + \Delta\,\frac{f''}{f}\right]\Delta\mathfrak{U}_n = \Delta g_n - (n^2 - 1)\mathfrak{U}_n\Delta\frac{f''}{f}$$

which assumes the values at $s=0$ and at $s=s_0$ given by the conditions (16) and (1). We may estimate the magnitude of $\Delta\mathfrak{U}_n$ in two steps.

I. First consider solutions of the equation

$$(18)\qquad (\Delta\mathfrak{U}_n)'' + (n^2 - 1)\left[\frac{f''}{f} + \Delta\frac{f''}{f}\right]\Delta\mathfrak{U}_n = 0.$$

Since the solutions of differential equations (under suitable conditions) depend continuously upon the coefficients, as $\Delta(f''/f)\to 0$ the solutions of the equation (18) must approach certain solutions of the equation

$$(19)\qquad (\Delta\mathfrak{U}_n)'' + (n^2 - 1)\frac{f''}{f}\Delta\mathfrak{U}_n = 0.$$

This equation is satisfied also by the complete stress functions $\mathfrak{U}_{n0}$ for the original shell, so its general solution we may write down in terms of functions we already know. We then attempt to estimate the maximum deviation of solutions of the equation (18) from these known functions.

II. We consider the equation

$$(20)\qquad (\Delta\mathfrak{U}_n)'' + (n^2 - 1)\frac{f''}{f}\Delta\mathfrak{U}_n = \Delta g_n - (n^2 - 1)\mathfrak{U}_n\Delta\frac{f''}{f}\cdot$$

Since we are presumed to know the complete primitive of the equation (19), and since the right-hand side of the equation (20) involves only known functions, we may write down the solution of equation (20) as a simple quadrature [MT, p. 127]. Using the result of step I, we may then estimate by how much this integral may deviate from the corresponding exact integral of the equation (17).

The prosecution of step II offers no difficulty whatever, but the result of step I is so complicated as to be useless. We shall set up the problem, however, because essential qualitative information may be gained from it. Suppose we have a function $Y(s)$ satisfying the differential equation

$$(21)\qquad Y'' + (\phi + \epsilon)Y = 0$$

and assuming prescribed values at $s=0$ and $s=s_0$. By how much can

$Y(s)$ deviate from the solution of the differential equation

$$y'' + \phi y = 0 \tag{22}$$

which satisfies the same boundary conditions? We solve the equation (21) by iteration, using the solution of the equation (22) as the first approximating function:

$$Y_1 = y$$

$$\begin{aligned} Y_{n+1} = \left(1 - \frac{s}{s_0}\right) Y(0) + \frac{s}{s_0}\left[Y(s_0) + \int_0^{s_0} d\xi(s_0 - \xi)(\phi + \epsilon) Y_n\right] \\ - \int_0^{s} d\xi(s - \xi)(\phi + \epsilon) Y_n. \end{aligned} \tag{23}$$

The bound for $|Y-y|$ is not simple, as it is in the familiar case illustrated in the text books,[7] first because of our two-point boundary condition, and second because the functions ϕ and ϵ in our case are f''/f and $\Delta(f''/f)$ respectively, which are usually [MT, p. 132] singular at the apex of a closed dome. A bound exists, however, because it can be shown [MT, pp. 132–137] that the zero of Y_n at $s = s_0$ is strong enough to keep $(s_0 - \xi)(\phi + \epsilon) Y_n$ bounded.

Now it can be shown [MT, p. 164] that Y_0, and hence Y_n, is of one sign for $0 \leqq s \leqq s_0$, providing f'' is of one sign. Then from the equation (23) we may see that even if $\epsilon \neq 0$ only in one small interval, the value of Y_n, and hence finally of Y, will be changed at every point except $s = 0$ and $s = s_0$ because of the presence of the integral from 0 to s_0, and further that if ϵ/ϕ is large in this interval the change in Y will be correspondingly large. Hence even if $\Delta(f''/f)/(f''/f)$ is large nowhere except in a small region near the apex, there will be a significant change in U_n everywhere.

If now the apex is representable in the form (4) with μ equal to 1/2, and if $g(s)$ is imbedded in a family of continuous functions $g(s, b)$ of the parameter b such that $g(s, 0) = g(s)$ and $f(s) + \Delta f(s) = (s - s_0)^{1/2} g(s, \Delta b)$, then it can be shown that

$$\begin{aligned} \frac{\Delta(f''/f)}{f''/f} = \frac{\dfrac{\partial}{\partial b}\left(\dfrac{g'}{g}\right)_{b=0} + (s - s_0)\dfrac{\partial}{\partial b}\left(\dfrac{g''}{g}\right)_{b=0}}{-1 + 4(s - s_0)\left(\dfrac{g'}{g}\right)_{b=0} + 4(s - s_0)^2\left(\dfrac{g''}{g}\right)_{b=0}} (s - s_0)\Delta b \\ + O(\Delta b^2). \end{aligned} \tag{24}$$

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[7] L. Bieberbach, *Theorie der Differentialgleichungen*, 3d ed., Berlin, 1930, pp. 39–41.

The term linear in Δb vanishes when $s=s_0$, so that near the apex the effect of the function ϵ in equation (23) will be small, and consequently $|Y-y|$ will be small. If on the other hand the exponent in the form (4) is 1, and if we write $h(s, b)$ for $(s-s_0)g(s, b)$, we find that

$$\frac{\Delta(f''/f)}{f''/f} = \left[\frac{\partial}{\partial b}\log h'' - \frac{\partial}{\partial b}\log h\right]_{b=0}\Delta b + O(\Delta b^2), \tag{25}$$

and at the apex this quantity may be infinite. Consider, for example,

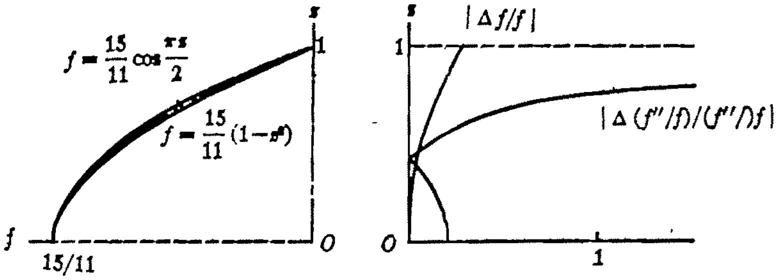

FIG. 1

the two shells previously treated in some detail by Flügge[8] and me [MT, pp. 155–158]:

$$f = A\cos\frac{\pi s}{2s_0}, \qquad f+\Delta f = A\left(1-\frac{s^2}{s_0^2}\right). \tag{26}$$

These two meridian curves are those members of the family

$$h(s, b) = A\left[bk\left(1-\frac{s^2}{s_0^2}\right) + (1-kb)\cos\frac{\pi s}{2s_0}\right] \tag{27}$$

which correspond to the values 0 and $1/k$ respectively for b. If k is chosen very large, Δb may be replaced by b and the term $O(\Delta b^2)$ may be neglected in the formula (25), and we find that the value of that expression becomes infinite at the apex. In Figure 1 are plotted the two meridian curves and the exact values of $|\Delta f/f|$ and $|\Delta(f''/f)/(f''/f)|$. From the equation (27) it is possible to show that $\partial[\Delta(h''/h)/(f''/f)]/\partial b$ is infinite at the apex both when $b=0$ and when $b=1/k$. The value .273 approached by $\Delta f/f$ at the apex represents a local poorness of fit which should affect but little the correct stress resultants at points far distant from the apex; the two curves are very closely matched, yet from equation (23) it is easy to see that the large values of $|\Delta(f''/f)/(f''/f)|$ in the upper part of the two shells will produce large differences everywhere in the stress resultants computed as solutions of the partial differential equations of the membrane theory.

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[8] Op. cit. pp. 47–49.

6. **Evaluation of the membrane theory.** The formulas (17) and (23) show that a change in the quantity f''/f anywhere along the meridian curve of a shell of revolution will affect the membrane stress resultants everywhere. Since at the apex of a closed shell that quantity usually becomes infinite, we reasonably expect large changes in it corresponding to small changes in the meridian curve. Formula (14) shows that if the apex is sphere like the ratio $\Delta(f''/f)/(f''/f)$ will vanish at the apex to the first order in the approximation parameter Δb so that membrane stress resultants in shells with flat tops will not exhibit undue response to changes in the meridian curve. Formula (25) and the succeeding example show that for a pointed dome this ratio may become infinite, and hence that the stress resultant distribution in pointed shells may vary disproportionately in response to slight changes of the meridian curve.

We may summarize both the preceding results and my previous general discussion of the membrane theory [MT, pp. 108, 123]:

1. For an open shell or for a closed shell with a *flat, sphere-like apex* ($\mu=1/2$), there will exist for a given middle surface and given shell loading a range of thicknesses for which the stress-resultant distribution computed from the membrane theory will be a correct first approximation to that computed from the stress distribution given by the three-dimensional theory, providing the load distribution is continuous and the support conditions are consistent with boundary conditions admissible in the membrane theory.[9] In this range slight local changes in the radius and in the curvature of the meridian curve will cause only slight changes in the stress resultants.

2. In a closed dome with a *pointed apex* ($\mu=1$), a slight local change in the meridian curve of the apex may produce a very great change in the membrane stress resultants throughout the shell. Hence the membrane theory is not reliable in treating problems of nonuniform load and support for pointed domes.

It is possible to explain this unsatisfactory behavior of the membrane stress resultants in pointed shells. First, the apex of any pointed shell of any thickness is a point where the basic assumption of shell theory, namely "thickness/minimum radius of normal curvature $\ll 1$," is invalid. We are not interested in the apex for its own sake, but rather we wish to find formulas giving valid stress resultants at all points sufficiently far from it [MT, pp. 123–124]. Now examples of cylindrical shells given by Neményi[10] show that the effect of other types of singularities, such as discontinuities in the thickness or in the loads, is localized and insignificant at distant points if the stress and moment resultants are computed from the equations of the bending theory. Perhaps the apex of a pointed dome represents an equally local source of disturbance in the bending theory, but equation (23) shows that this singularity affects the stress resultant distribution computed from the equations of the membrane theory not only locally but throughout the shell.

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[9] Flügge, op. cit. pp. 119–120, makes an equivalent statement, drawn apparently from experience with examples, concerning shells of circular cylindrical form.

7. Boundary conditions at the apex. As mentioned in §1, we have employed the ring limit condition at the apex in the foregoing discussion. In the literature of shell theory it is customary to apply instead the more artificial requirement that the stress resultants remain finite at a closed apex. Flügge in proposing the ring limit condition claimed to prove that a membrane stress resultant system satisfying it would automatically remain finite,[11] but his analysis contains errors, and I showed that in the case of pointed shells the condition of finiteness could never be satisfied for terms in which $|n| \geqq 2$, but that the ring limit condition could always be satisfied by shells with meridian of form (4). Infinite stress resultants at the apex are in no way objectionable, since the apex is an idealization of no particular interest, behavior near which we are forced to consider only so as to obtain results valid for reasonable distances away. The present paper, however, shows that the membrane theory with the ring limit condition is not satisfactory for problems of pointed shells unsymmetrically loaded or supported.

Now for pointed shells the annular section becomes undefined in the limit as the apex is approached [MT, p. 109]. Thus the ring limit condition is not physically natural. In re-proposing it originally I observed that it nevertheless led to a natural sort of boundary condition for solutions of the fundamental differential equation (2), that solutions satisfying it existed, and that it kept the membrane theory statically determinate for the same range of problems concerning pointed shells as for flat-topped shells.

Nearly two years ago Professor Stoker suggested to me that finiteness of the strain energy at the apex would be a physically more reasonable boundary condition, and I am in full agreement. There are two difficulties in the way of using this condition however.

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[10] Loc. cit.

[11] Op. cit. p. 40.

The first one is mainly formal. The strain energy involves strains as well as stress resultants, so that to compute it we must solve the displacement equations as well as the equilibrium equations, the problem thus being no longer statically determinate. There seems to be no particular reason why two shells having identical loads and supports, the one having a flat apex and the other a pointed one, should lead to basically different boundary problems.

The second difficulty is much more formidable, however. To explain it, we must recall the derivation of the basic equations. In the customary presentation the strains are approximated by their values at the middle surface,[12] and the strain energy is a function of these strains. Now in my thesis[13] I showed that strains so approximated and employed in the usual macroscopic stress-strain relations[14] cannot satisfy the conditions of compatibility, even approximately. Hence I proposed a new derivation (see §8 below) of the fundamental differential equations [MT, pp. 102–108, 118–122], employing no approximate formulas for the strains and no macroscopic stress-strain relations; this derivation presupposes the existence of appropriate solutions of the full equations of the three-dimensional theory, including the conditions of compatibility, and from the microscopic stress-strain relations and the usual additional assumptions of shell theory (as listed in §8 below) deduces relations connecting the stress resultants and the displacements of the middle surface. The strain energy does not appear, and I do not know how to find a correct expression for it. Thus it is not clear how the condition of finiteness of the strain energy can be applied, even when general solutions of all the differential equations are known.

8. A possible theory of average stresses. A theory of shells giving results which are better indications of the correct three-dimensional stresses in problems concerned with pointed shells nonuniformly loaded or supported might be obtained by abandoning the stress resultants, which are defined, for example,

$$(28) \qquad N_\phi = \int_{-t/2}^{t/2} \tau_{\phi\phi}\left(1 + \frac{x}{r_2}\right) dx,$$

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[12] Flügge, op. cit. pp. 51–52.

[13] Manuscript in Princeton Library.

[14] Flugge, op. cit. p. 50.

and returning to the averages originally introduced by Aron,[15] for example,

$$T_\phi = \int_{-\delta/2}^{\delta/2} \tau_{\phi\phi} dx. \tag{29}$$

At a sphere-like apex $r_2 \neq 0$, but at a pointed apex $r_2 = 0$ and hence N_ϕ must become infinite, even if $\tau_{\phi\phi}$ has no singularity at all. T_ϕ, on the other hand, can never become infinite unless $\tau_{\phi\phi}$ becomes infinite and is thus a better guide to the behavior of the three-dimensional stress system. Now

$$\left| \frac{N_\phi - T_\phi}{T_\phi} \right| \leq \frac{\delta}{r_2} \cdot \frac{\int_{-\delta/2}^{\delta/2} | \tau_{\phi\phi} | dx}{\int_{-\delta/2}^{\delta/2} \tau_{\phi\phi} dx}, \tag{30}$$

so that in regions far distant from the apex N_ϕ and T_ϕ differ by a quantity of the order neglected in the membrane theory. Hence at points far distant from the apex the differential equations of the average theory may be correctly approximated by those of the ordinary membrane theory, but near the apex the two systems of partial differential equations will be quite different because the difference δ/r_2 becomes very large. The correct theory of averages[16] would then agree with the present membrane theory in cases when the latter gives correct results, but in problem of pointed domes non-uniformly loaded or supported it might avoid the membrane theory's unrealistic response to insignificant changes in the meridian curve.

It is not obvious, however, how a correct theory of averages could be formulated.

Since the equilibrium equations for the bending theory are exact and not approximate equations,[17] the best way to derive the equilibrium equations of the average theory would begin by expressing the stress and moment resultants in terms of the averages. If, for example,

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[15] H. Aron, *Das Gleichgewicht und die Bewegung einer unendlich dünnen, beliebig gekrümmten elastischen Schale*, J. Reine Angew. Math. vol. 78 (1874) pp. 136–174.

[16] Some writers, for example, Timoshenko, *Theory of plates and shells*, New York and London, 1940, p. 352, make the error of initially neglecting the difference between averages and resultants, fancying they thereby simplify the derivation of the ordinary equations of the bending theory. There results a theory formally identical with the bending theory, but deduced with the aid of unnecessary approximations, for I have shown in [MT, pp. 118–122] that the equations of equilibrium of shells, expressed in terms of resultants, are exact consequences of the three-dimensional equilibrium equations, and that the displacement equations become poorer approximations to the correct displacement equations when averages are used in place of resultants. See also the observation of Flügge, op. cit. pp. 4–5.

$$P_\phi = -\int_{-\delta/2}^{\delta/2} x\tau_{\phi\phi}\,dx, \tag{31}$$

then from equations (28), (29), and (31) we have the relation

$$N_\phi = T_\phi - (1/r_2)\,P_\phi. \tag{32}$$

In order to obtain exact expressions like this one for all 10 stress and moment resultants, it is necessary to use in all 13 different averages like those given by the definitions (29) and (31). By substituting all 10 of the relations of the type (32) into the equilibrium equations of the bending theory we obtain 5 exact equilibrium equations for the average theory.

The real difficulty lies in finding correct expressions giving the stress averages in terms of derivatives of the displacements of the middle surface. For the ordinary theory of shells I have pointed out a lucid way [MT, pp. 120–121] to deduce these relations: (I) to express the displacements of an arbitrary point in terms of the displacements of the projection of that point on the middle surface, then (II) to substitute these formulas into the expressions giving the three-dimensional strains in terms of the three-dimensional displacements, then (III) to substitute these values for the strains into Hooke's law and obtain the stresses, and finally (IV) to put the resulting values of the stresses into the definitions of type (28) and evaluate the integrals. Practical results are obtained in the bending theory by the aid of the characteristic assumptions of shell theory, namely[18]

A. $|\delta/R| \ll 1$, where δ is the thickness and R the minimum radius of normal curvature of the middle surface.

B. $|\tau_{nn}/E| \ll |\epsilon_{nn} + \nu\Delta/(1-\nu)|$, where E and ν denote Young's modulus and Poisson's ratio respectively, and Δ is the cubical dilatation.

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[17] See the preceding note. The equilibrium equations of the bending theory are therefore exact equilibrium equations in the strained coordinate system. To obtain a manageable theory we wish to be able to use these equations in the unstrained coordinates, so just as in three-dimensional elasticity we add the assumptions of very small displacements, so that the descriptions of the deformation given in the strained and unstrained coordinate systems will coalesce.

[18] [MT, pp. 104–105]. See also E. Reissner, *A new derivation of the equations for the deformation of elastic shells*, Amer. J. Math. vol. 63 (1941) pp. 177–184.

C. $|\epsilon_{n\phi}| \ll \delta/R$, $|\epsilon_{n\theta}| \ll \delta/R$, $|\epsilon_{nn}| \ll \delta/R$, $|\partial\epsilon_{nn}/\partial\phi| \ll 1$, $|\partial\epsilon_{nn}/\partial\theta| \ll 1$.

The assumptions A and C simplify the results of step I and hence of step II, the assumption B simplifies the result of step III, and the results of step IV become then a set of power series expansions in δ/R which are valid up to and including terms of 3rd order.

If the average theory is to be correct even when r_2 is arbitrarily small,[19] assumptions A and C must be modified. If one replaces R by r_1 in these assumptions, they will then be correct for a range of problems dealing with thin shells of pointed apex, but unfortunately they no longer effect a simplification of step I sufficient to enable us to carry out the succeeding steps and arrive at manageable results.

The dominant characteristic of the ordinary membrane theory is that it is statically determinate: while assumptions A, B, and C are used in its derivation, the end result is a system of three differential equations involving as unknown functions only N_ϕ, N_θ, and $N_{\theta\phi}$ and making no mention of the displacements, which can be calculated from a second set of three partial differential equations as soon as the stress resultants are known. One estimates the orders of the stress and moment resultants from the relations expressing the stress-resultants in terms of the displacements of the middle surface, finding, for example, that $N_\phi = O(\delta/R)$, $M_\phi = O(\delta^3/R^3)$, so that a first order theory neglects bending moments. Since, as I have said above, it is not evident how to construct the two-dimensional stress-strain relations in the average theory when r_2 is allowed to be arbitrarily small, we do not have the apparatus for deciding whether or not there is a simple statically determinate theory of averages which will give a correct first approximation to the stress averages in pointed shells.

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[19] When $r_2 = 0$ the element of arc length in revolution coordinates [MT, p. 99] is no longer defined and the various differential formulas of elasticity therefore become indeterminate. Naturally we do not expect our results to be valid *at* the apex, but we wish our differential equations to be valid *near* it, that is, when r_2 is small but not actually zero.

ON THE EXISTENCE OF STEADY GAS FLOW IN PLANE ISOTHERMAL STREAMLINE PATTERNS

R. C. PRIM

In this paper we shall investigate the possibility of steady, plane gas flows having streamlines which can be mapped conformally onto a family of parallel lines. We shall limit our consideration to flows of an ideal gas (that is, a nonviscous, thermally nonconducting gas with constant specific heats) in the absence of body forces. The investigation will include rotational as well as irrotational flows.

We shall employ the formulation of the basic gas flow equations due to Munk and Prim [1].[1] This formulation includes the most general steady flows of an ideal gas in the absence of body forces.

$$(1) \qquad \operatorname{div}\left[(1-\mathbf{w}^2)^{1/(\gamma-1)}\mathbf{w}\right] = 0,$$

is the continuity equation and the dynamic equation is

$$(1a) \qquad (\mathbf{w}\cdot\operatorname{grad})\mathbf{w} + \frac{\gamma-1}{2\gamma}(1-\mathbf{w}^2)\operatorname{grad}\ln p = 0$$

whence the integrability condition

$$(2) \qquad \operatorname{curl}\left[\frac{\mathbf{w}\times\operatorname{curl}\mathbf{w}}{1-\mathbf{w}^2}\right] = 0,$$

where γ denotes the adiabatic exponent, p the pressure, and $\mathbf{w}$ the reduced velocity vector related to the actual velocity vector $\mathbf{v}$ by

$$\mathbf{w} = \frac{\mathbf{v}}{v_{\text{ultimate}}} = \frac{\mathbf{v}}{(2\gamma/(\gamma-1))(p/\rho) + v^2}$$

where ρ denotes the density.

Equations (1) and (2) will be referred to a plane, isothermal net ξ, η in which the squared element of arc length is given by

$$(ds)^2 = g(\xi,\eta)\left[(d\xi)^2 + (d\eta)^2\right]$$

where limitation of the net to a plane requires

$$(3) \qquad \frac{\partial^2 \ln g}{\partial\xi^2} + \frac{\partial^2 \ln g}{\partial\eta^2} = 0.$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

The curves $\xi=$ const. will be taken as streamlines. Thus specialized, (1) and (2) become [2]

$$\frac{\partial}{\partial\eta}\left[gw(1-w^2)^{1/(\gamma-1)}\right]=0$$

and

$$\frac{\partial}{\partial\eta}\left[\frac{w^2}{1-w^2}\frac{\partial}{\partial\xi}\ln gw\right]=0,$$

or

$$gw(1-w^2)^{1/(\gamma-1)}=e^{a(\xi)} \tag{4}$$

and

$$\frac{w^2}{1-w^2}\frac{\partial}{\partial\xi}\ln gw=b(\xi) \tag{5}$$

where $a(\xi)$ and $b(\xi)$ are arbitrary functions possessing derivatives of fourth and third order respectively. Our problem is to find what functions $w(\xi,\eta)$ and $g(\xi,\eta)$ are consistent with (3), (4) and (5).

Elimination of gw between (4) and (5) and of g between (3) and (4) results in two partial differential equations restricting w:

$$\frac{\partial w^2}{\partial\xi}=m(w^2,\xi) \tag{6}$$

and

$$\frac{\partial^2 w^2}{\partial\xi^2}+\frac{\partial^2 w^2}{\partial\eta^2}=f(w^2)\left[\left(\frac{\partial w^2}{\partial\xi}\right)^2+\left(\frac{\partial w^2}{\partial\eta}\right)^2\right]+k(w^2)a''(\xi) \tag{7}$$

where

$$\begin{aligned} m(w^2,\xi)&=\frac{1-w^2}{\beta w^2}\left\{b(\xi)-\left[b(\xi)+a'(\xi)\right]w^2\right\},\\ f(w^2)&=\frac{(2\beta+1)w^4-2w^2+1}{w^2(1-w^2)\left[1-(2\beta+1)w^2\right]},\\ k(w^2)&=\frac{2w^2(1-w^2)}{1-(2\beta+1)w^2}, \end{aligned} \tag{6a}$$

and

$$\beta=1/(\gamma-1)\cdot$$

Use of (6) to eliminate $\partial w^2/\partial\xi$ and $\partial^2 w^2/\partial\xi^2$ from (7) yields:

$$\frac{\partial^2 w^2}{\partial\eta^2} = f\left[m^2 + \left(\frac{\partial w^2}{\partial\eta}\right)^2\right] + ka'' - m\frac{\partial m}{\partial w^2} - \frac{\partial m}{\partial\xi}. \tag{8}$$

Evaluation of $\partial^3 w^2/\partial\xi\partial\eta^2$ from (8) and $\partial^3 w^2/\partial\eta^2\partial\xi$ from (6) leads to the requirement:

$$\left(\frac{\partial w^2}{\partial\eta}\right)^2 = \frac{\begin{array}{c} m^2\partial^2 m/\partial w^{22} + 2m\partial^2 m/\partial\xi\partial w^2 + \partial^2 m/\partial\xi^2 - fm(2\partial m/\partial\xi \\ +m\partial m/\partial w^2) - f'm^3 - mk'a'' - ka''' + k(\partial m/\partial w)a'' \end{array}}{-\partial^2 m/\partial w^{22} + f\partial m/\partial w^2 + f'm}. \tag{9}$$

Case I: $m \neq 0$. If $m \neq 0$, (9) determines $\partial w^2/\partial\eta$ as a function of w^2 and ξ, for brevity

$$\frac{\partial w^2}{\partial\eta} = l(w^2, \xi). \tag{10}$$

Evaluation of $\partial^2 w^2/\partial\xi\partial\eta$ from (10) and $\partial^2 w^2/\partial\eta\partial\xi$ from (6) then leads to the relation

$$\frac{\partial m}{\partial w^2} l - m\frac{\partial l}{\partial w^2} - \frac{\partial l}{\partial\xi} = 0$$

from which it follows that

$$w = w(\xi), \tag{11}$$

where $w(\xi)$ is arbitrary by virtue of the functions $a(\xi)$ and $b(\xi)$. (11) and (4) imply

$$g = g(\xi)$$

and (3) then requires

$$\ln g = \mathcal{A}\xi + \mathcal{B}.$$

This restriction of g limits the streamlines to concentric circles or parallel straight lines.

Case II: $m = 0$. If $m = 0$, it is implied (6a) that $b(\xi) = a'(\xi) = 0$ and from (6)

$$w = w(\eta).$$

Letting $a(\xi)$ be an arbitrary constant, (4) implies

$$g = g(\eta)$$

and (3) then requires

$$\ln g = \mathcal{D}\eta + \mathcal{E}. \tag{12}$$

This restricts the streamlines to a radial pencil of straight lines or parallel straight lines.

The function $w(\eta)$ is not arbitrary, but is given implicitly by (12) and (4). By virtue of $b(\xi)$ being zero, the w field is here irrotational.

This same geometric restriction applies also to the singular, physically trivial, case when $w^2 = 1$. This is easily shown by use of (1a).

We have now established the theorem:

The only isothermal streamline patterns possible in steady plane flow of an ideal gas without body forces consist of concentric circles, radial straight lines, and parallel straight lines.[1]

This theorem was established from purely local considerations and, hence, remains valid for the separate regions of flow fields containing shocks or other discontinuities. The proof given presupposes the existence of (piecewise) continuous third order derivatives of w. It is of interest to note that the class of irrotational gas flows includes exactly the same flow patterns of the type considered as does the more general rotational class.

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3. D. Gilbarg, *Flow patterns common to certain classes of plane fluid motions*, Journal of Mathematics and Physics, Massachusetts Institute of Technology vol. 26 (1947).

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[1] It has been called to the author's attention that D. Gilbarg has independently established this theorem for the special case of irrotational flow [3].

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MATHEMATICAL METHODS IN ANCIENT ASTRONOMY

O. NEUGEBAUER

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1. Chronological remarks. The main source for our knowledge of ancient astronomy is Ptolemy. His *Mathematical composition*, commonly known as the *Almagest*, quotes observations of his own ranging from 127 to 142 A.D. [35; 36; 37].[1] This work seems to be the earliest of a whole series of fundamental works, such as his *Geography* [41], the *Tetrabiblos* [39; 40], and so on, whose influence on mediaeval thought cannot be overrated.

Questions of historical priority will not be discussed here. Nevertheless it must be emphasized that Ptolemy relied heavily on methods developed by his predecessors, especially Hipparchus. Indeed, almost all our information about the latter's work is based on references in the Almagest. However fragmentary our knowledge of Hipparchus' astronomy may be, it is evident that it represents a milestone in the development of mathematical astronomy. Observations of Hipparchus quoted by Ptolemy extend from 162 to 127 B.C.

Finally, we have original Babylonian ephemerides for the moon and the planets covering, with only minor gaps, the years from 227 to 48 B.C. (Kugler [19; 20], Schnabel [47], Schaumberger [20], Neugebauer [31]). Nothing is known about the exact date or origin of the Babylonian methods though it might be a fair guess to assume a date between 400 and 250 B.C.

To our knowledge, Egypt exercised no positive influence on the development of mathematical astronomy. This is in perfect accord

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

with the fact that Egyptian mathematics never went beyond an extremely elementary level, totally unfit for the description of astronomical phenomena.

2. **Introduction.** It is not the scope of this paper to give even a sketch of the historical development of ancient astronomy. I shall only try to illustrate the close relationship between mathematics and astronomy, a relationship which goes much farther than one might assume at first sight. I shall mention only three problems which seemingly would belong to purely observational astronomy but actually are essentially dependent upon mathematical theories. These problems are (a) the determination of the apparent diameter of the moon, (b) the determination of the constant of precession, (c) the determination of geographical longitude. The first two problems were not solved by direct measurement but by relating them to the theory of the motion of the moon. It requires the whole mechanism of the lunar theory to compute the coordinates of the moon for given eclipses, especially the distance of the center of the moon from the center of the shadow. It is only after these elements were found according to the mathematical theory that the observed magnitudes of the eclipses are used to find the apparent diameter of the moon (*Almagest* V, 14). Similarly, the longitudes of fixed stars are not measured directly but are referred to the moon, and thus eventually to the sun, by means of occultations or close conjunctions. Again the whole lunar theory is required to find the common longitude of moon and star (*Almagest* VII, 3). Finally, the determination of geographical longitude is based on the simultaneous observation of a lunar eclipse, the circumstances of which must be determined from theory. This last problem involves, however, another theoretical consideration. The ancients measured time not by means of clocks of uniform rate but by sun dials and waterclocks which showed "seasonal" hours. Seasonal hours can be simply described as an extremal form of "daylight saving time" because each hour is always the 12th part of the actual length of daylight. Thus the time reckoning is fully adjusted to the variation of the seasons. For civil life, this undoubtedly has its great advantages. For astronomical computations, however, the reduction of seasonal hours to equinoctial hours requires a theory of the dependence of sun dials upon the geographical coordinates and the longitude of the sun.

These examples will suffice to make it understandable that two groups of mathematical theory play a paramount role in ancient astronomy. On the one hand, a detailed celestial mechanics is needed, especially for the theory of the moon; on the other hand, auxiliary problems must be solved, including the theory of celestial and terrestial coordinates, their transformation from one system into another, and their application to the theory of various types of sun dials. In short, we can say that kinematics and spherical astronomy play a much greater role than empirical observations. The ancient astronomers were fully aware of the fact that the low accuracy of their instruments had to be supplemented by a mathematical theory of the greatest possible refinement. Observations are more qualitative than quantitative: "*when* angles are equal" may be decided fairly well on an instrument but not "*how large* are the angles," says Ptolemy with respect to the lunar and solar diameter (*Almagest* V, 15; Heiberg p. 417). Consequently, period relations over long intervals of time and lunar eclipses are the main foundations so far as empirical material is concerned; all the rest is mathematical theory. We shall see that this holds for Greek as well as for Babylonian astronomy.

The fact that ancient astronomy is to a large extent "mathematics" has far-reaching consequences for the history of civilization. The Middle Ages inherited an astronomical system, and with it a picture of the structure of the universe, of a consistency and inner perfection which hardly seemed open to improvement. The bearers of the Christian civilizations, at the very beginning, had lost contact with Hellenistic science; hence the astronomy of Western and Central Europe relapsed for many centuries into a primitive stage of knowledge where a few simple period relations sufficed as the basis of the computation of Easter and similar problems. Though this process was to some extent delayed by the continued use of astronomical tables for astrological purposes, the destruction of the ancient tradition would have been complete had not Greek astronomy found a new and most interesting development among Hindu astronomers. When the Arab conquest reached India, Greek astronomy soon saw a triumphant revival everywhere in the Moslem world, thus preparing the basis for the new development of astronomy and mathematics in the Renaissance. The "Ptolemaic system" has often been blamed for the preservation for almost 1500 years of a narrow, yet much too complicated, picture of the world.[3] It is only fair to underline the fact that this system preserved for the same length of time the tradition of mathematical methods which became most powerful tools in the hands of Copernicus, Tycho Brahe and Kepler.

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[3] It should be remarked that "Ptolemaic system" is often used in a rather unhistoric fashion. Actually the Aristotelian version of the homocentric spheres of Eudoxus determined the cosmological ideas of the philosophers and theologians of the Middle Ages. Thus a system 500 years older than the Almagest should take most of the blame.

A. CELESTIAL MECHANICS

3. Greek astronomy. The cornerstone of Greek celestial mechanics is an essentially "dynamical" principle, however metaphysical its actual formulation in Greek philosophical literature may sound. It is the idea that the circular movement of celestial bodies is the only movement which can last eternally. This "principle of inertia" is the guiding principle of all Greek astronomical theories. The fortunate accident that the orbits in our planetary system deviate very little from circles made it possible to construct geometric models whose gradual improvement corresponds to the addition of new Fourier terms, each of which has a certain physical significance. Modern scientists have often declared simplicity to be the criterium of truth, and a whole philosophy of "economy of reasoning" has offered its guidance to the researcher. This school of thought could rightly claim the early Greek astronomers as its first followers. Nothing "simpler" and more natural could have been assumed than the preference of celestial bodies for circular movements. And the remarkable successes of this assumption could only strengthen confidence in its correctness. Nobody could foresee that the simplicity of the circular movements is due to an accidental distribution of masses or that this also causes the simplicity of Newton's law, conveniently hiding from us the effects of a general gravitational space.

Here is not the place to describe the development of astronomical hypotheses, based on the combination of circular movements, or, originally, of movements of spheres, following an ingenious idea of Eudoxus. This development is described in masterly fashion by Duhem in his *Système du monde* [18]. We shall here restrict ourselves to a short discussion of some points in the theory of eccenters and epicycles in its application to the lunar movement. For the theory of Mercury and Venus see Boelk [3] and Schumacher [48].

Assuming that all circular movements proceed at constant angular velocity, it is obvious that an observer who is not located at the center gets the impression of a variable velocity. We know that already Apollonius, about 200 B.C., knew that an eccentric movement can be replaced by an epicyclic movement, where the center of the epicycle moves on the deferent with the mean angular motion around the observer whereas the object moves on the circumference of the epicycle with the same angular velocity in the opposite direction.

The radius of the epicycle is identical with the eccentricity of the eccenter. Similar relations hold for more general cases and it is therefore a matter of choice which hypothesis is used in a specific case (*Almagest* XII, 1 and III, 3). We know, for example, that Hipparchus used an epicycle for the description of the solar anomaly (Theon Smyrnaei, *De astronomia*, XXXIV [50]) while Ptolemy preferred the eccenter because of its greater simplicity, using only one motion (*Almagest* III, 4).

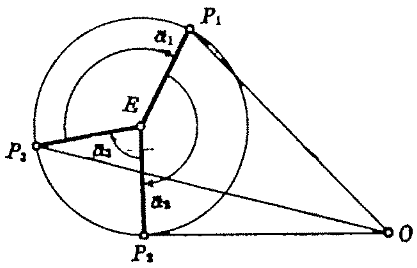

FIG. 1

The determination of the parameters of models of this type requires great ingenuity. As an example I might quote the case of the lunar theory in its simpler (Hipparchian) form which only accounts for the "first anomaly," that is, the eccentricity with uniformly progressing apsidal line. To this end an epicyclic model is assumed and three lunar eclipses are observed, giving the true longitudes λ_1, λ_2, λ_3 at given moments t_1, t_2, t_3. The mean motions in mean longitude ($\bar{\lambda}$) and mean anomaly (α) are known from period relations. Thus it is possible to find the positions of the moon on its epicycle, expressed by the differences of their anomalies (cf. Fig. 1). Furthermore the corresponding mean longitudes $\bar{\lambda}_i$ can be found for each t_i. The differences $c_i = \lambda_i - \bar{\lambda}_i$ are the corresponding values of the equation of center and can be interpreted as the angles under which the radii EP_i appear from the observer (*Almagest* IV, 6). The solution of this problem, undoubtedly known to Hipparchus and discussed in detail by Ptolemy, is often needed in surveying; one then speaks of the "Pothenot" or "Snellius" problem (Tropfke [51, V, 97]; the identity of these problems has been seen by Delambre [4, II, 164] and Oudemans [33]). As a result the radius of the epicycle can be found in terms of the radius of the deferent. Thus the eccentricity of the lunar orbit is known.

Before we proceed to Ptolemy's addition to Hipparchus' theory of the moon we might for a moment discuss the often underlined contrast between the simplicity of Kepler's theory and the clumsiness of eccenters and epicycles. Actually one compares here theories of different level. If one accepts the statement that the earth or the moon travels on an ellipse, one disregards perturbations and describes the true longitude in first approximation by

$$\lambda = \bar{\lambda} + 2e \sin \alpha,$$

where α is the mean anomaly. But exactly the same relation holds for an eccenter, or the equivalent epicycle, if its eccentricity is $2e$. If we want to take the movement of the apsidal line into account, again both models are equivalent because a uniform rotation of the apsidal line must be added in both cases. The error of the ancient theory does not at all lie in its unnecessary complication but in its simplicity, which leads only to correct longitudes whereas the distances are very badly represented, especially near the apogee. Because the observation of distances requires the measurement of very small angular differences for parallax or apparent diameters, this part of the theory was very difficult to check. Consequently one distrusted the observation of small angles even in cases where one should have found the discrepancy and preferred a simpler model to added corrections. For the longitudes, however, the ancient theory is exactly as simple and as efficient as Kepler's theory within the same first order approximations.

As was mentioned in the preceding remarks, the fundamental parameters of the lunar theory were obtained from eclipses, thus for syzygies. Already Hipparchus started to test the theory also for intermediate quadratures, and observations of his, quoted by Ptolemy (especially *Almagest* V, 5; Heiberg p. 369), showed that the "mean apogee" of the epicycle, from which α has to be counted, cannot be considered fixed. Ptolemy further investigated this situation and constructed a model which coincides with the simple theory in the syzygies and shows the proper fluctuation of the longitudes in the quadratures, depending on mean anomaly α and double mean elongation 2ϵ. The corresponding term $a \cdot \sin(2\epsilon - \alpha)$ in the modern theory is called "evection." Because the value of the constant a is about 1;16° whereas $2e = 6;17°$, we have for the syzygies and quadratures respectively[3]

$$\lambda = \bar{\lambda} + 6;17° \sin \alpha \mp 1;16° \sin \alpha$$

which shows that the maximum of the equation of center varies between about 5;1° and 7;33°. The Ptolemaic values are 5;1° and 7;39° respectively (*Almagest* IV, 10, and V, 8). Also for intermediate elongations and anomalies Ptolemy's representation of the longitudes is very good (Kempf [15]). His model is described by Fig. 2. In order to obtain the necessary increase of the equation of center from conjunction towards quadrature, he moves the lunar epicycle closer to the observer O, by letting Δ travel on a circle around O such that Δ has the elongation 2ϵ from the mean moon E. Simultaneously the oscillations of the anomaly are obtained by introducing a "mean apogee" $\bar{A}$ from which α is to be counted. The position of $\bar{A}$ is defined by means of the rule that $\bar{A}E$ always points towards the point N which is diametrically opposite to Δ on the circle with center O.

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[3] I use the notation $a,b;c$ for $a \cdot 60 + b + c \cdot 60^{-1}$.

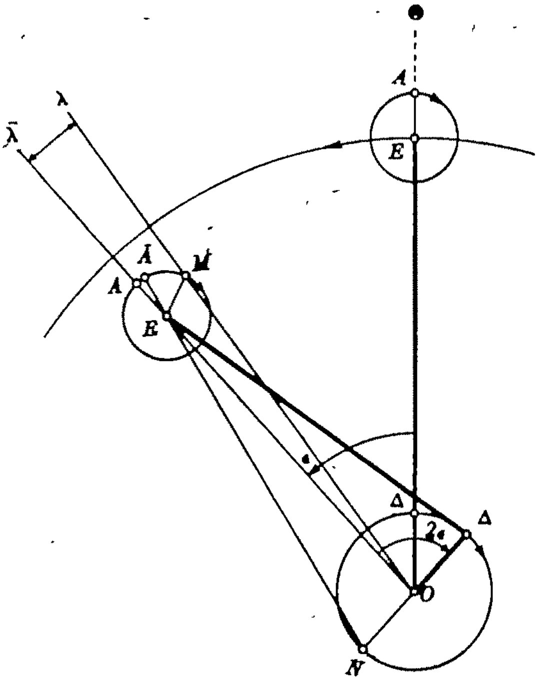

FIG. 2

If we want to compare this model with a geometric representation of the modern theory, using the same degree of approximation, we have to represent the evection as follows (Möbius [24]). The center C of the elliptic orbit rotates around the mean center $\bar{C}$ on a circle of radius $\bar{e}/5$ (where $O\bar{C}=\bar{e}$) with an angular velocity which is twice the velocity of the elongation of the mean sun from the mean apsidal line of the lunar orbit. Hence not only the apsidal line but also the eccentricity varies according to position and length of OC. Translating these movements into the language of eccenters and epicycles one obtains a model which closely resembles Ptolemy's construction, though with the essential difference that the radius of the epicycle must be made a function of α. Again one sees that Ptolemy's model is not too complicated but too simple because it represents only the longitudes correctly at the expense of the distances.

4. **Babylonian astronomy.** The final goal of Greek and Babylonian astronomy is, of course, identical. Starting from a few empirical elements one wishes to be able to compute the positions of the celestial bodies for any given moment. One may say with equal right that the progress, or the error, of the Greek method consisted in the invention of an intermediate step, namely the construction of a "dynamical" model, based on circular movements. From this model the Greeks derived their numerical tables exactly as a modern "Nautical Almanac" is derived from computations which are determined by consequences of dynamical rules and certain empirically determined initial values. As far as we can say from the material available to us, it seems that no such theoretical model existed in Babylonian astronomy. One apparently tried to obtain, on purely mathematical grounds, the rules for the computation of the tables from the empirical data. To use modern terminology the Babylonian procedure is very close to harmonic analysis. Given periodic phenomena of a rather complex character; find simple periodic functions, whose combination describes, within given limits of accuracy, the observed phenomena. While in our harmonic analysis the basic periodic functions admit, at least in principle, a direct geometric interpretation by simple harmonic oscillations, no such interpretation is obvious for the functions used in Babylonian astronomy. We do not know what were the Babylonian concepts about the physical structure of the universe. It seems safe to say that whatever concepts might have existed they were not directly reflected in the mathematical methods for the lunar and planetary movements.

The fundamental tools of Babylonian astronomy are periodic difference sequences of first and higher order. The simplest case is represented by tables in which each line represents a certain moment, these moments being equidistant, for example, one mean synodic month apart. The tabulated values can therefore be represented as a function $f(n)$ for integer values of the argument. Linear interpolation then leads to a continuous function $f(n)$ whose graph looks like Fig. 3. We shall call such functions "linear zigzag functions." The period of the function $f(x)$ may be called P and it is obvious that

$$P = 2\Delta/d$$

where Δ is the amplitude and d the slope of the increasing branch.

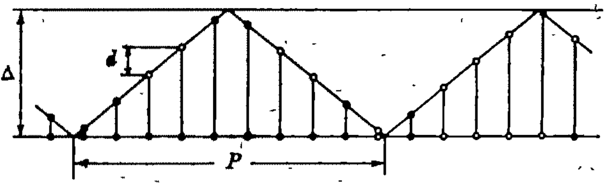

FIG. 3

The tabulated values $f(n)$ have, of course, in general a much larger period than P because P is in practice never an integer. If

$$P = 2\Delta/d = \pi/Z$$

where π and Z are relative prime integers, we see that $\pi = PZ$ is the smallest integer after which the sequence $f(n)$ repeats all its values. We thus call π the "number period," Z the "wave number," that is, the number of waves of length P contained in π. Ordinarily π is a large number. If we, for example, know that one year contains 12;22,8 mean synodic months we have 12;22,8 = 46,23/3,45 which shows that π = 46,23 = 2783 and Z = 3,45 = 225. This means that "2783 mean synodic months = 225 years" is the smallest period relation on which a zigzag function with P = 12;22,8 is built.

Functions of this type are especially used in the lunar theory. Two coexisting "systems" are known. One, called "System B," uses for the variable solar velocity a linear zigzag function; "System A," however, assumes a constant solar velocity for an arc of the ecliptic, with a discontinuous change to another value, constant on the remaining arc. The assumption of system B seems much more natural than the very bold assumption of sudden jumps in the solar velocity. Actually, however, the theory of system B is much more involved in its purely mathematical consequences. The reason can be outlined as follows. Let $f(t)$ be a periodic function of t, for example, the latitude of the moon, represented by a linear zigzag function of period p_0. An "ephemeris" is a table of values of $f(t)$ for equidistant values of t, for example, for all mean conjunctions. If the solar velocity varies from a value w to W then the lunar velocity must also vary from

$360+w$ to $360+W$ in order to maintain the same mean distance between conjunctions. On the other hand the anomaly of the sun must not influence the mean distance between consecutive nodes. Developing the consequences of this requirement one finds that the tabulated values of $f(t)$ again form a linear zigzag function but with difference d where the solar velocity is w, and with difference D where the solar velocity is W. It then holds that

$$D - d = \frac{\Delta}{3{,}0\, p_0} \cdot \frac{Y}{Y+1} (W - w)$$

where Δ is the amplitude of $f(t)$ and Y the length of the solar year. This formula shows immediately the essential difference between the systems A and B. If the solar velocity is restricted to two values only, fixed in their relation to the ecliptic (system A), then two differences suffice for the computation of $f(t)$. If, however, the solar velocities form a linear zigzag function (system B) then the same holds for the differences of $f(t)$ and $f(t)$ is a difference sequence of second order (Neugebauer [27]). Because we are actually dealing in these tables with arithmetical functions, instead of continuous functions, it is not at all trivial to determine the parameters of the zigzag function $g(n)$ of the differences of $f(n)$. The solution of this problem is found by considering mean periods and mean differences, taken over a whole number period. The mean value μ_g of g is then defined by

$$\mu_g = 2\Delta \frac{1 - p_0}{p_0} = d_0$$

where d_0 is the mean slope of $f(n)$. Arithmetical problems of this type were obviously the reason for assuming in system A a discontinuous change in the solar velocity at two points only instead of adopting the model of system B with its much greater mathematical complications.

In dealing with difference sequences of second order, Babylonian astronomers came close to problems whose importance was again seen only in the early days of the development of calculus. As an example may be quoted two sequences in system B, called H and J, whose astronomical significance is here without interest. H is a linear zigzag function with minimum 0 and maximum Δ_H. The values of H are the differences of J. The mean value of J is 0 and its amplitude is determined in such a way that increasing branches change to decreasing branches and vice versa whenever H is zero. In other words H behaves like the derivative of J. Fig. 4 shows in the lower part the function H, in the upper part the function J. I have chosen an example where an error has occurred at the decreasing branch of the second wave of J. The correct value (dotted line) would have been negative. Instead, the computer took the positive sign and continued thereafter according to the rule. Our graph shows how strongly the shape of the curve is affected by this error. On the other hand, it underlines the agreement between the extrema in J and the zeros in H in the normal case though we are dealing only with discrete values and not with continuous functions. The question arises how one could determine the amplitude Δ_J of a sequence of discrete values to a given function H such that nevertheless a behavior of this type was achieved. The answer can be given as follows. We simply require that the relation between H and J be correct "in the mean" over the whole number period τ of H. In an interval of length τ the function J must oscillate, for example, from maximum to maximum $Z/2$ times because two waves of H correspond to one oscillation of J. The total change of values of J, caused by H, is the total of all differences, thus $\sum H = \mu\tau$ where μ is the mean value of H. This total change for a single wave of J is $2\Delta_J$. Thus we have

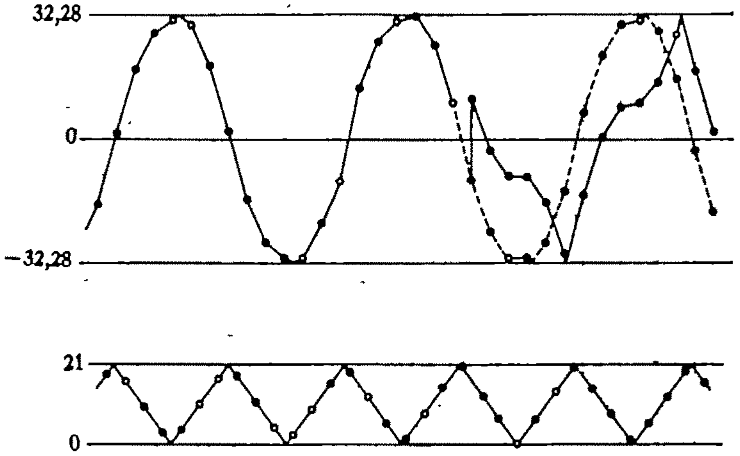

FIG. 4

$$2\Delta_J \cdot Z/2 = \mu\tau \quad \text{or}$$
$$\Delta_J = \mu\tau/Z.$$

This is actually the relation which is satisfied by the parameters of the sequences found in lunar ephemerides.

Many examples could be quoted for the extensive use which Babylonian astronomers made of difference sequences to represent periodic functions. We have examples of modified zigzag functions. The main part increases or decreases linearly. In order to avoid, however, the sharp change of direction at the extrema, sequences of second order which bridge the neighborhood of the extrema are constructed.

The idea of operating with interpolations obviously influenced the whole procedure of describing planetary movements. Apparently no attempt was made to find the longitude λ of a planet directly as function of t. The following very clever indirect method was invented. Instead of dealing with the total motion, single "phenomena" were considered as if they were independent celestial bodies. In the case of Mercury, to mention a specific example, four phenomena are considered: first and last appearance as morning star, and first and last appearance as evening star. Now, for example, the first appearance as evening star is taken independently, and rules are given to compute tables for the dates and longitudes of this single phenomenon. Here again, there exist different "systems" for the description of the dependence of these coordinates on the zodiac, that is, implicitly on the longitude of the sun. We have cases of step functions or linear zigzag functions, obviously several systems being used for the same planet. After the coordinates of one phenomenon were found for a period of time, the coordinates for the remaining phenomena were computed in a similar way, though generally with different functions, for the same period. The result is a complete list of all phenomena which now could also be read in their natural order. Finally, interpolation is used for the coordinates of the planet for intermediate dates.

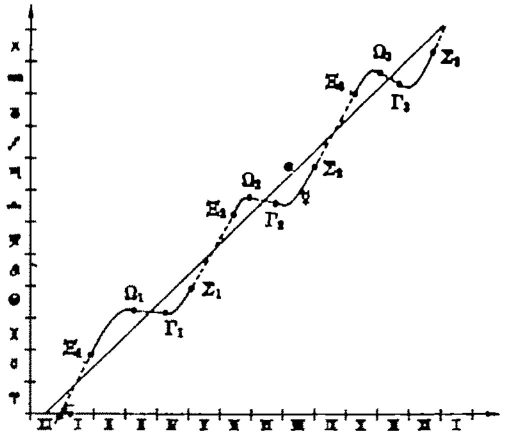

FIG. 5

This whole procedure can best be illustrated in an example for

Mercury (cf. Fig. 5). In the first step all points Ξ_i were computed. Three different zones of the zodiac are distinguished such that the distances between consecutive points Ξ depend upon the zone to which they belong. Different zones determine the distribution of the points Γ_i. Eventually, the points Ω_i and Σ_i can be found from the corresponding points Ξ_i and Γ_i respectively. Thus we end up with a sequence of irregularly distributed points near the straight line which represents the movement of the sun. The variability of the distances between these points causes also variable differences between consecutive points for different phenomena, for example, Ω and Γ. Because the planet is retrograde between Ω and Γ, this indicates different velocities of retrogradation. In order to find these velocities, schemes of interpolations are used. Fig. 6 shows an example; the retrograde motion is represented by the constant velocity $-0;6$ degrees per day. Another section of constant velocity $(+1;45^{\circ/d})$ is assumed from Σ to Ξ. In both sections the planet is near the sun and therefore invisible. For the other parts of the orbit, linearly variable differences are used (except for small irregularities near the end points, probably due to the practical requirement of relatively round numbers for the interpolation). As a result, day-to-day positions (Fig. 7) which represent a very satisfactory representation of the actual movement are obtained. It is clear that these methods require the knowledge of the summation formulae of arithmetic progressions.

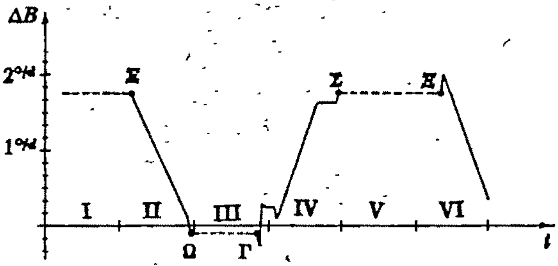

Fig. 6

These examples suffice to characterize the principle methods of Babylonian astronomy. For details of the planetary theory one may consult Kugler [20], Pannekoek [34], van der Waerden [52; 53]. The historical influence of these arithmetical methods is very great. They opened, for the first time, the way for a consistent numerical treatment of astronomical phenomena. This influence is still felt today in the use of sexagesimal units in the measurement of time and angles. In antiquity this influence was much more strongly visible in many cases which have left no trace in the modern development. Ptolemy's table of refraction, for instance, assumes that the angles of the reflected ray form a second degree sequence as function of the angle of the incident ray (Lejeune [21]). Babylonian methods are predominant in astronomical computations for astrological purposes for the obvious reason that they are easier to handle and do not require full understanding of complicated geometrical models. Greek papyri (Knudtzon-Neugebauer [17], to be supplemented by P. Ryl. 27 [14]) show the use of Babylonian period relations. Ancient and mediaeval geography was deeply influenced by concepts based on Babylonian methods. We shall return to these "linear methods" in the next section.

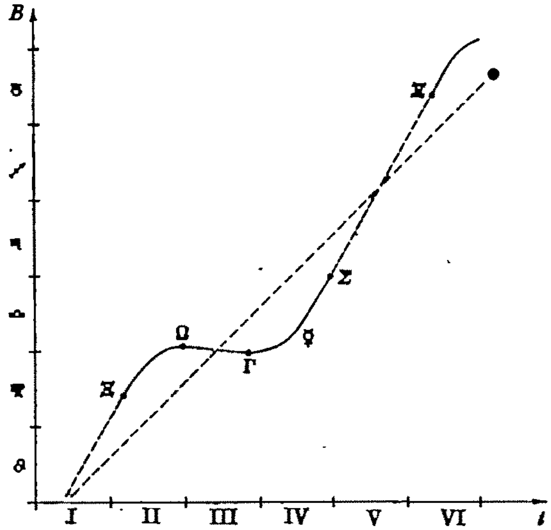

FIG. 7

B. NUMERICAL AND GRAPHICAL METHODS

5. Introduction; the linear methods. The fundamental role of the eccenters and epicycles for Greek astronomy and the dramatic history of the discussion of these geometric assumptions in the Renaissance have left us with the impression that Greek astronomy was mainly geometrical in character. Though it is evident that the Greek astronomers must have felt that their models somehow reflected physical facts, at least in theory the geometric models were only the tools for the computation of tables. Consequently we find many tables in the *Almagest* whose construction involved an enormous amount of numerical work. According to the contents they fall naturally into tables for spherical astronomy (chords, rising times, zenith distances, and so on), tables for the sun (mean motion, equation of center), for the moon (mean motion for longitude, anomaly, argument of latitude, anomaly, equation of center, evection, parallax, mean conjunctions, eclipses) and for the planets (mean motion, anomaly, stationary points, elongation, latitudes, heliacal risings and settings).

The Babylonian influence on Greek numerical methods is obvious from the use of the sexagesimal division of all units. Unfortunately this influence did not fully penetrate all computational steps. Thus often we find fractions expressed by means of unit fractions, replacing, for example, 0;42 by $1/2+1/5$. Though this notation never occurs in tables, it influenced the accuracy of single calculations because frequently results are rounded off in order to obtain simple unit fractions. Thus, for example, Ptolemy finds in *Almagest* IV, 6 the value 5;13 or 5;14 for the radius of the lunar epicycle (Heiberg [**35**, pp. 313 and 322]) but he later always uses 5 1/4(=5;15) as a round value. In general it must be said that the ancients were little concerned about the influence of rounding off and accumulated errors. Often the errors are of the same order of magnitude as the effect under consideration. Apparently it was only under the influence of modern analysis that we have learned to consider the evaluation of errors as an essential part of numerical methods. On the other hand it must be said that, for example, excellent approximations of square roots were developed very early. The Babylonians of the second millenium B.C. already used alternating geometric and harmonic means (Neugebauer [**26**, p. 33 ff.]) and many values in the *Almagest* can be explained by this method. It would not be surprising if this technique reached the Greeks together with the sexagesimal system and it might be significant that the value $2^{1/2}=1;21,50,10$ is not only found in cuneiform records (Neugebauer-Sachs [**32**, p. 32]) but is used by Ptolemy in the computation of the chord of 90° (Heiberg [**35**, p. 35, 15]).

The arithmetical methods of the Babylonian astronomers are also discernible in another field of ancient astronomy. We have already mentioned the "seasonal hours" for ancient time measurement which thus requires a knowledge of the law of variation in the length of daylight whenever astronomical computation with equinoctial hours was needed. For the Babylonian astronomers the length of daylight was of great importance also because one of the main goals of their lunar theory was the prediction of the evening of first visibility of the new crescent of the moon after conjunction. This evening defined the beginning of a new month and consequently the whole calendar depended on this problem. For its solution the knowledge of the moment of sunset is, of course, required. Because the length of daylight is the time from sunrise to sunset, one also may ask for the time it takes the semicircle of the ecliptic from λ to $\lambda+180$ to rise if λ is the longitude of the sun at the given moment. A month later the longitude of the sun will be given roughly by $\lambda+s$ where $s=30^\circ$ is the length of one zodiacal sign. Now the length of daylight is the rising time of the semicircle from $\lambda+s$ to $\lambda+s+180$. Its value can be obtained from the previous value by adding the rising time of the zodiacal sign $s+180$ and by subtracting the rising time of s. Hence we see that the length of daylight can be found if the rising times of ecliptic arcs are known. The rising time of a given arc of the ecliptic depends obviously on its variable inclination to the horizon. To obtain an insight into the relationship between rising times, seasons and geographical location can be called the central problems of early Greek spherical astronomy. Its complete solution is found in the tables of the *Almagest* where the rising times are given for every 10 degrees for all latitudes whose longest daylight varies between 12^h and 17^h in steps of $1/2^h$.

Ptolemy already makes full use of spherical trigonometry. The Babylonian astronomers, however, used also here arithmetical schemes to describe the values of the rising times of the zodiacal signs as function of the longitude. Two methods were developed: a crude approximation related to system A of the solar theory, and a more refined scheme in system B. Both are built on arithmetical progressions. The Greeks expanded these methods by varying the parameters linearly, thus introducing geographical zones of given length of daylight, known as "climates." This concept remained fundamental for ancient and mediaeval mathematical astronomy (Honigmann [13], Neugebauer [29]). Ptolemy himself uses the rising times of system A in the Tetrabiblos ([40, I, 20 p. 94–95]; [39, I, 21 p. 46]) and thus contributed to securing the survival of Babylonian methods for many centuries.

6. Spherical trigonometry. The lack of a convenient algebraic notation prevented the Greeks from condensing the solution of a general triangle into a single formula instead of solving two right triangles. Of greater consequence was their use of chords in a circle of radius 60 instead of the trigonometric functions. Consequently their plane trigonometry is expressed in the relations

$$(1)\qquad a = \frac{c}{120}\,\mathrm{crd}(2\alpha),$$

$$(2)\qquad b = \frac{c}{120}\,\mathrm{crd}(180 - 2\alpha),$$

$$(3)\qquad \frac{a}{b} = \frac{\mathrm{crd}(2\alpha)}{\mathrm{crd}(180 - 2\alpha)},$$

$$(4)\qquad c^2 = a^2 + b^2.$$

Tables for crd α are given, for example, in *Almagest* I, 11 in steps of 1/2° and with an accuracy of two sexagesimal places. Their computation is based on the so-called Ptolemaic theorem for a quadrilateral and its diagonals inscribed in a circle.

As plane trigonometry is described by the four above relations (1) to (4), so spherical trigonometry contains four similar relations which we may describe in our symbols by

$$(1a)\qquad \sin\alpha = f(a, c),$$

$$(2a)\qquad \cos\alpha = g(b, c),$$

$$(3a)\qquad \tan\alpha = h(a, b),$$

$$(4a)\qquad \cos c = \cos a \cos b.$$

Menelaus, about 100 A.D., already knew that a spherical triangle is determined by its angles (Spherics I, 18 Krause [18, p. 138]). Yet relations of the type

$$(5a)\qquad \cos a = \phi(\alpha, \beta),$$

$$(6a)\qquad \cos c = \psi(\alpha, \beta)$$

do not seem to have been discovered before the Arabs.

In general, it is my impression that spherical trigonometry was completed rather late in the development of Greek science. Its problems and methods were exclusively determined by astronomical needs and astronomical concepts. This is obvious in the earliest treatises, about 300 B.C., by Autolycus [2] and Euclid [9], but it also holds for Theodosius ([49], Schmidt [46]), who probably was a younger contemporary of Hipparchus (Ziegler [56]). The astronomical importance of parallel circles certainly contributed to obscuring for a long time the necessity of restricting oneself to great circles, a discovery which very well may have been Menelaus' great contribution. Even Ptolemy did not yet have a clear concept of the possibility of replacing spherical triangles by plane triangles. In the computation of the components of parallax in *Almagest* V, 19 (Heiberg [**35**, p. 456 f.]), he treats a right spherical triangle with one very small angle, but with two large sides, as if it were a plane triangle in order to find the second angle at the small base, though this is just the case where also the second angle is very close to 90°. It has often been conjectured that already Hipparchus was able to solve problems of spherical trigonometry. This seems rather implausible in view of the above-stated facts, and indeed we shall see (p. 1036) that there existed methods to avoid completely spherical trigonometry. Hence I see no reason for assuming that the central theorem of ancient and Arabic spherical trigonometry, the "Menelaus theorem," was known before Menelaus. That its two forms are not independent was remarked by Theon ([**43**, p. 569], Rome [**44**]).

7. **The "Analemma."** At least since the early part of the third century B.C., astronomers were able to predict solar and lunar positions with a high degree of accuracy. In particular, the longitude of the sun could be considered as known for any given date. Ecliptic coordinates, however, are not directly visible in the sky. The daily rotation moves all celestial objects around the pole of the equator, and thus relates the measurement of time to equatorial coordinates. Yet local noon is again determined by the sun and thus, by the relation of ecliptic coordinates to the local coordinates of the observer, horizon and meridian. All these coordinates play a role in the practical measurement of time by means of sun dials, the simplest form of which is the vertical "gnomon" on a horizontal plane. Finally, different observers had to establish their relative positions through the determination of their geographical coordinates. It is therefore no great wonder that we can observe that ancient astronomers concentrated a great deal of attention on the theory of spherical coordinates in their relation to celestial and terrestrial objects and the theory of sun dials.

Ptolemy's role in this branch of astronomy can be well appreciated because we have not only a work of his own, called the *Analemma* ([**38**, p. 187 ff.], Luckey [**22**]) but we also know a little about his predecessors. We can see that Ptolemy had all the essential methods, inherited from earlier times, but loaded down with historical relics which made their application unnecessarily clumsy. Ptolemy rationalized the whole procedure, down to the smallest details. First of all, he introduced coordinates whose mutual relations are independent of the geographical position. We consider the octant of the celestial sphere which contains the sun Σ, and whose vertices are the zenith Z, South S, and East E (cf. Fig. 8 left). From each of these points we draw a great circle to the sun, calling the respective arcs "descensivus" (= zenith distance), "horarius" and "hectemoros." Each pair of these arcs may be used to determine the position of the sun.

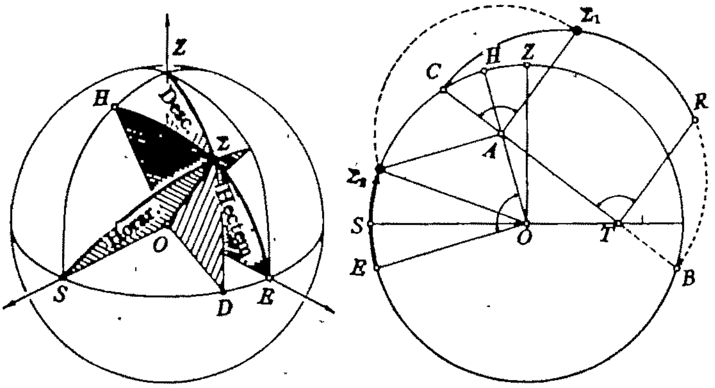

FIG. 8

This perfectly symmetric arrangement can be compared with the older system. There the position of the sun was given by the horarius and the arc of the equator from E to the plane of the horarius. Only for the equinoxes does this arc end at the sun (then being identical with the hectemoros), but ordinarily it meets the horarius in a point which is not only different from the sun but which also depends on the geographical latitude of the observer. Another pair of "old" coordinates was descensivus and "antiskion," that is, the azimuth of the direction of the shadow counted from S to D.

Ptolemy now treats in great detail the methods to determine his coordinates for a given solar position. The right part of Fig. 8 describes his solution for the hectemoros. Let the circle around O be the plane of the meridian. The geographical latitude of the observer determines the inclination of the equator, and from the solar tables we can find the longitude of the sun for the given moment. As we shall see presently, this also determines the position of the path of the sun, henceforth called its "daily circle." Its intersection with the plane of the meridian may be CB. Ptolemy now proceeds in the typical fashion of "descriptive geometry." He revolves the plane of the daily path of the sun about its trace CB into the plane of the meridian. C is the culminating point; vertically above A, we have the given position Σ_1 of the sun which rose at R, the arc BR being below the horizon. Because A is the orthogonal projection of the sun onto the plane of the meridian, we know that A belongs also to the plane of the hectemoros. Because also O belongs to both the plane of the meridian and the plane of the hectemoros we have in OAH the trace of the plane of the hectemoros in the plane of the meridian. Perpendicular to this trace is the line OE where E represents the East point of the horizon, revolved about OH into the plane of the meridian. The distance $A\Sigma_1$ is a true distance. Using it as a radius we find on the celestial sphere Σ_1, which gives the position of the sun in the plane of the hectemoros, turned into the plane of the meridian. Thus the arc $E\Sigma_1$ is the true "hectemoros" we wanted to find.

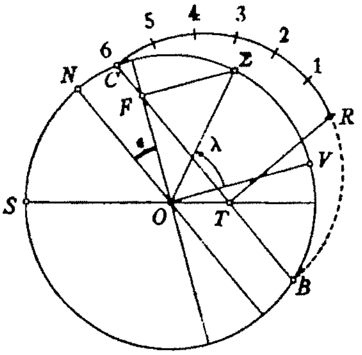

FIG. 9

A similar procedure can be followed for the other angles. The method consists, in general, in constructing first the Cartesian coordinates of the sun with respect to the fundamental planes of the octant ZSE and then revolving the Cartesian coordinate planes about the coordinate lines in order to get the angles in true size.

It follows from the preceding discussion that the whole construction of all Ptolemaic coordinates requires only the main circle with center O and the parallel circle whose diameter CB is determined by the time of the year, that is, by the longitude of the sun. Ptolemy thus proceeds to construct a nomogram whose central part is the great circle with center O. Added are half circles of the type BC corresponding to the solar path from month to month. This system of circles is mounted on a turntable with various scales for the reading of the angles. This makes it possible to bring the equator in the proper position towards a given horizon. Rectangular plates placed on the proper points of the scales then allow the direct determination of the angles required (cf. for details and for a figure Luckey [22]). A mechanized age can only be delighted with the extremal efficiency of this apparatus, of which nothing has come down to us except for an incomplete description in a Greek palimpsest and a poor Latin translation by William of Moerbeke.

Ptolemy is not the inventor of descriptive geometry. More than a century before his time, Vitruvius describes in his *De architectura* the Analemma in connection with the construction of sun dials ([54]). From Vitruvius we also learn how to determine graphically the path of the sun for a given longitude λ. Let the circle around O again be the meridian (Fig. 9); then the trace ON of the equator is given for a given place. If ϵ denotes the obliquity of the ecliptic, we also know the trace OF of the ecliptic. Revolving the plane of the ecliptic into the plane of the meridian we find the vernal point V on the end point of the radius OV perpendicular to FO. Hence Σ is the sun for given λ, and FB the trace of its path. If we divide RC into six equal parts we have the position of the sun for each seasonal hour from sunrise (R) to noon (C).

The theory of sun dials leads to the solution of two problems in the history of mathematics, problems whose fame is inversely proportional to their interest: the date of Heron of Alexandria and the origin of the conic sections.

It is in itself of very little importance to establish accurately the date of a rather mediocre author whose role in the history of science is due only to the fact that so much else is lost. Yet it is somewhat unsatisfactory to know no more about an often quoted writer than that he lived sometime between -200 and $+300$. It is therefore a pleasant side result of the study of an analemma, described by Heron in his *Dioptra* ([12], Rome [42], Neugebauer [28]), to see that the elements which he quotes for a lunar eclipse fit exactly one and only one eclipse between -200 and $+300$, namely the partial eclipse (magnitude .8 digits) of A.D. 62 March 13. It is extremely plausible to assume that the writing of the Dioptra fell close to the occurrence of this eclipse. This is indeed the only excuse for selecting an example whose date coincides almost with equinox, because the main problem in the analemma discussed by Heron consists in reducing local seasonal time of one place (Rome) to the local seasonal time of another locality (Alexandria); this problem loses its importance only twice a year, namely at the equinoxes, when seasonal time and equinoctial time coincide. Only the desire to quote a real eclipse, which had occurred recently, can have led to quoting as an example a date which fell only a few days before equinox. If we thus must place Heron into the end of the first century A.D. we support at the same time an argument of J. Klein [16, p. 135, note] who conjectured that Heron and Diophantus were contemporary, both belonging to the time of Nero.

For us the interest in Heron's analemma consists in the direct relationship it establishes between astronomy and mathematical geography. The only method in antiquity to determine the geographical longitude of a place consisted in using lunar eclipses as time signals. The comparison of two local seasonal times is made by Heron by means of a hemispherical sun dial combined with an analemma of the Vitruvian type. Its main idea is to use the analemma for each place and thus to find the position of the local meridian with respect to the sun. Transferring the results into the hemispherical dial gives directly the angular difference between the meridians. This is one of several examples which show the intimate combination of geometrical construction and the direct use of globes or hemispheres in Greek astronomy. In general it may be said that Greek mathematics is often much less "pure" than is generally assumed.

This latter remark might be kept in mind when we try to relate one of the most interesting subjects of Greek geometry, the conic sections, to an astronomical origin. It is well known that these curves were defined by Apollonius (about 200 B.C.) as the intersection of a circular cone by planes of variable inclination, whereas the "old" geometers considered only right circular cones, intersected by a fixed plane perpendicular to one generating line. The different types of these curves were obtained by varying the angle at the vertex (cf., for example, Heath [1]). This definition suggests immediately that its origin is to be sought in the fixed right angle between the gnomon and the receiving plane of the shadow. Indeed, we have only to point the gnomon towards the culminating point to obtain exactly the configuration required (Neugebauer [30]). The sun travels on its "daily circle," its rays form the right circular cone, whose vertex is the tip of the gnomon. At noon the gnomon falls into one generating line and the receiving plane is perpendicular to it. The shadow describes a hyperbola. The natural question, namely, how these curves depend on the declination, is equivalent to asking how the angle at the vertex of the cone influences the shape of the curve. This is exactly the form in which the conic sections were studied by Menaechmus (about 350 B.C.).

8. **The "Planisphaerium."** The importance of spherical astronomy is reflected in the manifold of mathematical tools developed in order

to solve its main problems. Eventually spherical trigonometry superseded all other methods which were, in all probability, invented before spherical geometry was sufficiently far advanced to reach numerical results. At that early stage the methods of descriptive geometry, reflected in the *Analemma*, might seem the most natural way to transform arcs on the sphere into arcs of one plane. Another work of Ptolemy, the *Planisphaerium* ([38, p. 227–259]; Drecker [5]), shows, however, that also stereographic projection was known, in particular its important quality of mapping circles into circles, straight lines included. No proof is given for this fact by Ptolemy, a clear indication that he is operating on well known grounds, developed long before his time.

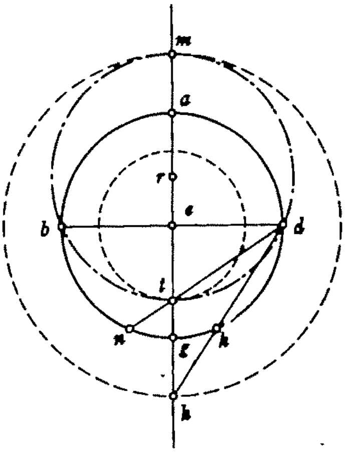

FIG. 10

The projection chosen maps the whole sphere onto the plane of the equator with the south pole as center of projection. In order to determine the center and the radius of the image of a circle, descriptive methods are again employed. As a simple example might be quoted the determination of the solstitial circles and of the ecliptic. Let $abgd$ be the equator (Fig. 10); we then consider the same circle also as a picture of the meridian, turned about the diameter ag into the plane of the equator. Consequently d represents the south pole. Let $gh = gn = \epsilon = 23;51°$. The point h is therefore a point of the diameter of the winter solstitial circle, n of the summer solstitial circle. Their projections from d are k and t respectively. Because all parallel circles have their center in e, the images of both circles are found. Because the ecliptic must be represented by a circle touching the solstitial circles in l and m respectively, also this circle is known. It is then proved that the straight line bd, b and d being the equinoxes, passes through e, a general relation which is often used in the following for intersecting great circles. The conformity of the mapping was apparently unknown.

The main goal of the whole procedure is the determination of the rising times for the zodiacal signs. In order to find the representation of these signs one has only to construct the parallel circles of given declination by the same method which was used for the solstitial circles. The declinations of given ecliptic points are considered known, but it is in principle important to remark that they can also be found by geometric construction, namely from the "daily circle" in the analemma (cf. above p. 1031). Finally it is easy to construct the circle which represents the horizon for a given latitude. The variable positions of the ecliptic with respect to the horizon at different times of the year are in our projection represented by different positions of the horizon circle with respect to the fixed image of the equator-ecliptic system. In order to find the rising times of a given arc of the ecliptic we have only to construct the two positions of the horizon passing through its end points. These two horizon circles intersect the equator in two points whose angular distance is the rising time in question. Because angles on the equator are represented without distortion, our problem is solved by this construction.

The above description shows that the "planisphaerium" could be used for a purely graphical or mechanical solution of problems of spherical astronomy. This was indeed the use made of this method especially by the Arabs whose "astrolabes" are based on the projections described here (Drecker [6] and [7], Michel [23]). In Ptolemy's treatise, however, a different attitude is taken. The geometric constructions are only used for transforming spherical problems into problems of plane geometry which then are solved numerically by means of plane trigonometry. Obviously we have here before us the method used before spherical trigonometry was invented, that is, before the Menelaus theorem was known.

Much speaks in favor of the assumption that the planisphaerium was the tool of Hipparchus (Delambre [4, II p. 453 ff.], Drecker [6, p. 16 ff.]). All computations are based on the latitude of Rhodes, where Hipparchus made his observations. Synesius of Cyrene ascribes the invention of the "astrolabe" to Hipparchus (FitzGerald [10, p. 263]); this statement is certainly to be taken seriously in view of the fact that Synesius was a pupil of Hypathia, who collaborated with her father Theon on the commentaries to the Almagest (Rome [43,

p. LXXXIII]). Finally, Hipparchus determines the positions of stars by a combination of ecliptic and equator coordinates (Vogt [55]); he takes the longitude of the point where the circle of declination through the star meets the ecliptic and then uses the remaining declination as second coordinate. This system finds its direct explanation in the planisphaerium: the first coordinate is given on the image of the ecliptic whereas the circles of declination are mapped into radii.

9. **Map projection.** The work of Mžik and Hopfner [25] has given us a good understanding of Greek mathematical geography as contained in the first book of Ptolemy's *Geography*. I shall give a short summary, using modern terminology, which will suffice to show how far the Greeks had come in the general problem of mapping a sphere onto a plane. From the earlier development we only know that Ptolemy's predecessor, Marinus (about 100 A.D.), used a cylinder projection which can be described by

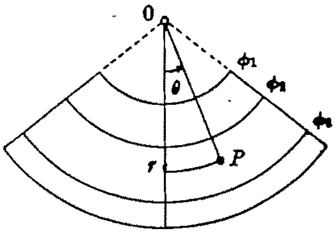

FIG. 11

$$y = \phi, \qquad x = \lambda \cos 36^\circ$$

where x and y are the Cartesian coordinates of the map, ϕ the geographical latitude, λ the geographical longitude, and $\phi = 36^\circ$ the latitude of Rhodes. Obviously this projection preserves latitudes on all meridians and longitudes for the parallel of Rhodes.

Ptolemy introduced two types of conic projection. In the first type he maps meridians on radii, parallels of latitude on circles with center O (Fig. 11). In order to determine the parameters of this mapping three conditions are imposed. We introduce polar coordinates r and θ, and call $\bar{\phi} = 90 - \phi$ the colatitude.

Condition 1: preservation of length on all meridians

$$r = \bar{\phi} + c. \tag{1}$$

Condition 2: preservation of length for the latitude $\phi_2 = 36^\circ$ of

Rhodes. Thus

$$r(\bar{\phi}_2)\cdot\theta = \sin\phi_2\cdot\lambda$$

or from (1)

$$\theta = \frac{\sin\bar{\phi}_2}{\bar{\phi}_2 + c}\cdot\lambda. \tag{2}$$

Condition 3: preservation of the ratio of lengths on the parallels of Thule ($\phi_1 = 63$) and on the equator ($\phi_2 = 0$). Hence c can be determined from

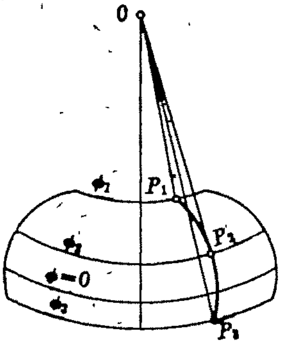

FIG. 12

$$\frac{\sin\bar{\phi}_1}{\sin\bar{\phi}_2} = \frac{\bar{\phi}_1 + c}{\bar{\phi}_2 + c}. \tag{3}$$

It is obvious that this projection is an improvement of Marinus' simple cylinder projection.

The second type of conic projection, proposed by Ptolemy, assumes that both meridians and parallels of latitude are represented by circles. We again use polar coordinates (Fig. 12) but only the circles $r = \text{const.}$ now have geographical significance as images of circles of constant latitude. The radii, however, are no longer meridians. Nevertheless it is required that length is preserved on all radii. Thus we have

$$r = \bar{\phi} + c_0 \tag{4}$$

where c_0 is an arbitrary parameter which gives the value $r = r_0$ of the image of the equator. In order to determine the circles which represent the meridians we need three points. Thus we can impose the following three conditions: preservation of length on each of the following three parallels $\phi_1 = 63$ (Thule), $\phi_2 = \epsilon = 23;50$ (Syene in Egypt), and $\phi_3 = -16;25$ ($\phi = 16;25$ is the latitude of Meroe on the Upper Nile). Thus

$$r(\bar{\phi}_i)\cdot\theta = \sin \bar{\phi}_i. \tag{5}$$

For a given value of λ we can find three values of θ by substituting in (5) the values $\bar{\phi}_1 = 27$, $\bar{\phi}_2 = 66;10$ $\bar{\phi}_3 = 106;25$. The values $r(\bar{\phi}_i)$ are similarly given by (4); thus we are able to construct the circle $P_1P_2P_3$ which represents the meridian λ. If (5) were assumed to hold for all latitudes, the meridians would no longer be circles but we would obtain the Bonne projection, which is preserving length in radial direction and on all parallels.

In concluding, I want to repeat that the topics discussed here were not intended to give a complete picture of mathematical problems which originated from problems of astronomical character. Nevertheless, they may suffice as an illustration of the close relationship between mathematics and astronomy in antiquity, a relationship which thereafter never lost its importance. "Astronomy proceeds to its demonstrations in no uncertain way, for it uses as its servants geometry and arithmetic, which it would not be improper to call a fixed standard of truth" (Synesius of Cyrene in his letter to Paeonius, about 400 A.D.; FitzGerald [10, p. 262]; cf. *Almagest* I, 1; Heiberg [1, p. 6, 17 ff.]).

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BROWN UNIVERSITY

THE SUMMER MEETING IN MADISON

The fifty-fourth Summer Meeting and thirtieth Colloquium of the American Mathematical Society was held at the University of Wisconsin, Madison, Wisconsin, Tuesday to Friday, September 7–10, in conjunction with the summer meetings of the Mathematical Association of America, the Institute of Mathematical Statistics and the Econometric Society. The Mathematical Association of America met on September 6–7, the Institute of Mathematical Statistics and the Econometric Society on September 7–10. About 700 people attended these meetings, among whom were the following 431 members of the Society.

C. R. Adams, R. B. Adams, V. W. Adkisson, J. E. Adney, J. G. Adshead, R. P. Agnew, A. A. Albert, F. E. Allen, W. R. Allen, C. B. Allendoerfer, R. V. Andree, Richard Arens, K. J. Arnold, M. C. Ayer, W. L. Ayres, H. M. Bacon, R. H. Bardell, A. S. Barr, J. J. Barron, R. C. F. Bartels, Walter Bartky, M. A. Basoco, Leon Battig, H. P. Beard, H. M. Beatty, E. F. Beckenbach, E. G. Begle, J. H. Bell, A. A. Bennett, A. F. Bentley, Stefan Bergman, D. L. Bernstein, Lipman Bers, H. R. Beveridge, F. C. Biesele, R. H. Bing, M. C. Blackstock, A. L. Blakers, J. W. Bradshaw, Richard Brauer, R. W. Brink, Foster Brooks, M. C. Brown, R. H. Bruck, H. E. Buchanan, R. C. Buck, P. B. Burcham, F. J. H. Burkett, L. J. Burton, L. E. Bush, Hobart Bushey, J. H. Bushey, W. H. Bussey, A. S. Cahn, S. S. Cairns, W. D. Cairns, R. H. Cameron, C. C. Camp, H. E. Campbell, R. E. Carr, W. B. Carver, E. D. Cashwell, T. E. Caywood, Subrahmanyan Chandrasekhar, K. Chandrasekharan, J. O. Chellevold, Herman Chernoff, Sarvadaman Chowla, R. V. Churchill, C. E. Clark, F. M. Clarke, H. E. Clarkson, Nathaniel Coburn, C. J. Coe, H. J. Cohen, I. S. Cohen, L. A. Colquitt, B. H. Colvin, E. G. H. Comfort, H. H. Conwell, G. R. Costello, V. F. Cowling, C. C. Craig, J. S. Cronin, A. B. Cunningham, J. H. Curtiss, E. H. Cutler, G. B. Dantzig, J. A. Daum, P. H. Daus, Robert Davies, D. R. Davis, James Elmer Davis, L. A. V. DeCleene, D. B. DeLury, R. F. Deniston, A. H. Diamond, L. L. Dines, G. P. Dinneen, J. M. Dobbie, C. L. Dolph, J. L. Doob, H. L. Dorwart, D. M. Dribin, R. J. Duffin, Nelson Dunford, W. L. Duren, W. H. Durfee, Ben Dushnik, P. S. Dwyer, W. F. Eberlein. M. C. Eide, Samuel Eilenberg, Churchill Eisenhart, M. P. Emerson, J. H. Engel, Paul Erdos, W. S. Ericksen, H. P. Evans, H. S. Everett, G. M. Ewing, A. B. Farnell, J. V. Finch, C. H. Fischer, S. G. Fleming, L. R. Ford, W. C. Forman, J. S. Frame, Evelyn Frank, J. E. Freund, W. H. J. Fuchs, R. E. Fullerton, David Gale, H. L. Garabedian, R. E. Gaskell, H. M. Gehman, H. H. Germond, B. E. Gillam, M. A. Girschick, Wallace Givens, A. M. Gleason, Casper Goffman, V. D. Gokhale, R. A. Good, R. E. Goodman, Cornelius Gouwens, A. A. Grau, L. M. Graves, W. L. Graves, Edison Greer, L. W. Griffiths, V. G. Grove, S. G. Hacker, Elizabeth Hahnemann, D. T. Haimo, Franklin Haimo, Edwin Halfar, N. A. Hall, P. R. Halmos, E. S. Hammond, A. G. Hansen, W. L. Hart, G. E. Hay, Camilla Hayden, N. A. Haynes, C. T. Hazard, Erik Hemmingsen, D. E. Henriques, J. G. Herriot, I. N. Herstein, Fritz Herzog, A. D. Hestenes, Edwin Hewitt, E. H. C. Hildebrandt, T. H. Hildebrandt, Einar Hille, J. J. L. Hinrichsen, G. P. Hochschild, D. L. Holl, M. W. Hopkins, Harold Hotelling, K. O. Househam, A. S. Householder, C. C. Hsiung, R. C. Huffer, E. M. Hull, Ralph Hull, M. G. Humphreys, Witold

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On Tuesday afternoon, Wednesday, Thursday and Friday mornings, Professor Richard Brauer of the University of Michigan delivered the Colloquium Lectures on *Representations of groups and rings*. The presiding officers at these lectures were Professor C. C.

MacDuffee, President Einar Hille, Professor Saunders MacLane, and Professor A. A. Albert respectively.

The Society held a joint session with Section A of the American Association for the Advancement of Science on Wednesday morning, at which time Professor R. L. Moore of the University of Texas delivered a retiring address (as Vice President of the American Association for the Advancement of Science and Chairman of Section A) entitled *Spirals*. Professor R. L. Wilder presided.

On Thursday afternoon Professor J. W. T. Youngs of the University of Indiana lectured on *Topological methods in the theory of Lebesque area*, with Professor Tibor Rado in the chair.

Presiding officers for the sessions of contributed papers were as follows. Tuesday afternoon: Algebra, Professor George Whaples; Analysis, Professor J. L. Walsh; Probability and Statistics, Professor P. S. Dwyer. Thursday morning: Analysis, Professor L. M. Graves; Topology, Professor R. L. Wilder. Thursday afternoon: Applied Mathematics, Professor R. V. Churchill; Algebra, Professor Ralph Hull. Friday morning: Analysis, Professor G. B. Price; Topology, Geometry and Analysis, Professor J. W. T. Youngs; Unclassified Late Papers, Professor Gordon Pall.

Except for one sectional session in Room 102, Biology Building, all sessions were held in Bascom Hall, the general sessions in Room 272 and sectional sessions in Rooms 272, 165 and 312.

Registration headquarters were in Tripp Hall. Rooms were available for those attending the meetings in Tripp and Adams Halls, on the edge of Lake Mendota, and the dormitory pier was much in use for swimming. Meals were served in the Cafeteria of Van Hise Hall. Arrangements were made for tennis and golf. On Wednesday afternoon there were trips to the Forest Products Laboratory of the United States Government and to the offices of the United States Armed Forces Institute and a boat ride on Lake Mendota. On Tuesday and Wednesday evenings there were visits to the University observatory. All these activities were well attended. In addition there were during the week several spontaneously organized excursions to cheese factories within easy driving distance of Madison.

On Tuesday afternoon the ladies of the Mathematics Department served tea on the lawn behind Tripp Hall.

On Tuesday evening a capacity audience were the guests of the University of Wisconsin at a concert given in Music Hall by the Pro Arte String Quartet of the University of Wisconsin.

On Wednesday evening a dinner was held for the four organizations in the Crystal Ball Room of the Loraine Hotel, with an assembled group of over four hundred people. Dean Walter Bartky acted as toastmaster. Dean M. H. Ingraham welcomed the mathematical groups on behalf of the University of Wisconsin. Professor J. L. Walsh responded for the Society, Dr. H. M. Gehman for the Association, Professor P. S. Dwyer for the Institute and the Econometric Society. Before concluding the dinner, Dean Bartky warmly commended the efforts of the Local Committee on Arrangements and brought to their feet in turn for a hearty clap from the audience the three most valiant members of that Committee, namely Professors H. P. Evans (Chairman), B. H. Colvin and Elizabeth S. Sokolnikoff.

A group photograph of those attending the meeting was taken on Wednesday at 1:30 P.M.

A second informal luncheon meeting of the applied mathematicians of the midwestern area was held on Thursday in the small dining room of Van Hise Hall, with an attendance of about forty. Those present approved the principle of continuing such discussion groups, decided against formation of a formal organization and for continued cooperation with the Applied Mathematics Committee of the Society. They also added two new members to their informal committee of three, one addition being Associate Secretary J. W. T. Youngs.

On Thursday afternoon Professor G. B. Price proposed to those assembled for Professor Youngs' invited address a resolution, unanimously approved, expressing the appreciation of the members of the four organizations to the administration of the University of Wisconsin, the local committee and all who contributed to the success of the meetings.

A well attended picnic supper was held Thursday at Picnic Point on Lake Mendota.

On Thursday evening about two hundred people gathered in the Pine Room of Van Hise Hall for a beer party officially sponsored by the Institute of Mathematical Statistics but well attended by Society members. The atmosphere was one of song and friendly talk, and many expressed the hope that such an event would become a regular feature of the Summer Meetings.

The Council met in the lounge of Slichter Hall at 9:00 P.M. on Tuesday and at 10:00 P.M. on Wednesday.

The Secretary announced the election of the following sixty-three persons to ordinary membership in the Society:

Mr. Anderson B. Alexander, Purdue University;
Dr. Nachman (Natan) Aronszajn, Centre National de la Recherche Scientifique, Paris;
Mr. Augustus Francis Bausch, Princeton University;
Mr. Jack Bazer, New York, N. Y.;

Mr. John William Benedick, Manhattan College, New York, N. Y.;
Mrs. Maude Bryan Blondeau, University of Arkansas;
Miss Olga Bocanegra-Saldaña, University of Puerto Rico;
Mr. Clifford Breuner, Temple University;
Mr. Robert Lloyd Broussard, Louisiana State University;
Mr. James William Butler, Oak Ridge National Laboratories, Oak Ridge, Tenn.;
Professor Lucia Virginia Carlton, Centenary College, Shreveport, La.;
Mr. Salvatore Rosario Chiefa, Columbia University;
Mr. Alfred Halstead Cockshott, Manhattan College, New York, N. Y.;
Miss Natalie Coplan, National Bureau of Standards, New York, N. Y.;
Mr. John Willis Crispin, Jr., Wayne University;
Mr. Howard J. Dietsche, South Ozone Park, N. Y.;
Mr. Avron Douglis, Institute of Mathematics and Mechanics, New York University;
Dr. Helen Margaret Elliott, Harvard University;
Mr. Chester Feldman, University of Chicago;
Mr. Ivey Clenton Gentry, Duke University;
Mr. Lincoln J. Gerende, Naval Medical Research Institute, Bethesda, Md.;
Dr. Cástor Segundo Goa, Ministerio de Obras Publicas de Venezuela, Caracas, Venezuela;
Professor Ulysses S. Grant, University of California at Los Angeles;
Mr. Gerald B. Haggerty, Rhode Island State College;
Mr. Cyrus B. Hailperin, Duquesne University;
Miss Mary-Elizabeth Hamstrom, University of Texas;
Mr. Hugh G. Harp, Ohio State University;
Mr. Herbert H. Hinman, City College, New York, N. Y.;
Mr. Robert Ellsworth Holdman, New York, N. Y.;
Mr. Donald Bruce Houghton, Franklin Institute, Philadelphia, Pa.;
Professor Thomas Roland Humphreys, Bergen Junior College, Teaneck, N. J.;
Mr. Mario Leon Juncosa, Cornell University;
Mr. Richard Ellis Kloss, U. S. Weather Bureau, Raton, N. M.;
Mr. Leo Lapidus, University of Maine;
Miss Margaret Mary LaSalle, Louisiana State University;
Mr. William Vincent Leamon, Western Union Telegraph Company, New York, N. Y.;
Mr. James Edward Martin, University of Pittsburgh, Johnstown, Pa.
Mr. Emanuel Mehr, Johns Hopkins University;
Mr. Saad Luka Mikhail, University of Illinois;
Mr. Bertram William Miller, New York, N. Y.;
Mr. George Constantinovich Miloslavsky, Princeton Textile Printing Co., New York, N. Y.;
Mr. James Henry Mulligan, Jr., Allen B. DuMont Laboratories, Inc., Passaic, N. J.;
Mr. Donald Joseph Myatt, Clarkson College of Technology, Potsdam, N. Y.;
Professor Zeev Nehari, Washington University;
Professor Otton Martin Nikodým, Kenyon College;
Dr. Ilse L. Novak, Wellesley College;
Mr. Albert Raymond O'Connor, Manhattan College, New York, N. Y.;
Mr. Charles Oister, Jr., Keystone Junior College, La Plume, Pa.;
Professor Edgar Phibbs, University of Alberta, Edmonton, Alberta, Canada;
Mr. Frank Proschan, Arlington, Va.;
Dr. Clarence Ross, Dahlgren, Va.;
Mr. Walter Rudin, Duke University;

Professor Socrates Watter Saunders, Morgan State College, Baltimore, Md.;
Major Raymond Ira Schnittke, University of Chicago;
Mr. Keeve Milton Siegel, Rensselaer Polytechnic Institute;
Miss Roselyn Adeline Siegel, Institute for Numerical Analysis, University of California at Los Angeles;
Professor Donovan Bradshaw Sumner, Louisiana State University;
Mr. Paul McCullough Sutton, New York, N. Y.;
Professor Garnet Louis Tiller, Utica College, Syracuse University;
Professor Lambuth Reilly Towson, North Georgia College, Dahlonega, Ga.;
Miss Marion Treon, Murray State Teachers College, Murray, Ky.;
Mr. John Edmund Warren, Teleregister Laboratories, New York, N. Y.;
Mr. Vincent Andrew Zora, Blaw-Knox Construction Company, Pittsburgh, Pa.

It was reported that the following had been elected to membership on nomination of institutional members as indicated:

Institute for Advanced Study: Dr. Walter Rudolf Habicht, Dr. Luis Antonio Santoló.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: French Mathematical Society: Professor Marcel Emile Brelot, Institut Fourier, Grenoble; Professor Michel Loève, University of California; Dr. Oleg I. Yadoff, Columbia University; Wiskundig Genootschap te Amsterdam: Dr. Jan Arnoldus Schouten, Epe, Netherlands.

The following actions of the Council taken by mail vote since the April meeting were reported: that the Proceedings of the Second Symposium on Applied Mathematics be published under the auspices of the Society; that the Society contribute to the Canadian Journal of Mathematics an annual subvention of $1,000 for a period of three years, with presumption of renewal for two additional years; that Professor Oswald Veblen be named the nominee of the American Mathematical Society for President of the International Congress of Mathematicians to be held in Cambridge, Massachusetts, in September, 1950.

The following appointments by President Einar Hille of persons to represent the Society were reported: Professor G. C. Evans at the Pacific Regional Conference on UNESCO in San Francisco on May 13–15, 1948; Professor C. V. Newsom at the inauguration of T. Keith Glennan as President of Case Institute of Technology on May 29–31, 1948; Professor E. F. Beckenbach at the inauguration of Fred D. Fagg, Jr., as President of University of Southern California on June 11, 1948; Professor Marston Morse at Third National Congress of Unione Matematica Italiana in Pisa, Italy, on September 23–26, 1948; Professor Richard Courant on Advisory Board of Applied

Mechanics Reviews for three years beginning July 1, 1948.

The following additional appointments by President Hille were reported: Professor J. W. T. Youngs as an additional member of the Committee on Arrangements for the 1948 Annual Meeting; Professors S. S. Cairns (Chairman), G. B. Price and P. A. Smith as a Committee on Revision of By-Laws; Professor J. R. Kline (Chairman), Dean W. L. Ayres, Professor W. T. Martin as a Committee to nominate an Executive Director of the Society; Professor T. H. Hildebrandt as Chairman (to replace Professor M. H. Stone) and Professor L. V. Ahlfors as an additional member of the Committee on the Award of the Bôcher Prize; Dean Walter Bartky as Chairman of the Committee on Applied Mathematics; Professor R. V. Churchill as a member of the Committee on Applied Mathematics, for unexpired term of Professor J. L. Synge, August 1, 1948–December 31, 1949; Professor A. H. Taub as Chairman for one year and as a member for three years, beginning July 1, 1948, of the Editorial Committee for Applied Mathematics Symposium Proceedings.

The Secretary reported that the Society had been listed as a sponsor of the Symposium on Modern Calculating Machinery and Numerical Methods, sponsored by the University of California at Los Angeles and the National Bureau of Standards in Los Angeles on July 29–31, 1948.

The Council voted that the Society should agree to be listed as a sponsor of a meeting of the American Society for Engineering Education, to be held at Rensselaer Polytechnic Institute in June, 1949.

A bequest of $1,000 to the Society under the will of the late Professor J. K. Whittemore of Yale University was reported to the Council and the Secretary was authorized and requested to express its appreciation to Mrs. Whittemore.

A contract has been signed by the Society and the Office of Naval Research whereby a sum of $25,000 will be available to the Society for the project of providing translations of important mathematical articles in Russian or other unfamiliar languages and for the project of the preparation of a pamphlet to aid mathematicians in reading Russian mathematical articles.

The following dates of meetings in 1949 were approved: February 26 in New York City; February 26 in Chicago; April 1–2 in Durham, North Carolina; April 29–30 in New York City; April 29–30 in Lawrence, Kansas; April 30 at Stanford University.

On recommendation of the Committee on Places of Meetings, the Council voted to hold the 1950 Annual Meeting at the University of Florida and the 1951 Summer Meeting at the University of Minnesota.

It was voted that the practice of having a group photograph at Summer Meetings should no longer be mandatory.

Certain invitations to deliver hour addresses were announced: Professor Raphael Salem for the October, 1948, meeting in New York City; Professor Herbert Busemann for the November, 1948, meeting in Los Angeles; Professors A. S. Besicovitch and Lamberto Cesari for the 1948 Annual Meeting in Columbus, Ohio; Professor S. B. Myers for the February, 1949, meeting in Chicago; Professors E. J. Mickle and R. M. Thrall for the April, 1949, meeting in Lawrence, Kansas.

The Policy Committee for Mathematics reported regarding its activities in connection with the possible formation of an International Mathematical Union. The Council voted to reaffirm its assignment to the Policy Committee of the problem of the International Mathematical Union and its designation of that Committee as the sole agent to carry on negotiations and bring recommendations to the governing boards of the organizations represented in the Policy Committee, leading to the formation of the Union. It was reported that the Mathematical Association of America would be represented in the Policy Committee in the future and the Council voted to approve a change in the original plan for the Policy Committee, thus giving the Association a representation of three members.

For the Organizing Committee of the International Congress, Vice Chairman W. T. Martin reported that the Committee had decided to hold Conferences in the following fields: Algebra, Analysis, Topology, and Applied Mathematics. Secretary Kline, for the Financial Committee, reported that the Rockefeller Foundation had increased its pledge of $7,500 (made for the Congress scheduled for 1940) to $12,000, that the Carnegie Corporation had increased its pledge from $12,000 to $18,000 and that the Mathematical Association of America had increased its pledge from $600 to $1,000. It was reported that Professor J. L. Doob had been added to the membership of the Organizing Committee for the International Congress.

The Council adopted, with minor changes, the report of the Committee on Revision of By-Laws. This committee was appointed to revise the By-Laws in order (1) to incorporate changes made necessary by the adoption of the report of the Committee on Reorganization and (2) to make other changes in the By-Laws as seemed necessary to make the By-Laws conform to existing practices of the

Society. The President was authorized and requested to appoint a new committee to study suggestions for additional changes made by the Committee on Revision of By-Laws.

The Council voted to accept the invitation to hold the Third Annual Symposium on Applied Mathematics at the University of Michigan in the summer of 1949.

A proposal that the Society provide competent survey articles every four or five years in each of the various fields of mathematics was referred to the Committee on the Role of the Society in Mathematical Publication.

Professors R. P. Agnew and W. M. Whyburn were elected representatives of the Society on the Council of the American Association for the Advancement of Science for the year 1949.

Professors Marston Morse and Oscar Zariski were nominated as representatives of the Society in the Division of Mathematical and Physical Sciences of the National Research Council for a three-year period beginning July 1, 1949.

The Council voted to invite Professor Deane Montgomery of the Institute for Advanced Study to deliver a series of Colloquium Lectures at the first Summer Meeting after that of 1949.

Abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 422 was read by Professor Schafer, number 432 by Professor Durfee, number 452 by Professor Beckenbach, number 453 by Dr. Chandrasekharan, number 455 by Professor Cameron, number 460 by Professor Piranian, number 491 by Professor Scott, number 497 by Professor Todd, and number 528 by Professor Scheffé. Mr. Strehler was introduced by Professor C. C. MacDuffee and Mr. Pa by Professor G. T. Whyburn.

ALGEBRA AND THEORY OF NUMBERS

416*t*. A. A. Albert: *Absolute valued algebraic rings.*

An algebraic ring is a vector space A over a field F such that every element of A generates a finite dimensional algebra over F. A quadratic ring is a ring with a unity quantity such that the algebra generated by each element (not a scalar) is a quadratic field. We show that all alternative quadratic rings are finite dimensional, and use this result to prove that every absolute-valued real algebraic ring with a unity element is finite dimensional and thus is the real field, the complex field, the algebra of real quaternions, or the Cayley-Dickson algebra. The results may be extended immediately to division rings, that is, to algebraic rings whose nonzero elements form a quasigroup. (Received June 19, 1948.)

417. J. H. Bell: *Families of solutions of the unilateral matrix equation.*

A recent investigation of a problem described in the author's Bull. Amer. Math.

Soc. Abstract 47-3-115 leads to more general results regarding the existence of infinite families of solutions of the unilateral matrix equation $\sum A_i X^i = 0$, where the A_i and X have elements in a field of characteristic zero. It is found that the unilateral matrix equation will have an infinite family of solutions if and only if there are at least two distinct solutions which are similar. Also, all matrices, similar to a given solution X_1, will be solutions if and only if the minimum polynomial $\pi(\lambda)$ of X_1 is a common divisor of the elements of the matrix $A(\lambda) = \sum A_i \lambda^i$. In the proof of the latter result the following theorem arises: The equation $\sum A_i \cdot xX^i = 0$ has a solution if and only if the elements $A(\lambda)$ have a greatest common divisor which is not a unit. It is also proved that if the determinant of $A(\lambda)$ is identically zero and the unilateral equation has a solution, there is an infinite number of solutions. (Received May 4, 1948.)

418. J. W. Bradshaw: *Note on Kummer's second illustration.*

The author calculates the series $\sum (u_n/v_n)^n$, where $u_n = 1 \cdot 3 \cdots (2n-1)$, $v_n = 2 \cdot 4 \cdots (2n)$, by means of a continued fraction formula for the remainder after summing a fixed number of terms. He compares the method with the calculation in terms of gamma functions. (Received July 21, 1948.)

419t. A. T. Brauer: *Limits for the characteristic roots of a matrix.* III.

It was proved in the first two parts (Duke Math. J. vol. 13 (1946) pp. 387–395 and vol. 14 (1947) pp. 21–26) that the characteristic roots ω_ν of an arbitrary matrix must lie in certain n circles, and more exactly in certain $n(n-1)/2$ ovals of Cassini. In this paper much smaller regions are obtained which must contain all the ω_ν. These results are sometimes very sharp for numerically given matrices. This is shown by an example of a matrix of order 5 where for the absolute greatest root ω the interval $9.061 < \omega < 9.215$ is obtained while actually this root satisfies $9.187 < \omega < 9.188$. (Received July 26, 1948.)

420t. R. C. Buck: *Approximation in a class of polynomials.* Preliminary report.

The problem studied is that of the minimal value of $\|P\|$ for all polynomials P having integral coefficients and leading coefficient 1, under various choices of the norm $\| \ \|$. Results parallel those of Fekete (Math. Zeit. vol. 17 (1923) pp. 228–249). (Received July 29, 1948.)

421t. Leonard Carlitz: *Representations of arithmetic functions in* $GF[p^n, x]$. II.

In a previous paper (*Representations of arithmetic functions in* $GF[p^n, x]$, Duke Math. J. vol. 14 (1947) pp. 1121–1137), the writer proved that an arbitrary arithmetic function $f(A)$ can be written in the form $f(A) = \sum^* \alpha_{GH} \epsilon_{GH}(-A)$, the summation extending over a "fundamental" set of functions ϵ_{GH}, deg $A < r$. Now in certain applications considerable difficulty is caused by the presence of ϵ's with $k > r/2$ in this formula. It is now shown how in at least some problems this difficulty can be overcome by setting up a correspondence between ϵ's of degree $k > r/2$ and those with $k \leq r$. This correspondence is effected by means of the equation $G'H - GH' = \theta$, where θ is in $GF(p^n)$. Applications are made to the sums $\sum_B \epsilon_{GH}(B^m)$, where $r = mk$, and the summation is either over all primary B of degree k or all B of degree less than k. (Received July 6, 1948.)

422. Claude Chevally and R. D. Schafer: *The exceptional simple Lie algebras F_4 and E_6.*

There is given a characterization of the exceptional simple Lie algebras of dimension 52 and 78 over an algebraically closed field K of characteristic 0. Let $\mathfrak{J}$ be the exceptional simple Jordan algebra of dimension 27 over K. It is shown that the derivation algebra $\mathfrak{D}$ of $\mathfrak{J}$ is the algebra F_4 (of rank 4 and dimension 52) in Cartan's list of simple Lie algebras over K. For X in $\mathfrak{J}$, denote by R_X the corresponding right multiplication. Since $[R_X, R_Y]$ is a derivation, the set $\mathfrak{D}+\{R_T\}$ for T of trace zero is a Lie algebra, which is shown to be E_6 (the simple Lie algebra of rank 6 and dimension 78 over K). (Received July 23, 1948.)

423*t*. Sarvadaman Chowla: *A new proof of a theorem of Siegal.*

Let $\chi(n)$ denote a real *primitive* character (mod k) where $k>1$. Siegel's celebrated theorem asserts that if ϵ is any positive number, we have $\sum_1^\infty \chi(n)n^{-1}>k^{-\epsilon}$ $[k>k_0(\epsilon)]$. Siegel's proof (Acta Arithmetica vol. 1 (1936)) used class-field-theory. In this paper the author gives a proof which is based on simple properties of Dirichlet's L-functions defined by $L(s, \chi)=\sum_1^\infty \chi(n)n^{-s}$ which is valid for $R(s)>0$. (Received July 27, 1948.)

424*t*. Sarvadaman Chowla: *On the least prime quadratic non-residue of a prime.*

Let q denote the smallest positive prime for which $x^2\equiv q \pmod{p}$ has no solution. Vinogradoff proved that for large p, $q<p^{1/2e^{1/2}} \log^2 p$. The author proves the sharper result, $q<p^{\epsilon}$ $[q>q_0(\epsilon)]$ (where ϵ is an arbitrary positive number), *assuming*, however, the "*extended Riemann hypothesis*." (Received July 27, 1948.)

425. I. S. Cohen: *Prime factorization of ideals.*

It is proved that a commutative ring in which every ideal is a product of prime ideals is a direct sum of a finite number of rings each of which is either an integral domain with Dedekind-Noether ideal theory or else a local ring whose maximal ideal is principal and nilpotent. As a consequence, such rings are Noetherian. Among Noetherian rings, such rings can be characterized by the distributivity of the lattice of ideals, or by the validity of the relation $A:(B\cap C)=A:B+A:C$ among any three ideals, or by the validity of several other similar relations. (Received July 29, 1948.)

426. Ben Dushnik: *Isoperimetric sets.*

A set of different pairs of natural numbers—$p_1, q_1; \cdots ; p_n, q_n, n>1$—is called a set of "isoperimetric" pairs (briefly, an isoperimetric set) if, for every pair, (I) $p<q$; (II) p and q are relatively prime and $p+q$ is odd; (III) $pq+q^2$ is constant for all pairs. Concerning such sets, the following two questions are discussed. If $N>1$ is any natural number, does there exist an isoperimetric set with N pairs? If p is any natural number, does there exist a natural $q>p$ such that the pair p, q satisfies (II) above and belongs to an isoperimetric set? Both questions are answered affirmatively; as to the second one, there are indeed infinitely many q's with the desired property. The constant value of $pq+q^2$ is called the "index" of the set. The smallest index is 858, corresponding to the pairs 7, 26 and 17, 22; this answers a question by Dernham in Amer. Math. Monthly vol. 55 (1948) p. 248. (Received August 4, 1948.)

427*t*. Franklin Haimo: *Division topology in an Abelian group.* Preliminary report.

Let G be an abelian group in which the patterns of the nonzero elements form a directed set with no maximum. (For information on patterns, see Duke Math. J. vol. 15 (1948) pp. 347–356.) For an element g in G, let V_g be the set of all h in G with the same pattern as g or with a "better" pattern. Then the V_g serve as nuclei with respect to which G is a topological group, the topology being called the division topology of G. If G has torsion elements and a division topology, then the torsion part of G is primary. A division topology is also a division-closure topology but not a connected topology. The possibility of a group having both a division topology and a compact topology is discussed. (Received July 26, 1948.)

428. I. N. Herstein: *Divisor algebras.*

In this paper the divisor algebras of algebraic function fields of degree of transcendence one over algebraically closed fields of constants are characterized axiomatically. Prime divisors, equivalence and dependence are taken as primitive. The free Abelian group generated by the prime divisors is considered. The elements of this group are called divisors. The axioms are of such a nature that: (1) equivalence classes of divisors form projective geometries under the dependence relation; (2) group multiplication induces a collineation; (3) a general Pappus theorem is true; (4) in every integral line of divisors there is at least one multiple of every prime divisor. It is shown that with the appropriate definition, functions on the set of prime divisors to lines of these geometries can be introduced and their sums and products defined so that a field is obtained which is an algebraic function field of degree of transcendence one over the algebraically closed projective field constructed on any line of these geometries. (Received May 14, 1948.)

429*t*. A. P. Hillman: *Reduction to standard form of an aerodynamical matrix characteristic value problem.*

The system $\sum_{j=1}^{n} d_{ij}y_j = \lambda(y_i + \sum_{j=1}^{n} c_j y_j)$ $(i=1, \cdots, n)$ arises in the National Bureau of Standards Report *Flexural transients in model wing following vertical "landing impact" at point below center of gravity*, NBS Lab 65181 PR1, BuAer TED NBS 2410, June 1946, p. 8. The present paper reduces it, under the assumption that $\sigma = \sum_{r=1}^{n} c_r$ is different from -1, to the standard matrix characteristic value system $\sum_{j=1}^{n} a_{ij}z_j = \lambda z_i$ by the substitutions $a_{ij} = d_{ij} - c_j(1+\sigma)^{-1}\sum_{k=1}^{n} d_{ik}$, $y_i = z_i - (1+\sigma)^{-1}\sum_{k=1}^{n} c_k z_k$. (Received July 6, 1948.)

430*t*. R. E. Johnson: *The product of groups and rings.* Preliminary report.

For any two sets M and N, let (M, N) be the free abelian semi-group with generators (x, y), x in M, y in N. A multiplication is naturally introduced into (M, N) in case M and N are multiplicative sets. Any group (or ring in case M and N are multiplicative) that is a homomorphic image of (M, N) is called a product of M and N. The study of these products is reduced to the study of "admissible" subgroups of the free abelian group generated by (M, N). Results similar to those holding for tensor products and classical products of algebras are obtained. (Received July 27, 1948.)

431*t*. B. W. Jones: *A theorem on integral symmetric matrices.*

The following theorem is proved: Let A and B be symmetric integral nonsingular matrices with respective dimensions n and m $(n>m)$ and S an n by m matrix of rank m with rational elements such that s is the l.c.m. of the denominators and $S^TAS=B$.

Then there is an n by n matrix T with rational elements the prime factors of whose denominators all divide s, whose determinant is 1 and which takes A into an integral matrix A_0 which represents B integrally, that is, $U^TA_0U=B$ for some integral matrix U. Though the proof is independent of the theory of quadratic forms it immediately implies the following known result: If f and g are two quadratic forms with integral coefficients, having nonzero determinants and n and m variables respectively, $n>m$, and if there is an n by m matrix with rational elements whose denominators are prime to $2|f|$ and taking f into g, there is a form f_0 in the same genus as f whose matrix has integral elements and which represents g integrally. (Received June 7, 1948.)

432. B. W. Jones and William H. Durfee: *A theorem on quadratic forms over the ring of 2-adic integers.*

Let f, g and h be quadratic forms over the ring $R(2)$ of 2-adic integers, g and h having no variables in common with f and such that the symmetric matrices of f, g and h have their elements in $R(2)$. The authors show that if f is unimodular, then $f+2g$ equivalent to $f+2h$ implies g equivalent to h. The proof depends on the following lemma, of some interest in itself: If $T'FT\equiv I \pmod 2$, where F is the matrix of f and T is a square matrix over $R(2)$ of the same order, then there is an automorph D of F over $R(2)$ such that $2(T+D)^{-1}$ has its elements in $R(2)$. (Received June 1, 1948.)

433t. Joseph Lehner: *Further congruence properties of the Fourier coefficients of the modular invariant $j(\tau)$.*

Let $j(\tau)=x^{-1}+\sum_0^\infty c_nx^n$, $x=\exp 2\pi i\tau$, where $j(\tau)$, with $\operatorname{Im}\tau>0$, is the absolute modular invariant on the full modular group: $j(\tau')=j(\tau)$ for every $\tau'=(a\tau+b)/(c\tau+d)$ with integer a, b, c, d, and $ad-bc=1$. Continuing the results of a previous paper (Bull. Amer. Math. Soc. Abstract 54-7-231), the author proves: $c_n\equiv 0 \pmod{2^{3a+8}}$ if $n\equiv 0 \pmod{2^a}$, $c_n\equiv 0 \pmod{3^{2a+3}}$ if $n\equiv 0 \pmod{3^a}$, for a, $n=1, 2, 3, \cdots$. The modular equations for the univalent functions on the groups $\Gamma_0(p)$ (defined by $c\equiv 0 \pmod p$) are derived for the primes p for which $\Gamma_0(p)$ is of genus zero. Sufficient conditions are given for a modular function to possess the congruences of this and the previous paper. (Received August 9, 1948.)

434t. W. J. LeVeque: *A metric theorem on uniform distribution* (mod 1).

A new proof is given of a theorem of Koksma (Compositio Math. vol. 2 (1935) pp. 250–258) that the sequence x^n, $n=1, 2, \cdots$, is uniformly distributed (mod 1) for almost all $x>1$. This is contained in the following theorem, which is slightly weaker than Koksma's Theorem 3: Let $g(x, 1)$, $g(x, 2)$, $\cdots$, be any sequence of twice-differentiable functions of x, such that $h(x, j, k)=g'(x, j)-g'(x, k)$ is either nonincreasing or nondecreasing and is different from zero in (a, b) for $j\neq k$, and such that for $x=a$ and for $x=b$, $|h(x,j,k)|\geq C|j-k|^\epsilon$ for some $C>0$, $0<\epsilon\leq 1$. Then the sequence is uniformly distributed (mod 1) for almost all $x\in(a, b)$. This theorem is deduced from a lemma used by Kac, Salem and Zygmund (Trans. Amer. Math. Soc. vol. 63 (1948) pp. 235–244) in their investigations of quasi-orthogonal functions. (Received August 2 1948.)

435. C. C. MacDuffee: *Orthogonal matrices in four-space.*

Every real proper 4 by 4 orthogonal matrix can be written $A=R(\alpha)\cdot S^T(\beta)=S^T(\beta)\cdot R(\alpha)$ where α and β are unit quaternions, and $R(\alpha)$ and $S(\beta)$ are their first and second regular representatives. Every such product is orthogonal and proper. The coefficients of α and β can be obtained from the elements of A merely by the solution of linear and quadratic equations. (Received July 26, 1948.)

436. H. B. Mann: *On the field of origin of an ideal.*

Equality of ideals is defined as in Hecke: *Theorie der algebraischen Zahlen*, § 37. An ideal $\mathfrak{a}$ is said to originate in $\mathfrak{F}'$ over $\mathfrak{F}$, if it is not contained in any proper subfield of $\mathfrak{F}'$ which contains $\mathfrak{F}$. The numbers in $\mathfrak{a}$ which are also in $\mathfrak{F}$ form an ideal $\mathfrak{A}$ in $\mathfrak{F}$ called the ideal corresponding to $\mathfrak{a}$ in $\mathfrak{F}$. If $\mathfrak{A} = \mathfrak{a}^e\mathfrak{c}$, $(\mathfrak{a}, \mathfrak{c}) = 1$, then $\mathfrak{a}$ is called of order e with respect to $\mathfrak{F}$. If $e = 1$ then $\mathfrak{a}$ originates over $\mathfrak{F}$ in one and only one field. If $\mathfrak{P} = \mathfrak{p}^g$ is the highest power of the ideal $\mathfrak{p}$ which is prime ideal in any field over $\mathfrak{F}$ then $\mathfrak{P}$ originates in a unique field $\mathfrak{F}'$ over $\mathfrak{F}$, whilst $\mathfrak{p}$ itself originates in infinitely many fields over $\mathfrak{F}$. Every field $\mathfrak{F}''$ in which $\mathfrak{p}$ originates over $\mathfrak{F}$ must be of the form $\mathfrak{F}'(\alpha)$ where α satisfies an equation of degree rg (r integral) with integral coefficients a_i ($i = 0, 1, \cdots, rg$), $a_0 = 1$, in $\mathfrak{F}'$ for which $a_{ig+k} = 0$ $(\mathfrak{P}^{i+1})$, for $k > 0$ and $a_{rg} \neq 0$ $(\mathfrak{P}^{r+1})$. (Received July 1, 1948.)

437. Leo Moser: *Number of distances determined by n points.*

Let $f(n)$ be the minimum number of distances determined by n points in a plane. P. Erdös has shown that $f(n) > (n-1)^{1/2} - 1$ (Amer. Math. Monthly vol. 53 (1946) pp. 248–250). By considering the possible distribution of points in concentric rings, center at one of the points determining the least distance, it is shown that $f(n) > cn^{2/3}$, where c is a fixed constant. For the n points vertices of a convex polygon Erdös conjectured $f(n) \geq [n/2]$. It is shown that $f(n) \geq [(n+2)/3]$. (Received July 16, 1948.)

438t. H. T. Muhly: *The irregularity of an algebraic surface and a theorem on regular surfaces.*

It is shown by elementary methods that if $|A_m|$ is the completion of the m-fold of the system of hyperplane sections of a normal model U of a field Σ/k of algebraic functions of two variables then the deficiency $\alpha(m)$ of the characteristic series of $|A_m|$ has a constant value $\delta(U)$ when m is large (a result due originally to Castelnuovo). The character $\delta(U)$ of U is shown to be a relative invariant of U, in the sense that it is invariant under regular birational transformations of U. Moreover, the value of $\delta(U)$ is the same for all nonsingular models of Σ, and it is this value which is defined to be the irregularity of Σ. If $\delta(U) = 0$, then derived normal models U_h of U belonging to sufficiently high characters of homogeneity h have the property that their generic hyperplane sections are arithmetically normal. If O is the ring of homogeneous coordinates along U_h and if (y_0, y_1, y_2) is any triple of homogeneous elements of degree one in O such that O depends integrally on $k[y_0, y_1, y_2]$, then O has an independent integral base over $k[y_0, y_1, y_2]$. If x_1, x_2 is an arbitrary transcendence base for a regular field Σ/k, and if I is the integral closure in Σ of the ring $k[x_1, x_2]$, then there exist integers h such that I has an independent integral base over $k[x_1^h, x_2^h]$. (Received July 21, 1948.)

439. W. V. Parker: *Sets of complex numbers associated with a matrix.*

If A is a square matrix with complex elements and x and y are vectors such that $x\bar{x}' = y\bar{y}' = 1$, the set of complex numbers $xA\bar{y}'$ is the set of all numbers in or on the circle of radius ρ about zero in the complex plane, where ρ^2 is the greatest of the characteristic roots of $A\bar{A}'$. The set $xA\bar{x}'$ is a convex set known as the field of values of A and is the set of all diagonal elements of matrices UAU' where U is unitary. The set $xA\bar{y}'$, where $x\bar{y}'=0$, is the set of all non-diagonal elements of matrices $UA\bar{U}'$ where U is unitary. If A is Hermitian with characteristic roots $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ this latter set consists of all numbers in or on the circle of radius $(\lambda_n - \lambda_1)/2$ about zero in the complex plane. (Received July 26, 1948.)

440t. C. E. Rickart: *Isomorphism of groups of linear transformations.*

Consider a system $(\mathfrak{X}, D, \mathfrak{X}^*; \mathcal{G})$ in which $\mathfrak{X}$, $\mathfrak{X}^*$ are respectively right and left linear vector spaces over the division ring D, which are dual in the sense of Jacobson (Ann. of Math. vol. 48 (1947) pp. 8–21) and have dimension greater than 3, and $\mathcal{G}$ is a group of linear transformations on $\mathfrak{X}$ under the operation $A \circ B = A + B - AB$. Assume further that every $A \in \mathcal{G}$ possesses an adjoint (loc. cit. p. 16) and that $\mathcal{G}$ contains every finite-dimensional involution (T is an involution if $T \circ T = 0$) which possesses an adjoint. Let $(\mathfrak{Y}, E, \mathfrak{Y}^*; \mathcal{K})$ be a second such system and assume $\mathcal{G}$ and $\mathcal{K}$ isomorphic as groups. Then either $\mathfrak{X}$ and $\mathfrak{Y}$ (also $\mathfrak{X}^*$ and $\mathfrak{Y}^*$) are isomorphic as linear spaces or $\mathfrak{X}$ and $\mathfrak{Y}^*$ (also $\mathfrak{X}^*$ and $\mathfrak{Y}$) are anti-isomorphic (here D and E are anti-isomorphic instead of isomorphic). The proof uses methods developed by Mackey (Ann. of Math. vol. 43 (1942) pp. 244–260) in studying normed linear spaces. For the finite-dimensional case in which D and E are fields, the above follows from results of Schrier and van der Waerden (Abh. Math. Sem. Hamburgischen Univ. vol. 6 (1928) pp. 303–322.) (Received July 19, 1948.)

441. W. E. Roth: *On the eliminant of $f(x)$ and $x^n f(1/x)$.*

Elliot, in Proc. London Math. Soc. vol. 25, showed that Sylvester's dialytic eliminant of the two polynomials $f(x)$ and $x^n f(1/x)$ is the product $f(1)f(-1)F$ where F is the square of a rational function of the coefficients of $f(x)$. This theorem was again taken up by Taylor and by Muir. In the present paper we write $f(\lambda x) = a_0 + a_1 \lambda x + a_2 \lambda^2 x^2 + \cdots + a_n \lambda^n x^n$ and express the Sylvester eliminant of $f(\lambda x)$ and $x^n f(\lambda/x)$ as the product $f(\lambda)f(-\lambda)c^2D^2$ where c is a constant and D is a determinant of order $n-1$, whose elements are given in terms of $f(\omega^i \lambda)$, where ω is a primitive $2n$th root of unity. (Received June 30, 1948.)

442. B. M. Stewart: *A note on least common left multiples.*

Consider n-by-n matrices with elements in a principal ideal ring. C. C. MacDuffee (Bull. Amer. Math. Soc. vol. 39 (1933) p. 574) presents a method, due in essence to E. Cahen and A. Châtelet, for finding a least common left multiple M of two given matrices A and B, provided that both A and B are nonsingular. In this note a different proof shows that the stated method for finding an M is correct provided merely that the greatest common right divisor D of A and B is nonsingular. In the case that D is singular the stated method for finding an M may fail; in this case a modified method is suggested for finding an M by operating in a "simplest" way on the Hermite triangular forms of A and B. In every case M is unique up to a unimodular left factor. (Received April 27, 1948.)

443. R. R. Stoll: *Homomorphisms of semigroups.*

Homomorphisms of a semigroup S onto groups with a zero element adjoined are investigated. With no loss of generality it is assumed that S has a zero element. Then the above homomorphisms exist if and only if S contains a prime ideal. The main result obtained is a set of sufficient conditions for the existence of maximal elements among these homomorphic images. These conditions are then applied to the various classes of semigroups. Methods due to Dubreil (Memoires de l'Academie des Sciences de l'Institut de France (2) vol. 63 (1941) pp. 1–52) are used. (Received September 10, 1948.)

444. A. F. Strehler: *A result in "factorisatio numerorum."*

An explicit expression is found for the number-theoretical function $f(n)$, which is the number of different factorizations of the natural number n into factors each >1, the ordering of factors being considered. This formula is a summation over the μ-functions of Souriau (Revue Scientifique vol. 82 (1944) pp. 204–211), which are in turn functions of the exponents in the prime-power factors of n. The coefficients in this summation are closely related to the binomial coefficients: they are, in fact, sums of the diagonal arrays in the first k-rows of a Pascal triangle, and they themselves form a new Pascal-type triangle. The previously used expressions for the function $f(n)$ have been recursion formulas. An expression similar to that for $f(n)$ is derived for the closely related Moebius μ-function. (Received June 28, 1948.)

445. Olga Taussky: *On a theorem of Latimer and MacDuffee.*

The theorem (Ann. of Math. vol. 34 (1933) 313–316) concerns the 1-1 correspondence between the classes of matrices $S^{-1}AS$ and the classes of ideals in the ring of polynomials in A where A is a matrix solution of an algebraic equation $f(x)=0$. All the polynomials are assumed to have rational integers as coefficients, $f(x)$ is of degree n and A is an $n\times n$ matrix with integral elements which has $f(x)=0$ as minimum equation, S is a unimodular matrix with integral elements. An alternative proof in the case that $f(x)=0$ is irreducible is given. (Received July 24, 1948.)

446. Leonard Tornheim: *Construction of an integral basis of any algebraic number field by means of valuations.*

A construction for the valuations of an algebraic field has been given by S. MacLane (Trans. Amer. Math. Soc. vol. 40 (1936) pp. 363–395; Duke Math. J. vol. 2 (1936) pp. 492–510). By means of it some of the ideas of Berwick (*Integral bases,* 1927) are restated and results extended to give a construction of an integral basis of any algebraic number field without the exceptions he had and a basis for any ideal of the field. (Received July 27, 1948.)

447*t*. Bernard Vinograde: *The maximum of certain determinants.*

The maximum determinant of a positive definite matrix P, whose diagonal elements are ones, under the transformations TPT', where T is diagonal $(A, B, C, \cdots)$, $A, B, C, \cdots$ nonsingular of preassigned degrees, is attained when the corresponding blocks in TPT' are unit matrices. For the case of two blocks, this coincides with the condition under which linear functions of two sets of variates have maximum correlation, P being the variance-covariance matrix. (Received July 27, 1948.)

448*t*. N. A. Wiegmann: *Some theorems on infinite normal matrices.*

Some theorems and analogs of theorems which are known to be true for finite normal matrices are shown to hold for fully continuous infinite normal matrices as defined by Hellinger-Toeplitz (that is, for the matrix A formed by the coefficients of a fully continuous normal bilinear form $A(x, y)$). Among these are: If A, B and AB are fully continuous infinite normal matrices, then BA is a fully continuous infinite normal matrix; if A and B are normal, then AB is normal if and only if each commutes with the hermitian polar matrix (as defined for the infinite normal matrix) of the other; a set $\{A_i\}$ of fully continuous infinite normal matrices can be brought into diagonal forms by the same unitary transformation if and only if they commute in pairs. In addition, a canonical form under a unitary transformation is obtained for a set of fully continuous infinite normal matrices $\{A_i\}$ which is closed under multiplication; and an analog of the representation of the conjugate transpose of a finite normal matrix as a polynomial in the matrix is obtained for this type of infinite normal matrix. (Received July 21, 1948.)

449*t*. L. R. Wilcox: *A class of partially ordered sets.*

Systems are considered which exhibit properties of biological and other hierarchies with the order relation that of ancestry. These systems are defined by postulating a set L together with an unrestrictedly additive "parental" function Π on $\mathcal{L}$ to $\mathcal{L}$, where $\mathcal{L}$ is the set of all subsets of L. An "ancestral" function Π^* is defined by $\Pi^*(S) = \sum(\Pi^n(S); n = 1, 2, \cdots), S \subset L$, where Π^n is the nth iterate of Π. Axioms are as follows: (1) for no x in L is x in $\Pi^*([x])$; (2) if $\Pi([x])$ is not empty, it consists of exactly two elements; (3) the partial ordering introduced in L by Π^* satisfies an appropriate chain condition. A symmetric "parental" relation P is defined so that xPy means $[x, y] = \Pi([z])$ for some z in L. Necessary and sufficient conditions are derived so that the domain of P may be divided into two disjoint sets ("sexes") with properties suggested by the biological instances. These systems are not lattices, except in trivial cases. Structure of the partial ordering is studied, and it is found that non-isomorphic systems exist whose partial orderings are nevertheless isomorphic. (Received July 27, 1948.)

450. L. R. Wilcox: *A problem concerning binary relations.*

Let L be a set and P a symmetric binary relation on $L \times L$. The equation $R + R^* = P$ ($+$ being set addition) is considered for R a relation on $L \times L$ such that R and its transpose R^* have disjoint domains. It is found that a solution exists if and only if P is *alternating* in the following sense: Define, for any $S \subset L$, $\phi(S)$ as the set of all y in L for which x in S exists with xPy; define ϕ^n as the nth iterate of the set function ϕ. Then for every one element set $[x] \subset L$, and for $n = 1, 2, \cdots$, $\phi^{n-1}([x])$ and $\phi^n([x])$ are disjoint. If P is alternating, a constructive method for finding all solutions R is found. Define P_{rt} as the smallest reflexive and transitive relation containing P. Let E be the set of equivalence classes in L defined by P_{rt}, and $\mathcal{E}$ the set of all subsets of E. Then the set of all solutions R is in one-to-one correspondence with $\mathcal{E}$. (Received July 27, 1948.)

ANALYSIS

451*t*. Warren Ambrose: *The L_1-system of a unimodular group.* I.

Some theorems are proved about the algebraic system made up by L_1 with respect to the Haar measure, with the operation of convolution (which is only defined for certain pairs of elements) for multiplication. (Received June 1, 1948.)

452. E. F. Beckenbach and E. W. Graham: *On subordination in complex variable theory.*

A general result is established, including as special cases the theorem of Study concerning conformal maps on convex domains, the analogous theorem concerning star-shaped maps, the general theorem of L. R. Ford (Duke Math. J. vol. 1 (1935) pp. 103–104), and the fundamental subordination theorem, and including also new results of which the following is typical: Let the analytic function $w=f(z)$ map $|z|<1$ on a plane domain D, and let the map of $|z|<r$, $0<r<1$, be denoted by $D(r)$. If for some r_0, $0<r_0<1$, the convex hull of $D(r_0)$ is contained in D, then for all ρ, $0<\rho<1$, the convex hull of $D(r_0\rho)$ is contained in $D(\rho)$. (Received July 26, 1948.)

453. Salomon Bochner and K. Chandrasekharan: *Summations over lattice-points in k-space.*

The convergence properties of expansions in Bessel functions, like $\sum r_k(n, h)J_{\alpha+k/2} \cdot (2\pi(nx)^{1/2})n^{-(2\alpha+k)/4}$, where $r_k(n, h)=\sum \exp\{2\pi i(n_1 h_1+\cdots+n_k h_k)\}$ $(\sum n_i^2=n)$, occurring in lattice-point problems are studied. The results obtained generalize or sharpen those of G. H. Hardy, Proc. London Math. Soc. (2) vol. 15 (1916) pp. 192–213; J. R. Wilton, Proc. London Math. Soc. (2) vol. 29 (1929) pp. 168–188; A. L. Dixon and W. L. Ferrar, Quart. J. Math. Oxford Ser. vol. 5 (1934) pp. 48–63, 172–185. The method employed is based on the theory of Fourier series in several variables. (Received July 8, 1948.)

454*t*. R. H. Cameron: *A "Simpson's rule" for the numerical evaluation of Wiener's integrals in function space.*

In this paper the author gives an approximate formula expressing the Wiener integral $\int_C^w F(x)d_wx$ of a smooth functional $F(x)$ as an $(n+1)$-fold ordinary integral whose first n variables go from $-\infty$ to ∞ and whose last variable goes from -1 to 1. This last variable enters in a very peculiar way, and the integration with respect to it makes the formula exact for third degree polynomial functionals. For general functionals satisfying certain conditions of smoothness and order of growth at ∞, a specific estimate of the error is given, and it turns out to be $O(n^{-2})$. Thus, for such a functional, one would expect that tripling the multiplicity of the integral would improve the accuracy by about one decimal place. (Received July 27, 1948.)

455. R. H. Cameron and W. T. Martin: *The transformation of Wiener integrals by nonlinear transformations.*

Let C be the space of functions $x(t)$ continuous on I: $0\leqq t\leqq 1$, and vanishing at $t=0$. Let $y=Tx=x+\Lambda(x|\cdot)$ be a transformation which takes a Wiener measurable set $\Gamma\subset C$ into $T\Gamma\subset C$ in a 1-to-1 manner. Here $\Lambda(xt)$ is a functional defined for all functions $x\in\Gamma$ and all numbers t on I, and having its first variation $\delta\Lambda$ expressible in the form $\delta\Lambda(x|t|y)=\int_0^1 K(x|t, s)y(s)ds$. Then if $D(x)$ is the Fredholm determinant of $K(x|t, s)$ (with parameter $\lambda=-1$) and Λ satisfies certain smoothness conditions, the authors call $D(x)$ the "volume element ratio" and define the "measure element ratio" $J(x)=D(x)\int_0^1 (d/dt)[x(t)-y(t)]d[x(t)+y(t)]$. In terms of this measure element ratio the following formula for transforming Wiener integrals is established under suitable hypotheses: $\int_{T\Gamma}^w F(y)d_wy=\int_\Gamma^w F(Tx)|J(x)|d_wx$. (Received July 19, 1948.)

456*t*. E. F. Collingwood: *Exceptional values of meromorphic functions.*

Suppose $f(z)$ is nonrational and meromorphic in $|z|<R\leqq\infty$ and denote by $G(a, \sigma)$ a typical domain defined by the inequality $|f(z)-a|<\sigma$ (or $1/|f(z)|<\sigma$ if $a=\infty$). A domain $G(a, \sigma)$ is "bounded" if its closure is contained in $|z|<R$. A recent theorem

(H. L. Selberg, Comment. Math. Helv. vol. 18, p. 313) shows that if the valency of $f(z)$ in any domain $G(a, \sigma)$ does not exceed $p < \infty$ and if all these domains are "bounded" then $m(r, a) < p \log r + \log^+ 1/\sigma + O(1)$. Hence, if $T(r, f)$ is unbounded, $\delta(a) = \Delta(a) = 0$, where $\delta(a)$ is the deficiency of a and $\Delta(a) = \limsup_{r\to R} m(r, a)/T(r, f)$. In this paper the theorem is generalized in the following way. The number $\lambda(r)$ of the domains $G(a, \sigma)$ intercepted by $|z| = r$, if any, is finite; these domains are denoted by $G_\nu(r, a, \sigma)$ $(\nu \leqq \lambda(r))$ while $p_\nu(r, a, \sigma)$ is the valency of $f(z)$ in $G_\nu(r, a, \sigma)$ and, by definition, $P(r, a, \sigma) = \max_{\nu \leqq \lambda(r)} p_\nu(r, a, \sigma)$. $E(a, \sigma, p)$ is the set of values of r in $0 \leqq r \leqq R$ in which all the $G_\nu(r, a, \sigma)$ $(\nu \leqq \lambda(r))$ are "bounded" and $P(r, a, \sigma) \leqq p$. These definitions remain applicable when σ and p are made single-valued functions, $\sigma(r)$ and $p(r)$, of r. It is proved here that *for a nonincreasing* $\sigma(r) > 0$ *and* $0 \leqq p(r) < \infty$ *such that* R *is a limit point of* $E = E(a, \sigma(r), p(r))$ then $m(r, a) < p(r)(\log r + O(1)) + \log^+ 1/\sigma(r) + O(1)$ *for* $r \in E$); *while if* $f(z)$ *is regular in* $|z| < R \leqq \infty$ *then* $m(r, a) < (\pi/2)p(r) + \log^+ 1/\sigma(r) + O(1)$ *for* $r \in E$). A new class of inequalities relating the deficiencies $\delta(a)$ and $\Delta(a)$ with the limits of the ratios $\log^+ (1/\sigma(r))/T(r, f)$ and $P(r, a, \sigma(r)) \log r/T(r, f)$ or $P(r, a, \sigma(r))/T(r, f)$ follow from this theorem. (Received May 17, 1948.)

457. Jane S. Cronin: *Branch points of solutions of equations in Banach space.*

A generalization of the theory of ramifications of solutions of non linear integral equations, developed by E. Schmidt, is obtained by studying the local solutions x of the equation in Banach space (1) $(I - C)x + S(y) + T(x, y) = \theta$. I is the identity, C is linear, completely continuous, S satisfies a Lipschitz condition. T is kth-order $(k \geqq 2)$, and y is fixed. By applying the Riesz theory of completely continuous transformations, a continuous mapping M of complex Euclidean n-space into itself, where n is the dimension of the null space of $I - C$, is derived. The number of solutions of $M(z) = 0$ equals the number of solutions of (1). If $n = 1$, and a certain nonzero condition is satisfied, the topological degree of M at 0 is k; if $n = k = 2$, and certain nonzero conditions are satisfied, the degree of M is four. If $S(y) = -y$, $T(x, y) = T(x)$, and the Leray-Schauder degree is defined for $I - C + T$, the degree of M equals the Leray-Schauder degree times ±1. (Received July 30, 1948.)

458. W. F. Eberlein: *Weak almost periodic functions.*

Let G be a locally compact Abelian group and $C(G)$ be the Banach space of complex-valued bounded continuous functions on G. $x(t)$ in $C(G)$ is said to be weakly almost periodic (w.a.p.) if the set of translates of x is conditionally compact in the weak topology. The set of w.a.p. functions is a closed invariant *subring* of $C(G)$ and contains, in addition to the ordinary a.p. functions, all continuous positive definite functions and all continuous functions vanishing at infinity. Every w.a.p. function is uniformly continuous and possesses a mean value in the sense of von Neumann. The convolution of two w.a.p. functions is then defined and turns out to be an ordinary a.p. function. Every w.a.p. function has an expansion in terms of countably many characters for which the Parseval equation holds. When G is the real line the w.a.p. functions form a subclass of the Weyl W^2 class of generalized a.p. functions. (Received May 21, 1948.)

459*t*. Arthur Erdélyi: *Inversion formulae for the Laplace transformation.* Preliminary report.

Let $L_{s,t}$ be a numerically valued linear operator on a sufficiently large class of functions of the complex variable s. $L_{k,t}$ depends on the positive variable t and on the (discrete or continuous) parameter k. If $L_{k,t}$ can be applied under the integral sign in $f(s)=\int_0^\infty\phi(t)e^{-st}dt$ and if $N(u,t,k)=L_{k,t}[e^{-su}]$ is a singular kernel, that is, $\lim\int_0^\infty N(u,t,k)\phi(u)du=\phi(t)$ in some sense as $k\to\infty$, then we have the inversion formula $\phi(t)=\lim L_{k,t}[f(s)]$ for the Laplace transformation. Application of known singular kernels (and of others which can be obtained by Laplace's asymptotic evaluation of integrals, by the method of stationary phase, or otherwise) gives all the known inversions of the Laplace transformation and many new ones. The extension to the Laplace-Stieltjes transformation and the establishing of representation theorems runs on comparatively well known lines. The same principle can be applied for the inversion of other functional transformations. (Received July 10, 1948.)

460. Paul Erdős and George Piranian: *Topologizations of a sequence space by convergence fields of Toeplitz transformations.*

Two bounded sequences x and y of complex numbers are defined to be equivalent provided there exists a nonzero constant λ and a convergent sequence α such that $y_n=\lambda x_n+\alpha_n$ $(n=1, 2, \cdots)$. The space m/L of equivalence classes of bounded sequences is topologized on the basis of the convergence fields of certain selected Toeplitz transformations, and some properties of the topological space are investigated. The investigation yields theorems concerning Toeplitz transformations whose fields of convergence are small. The basic device used in the paper is the principle of aping sequences: If A is a regular Toeplitz transformation and ξ a bounded sequence which oscillates sufficiently slowly; and if x is a bounded sequence and y the transform of x by A; then $A(x\xi)=y\xi+\alpha$, where α denotes a null sequence, $x\xi$ the sequence whose general element is $x_n\xi_n$, and $A(x\xi)$ the transform of $x\xi$ by A. (Received June 1, 1948.)

461. Evelyn Frank: *On the continued fractions* $k_0\gamma_0+K_0^\infty(k_n(1-\gamma_n\bar{\gamma}_n)\cdot z/\bar{\gamma}_n z-1/k_{n+1}\gamma_{n+1})$.

In this paper it is shown that every power series (1) $\sum_{n=0}^\infty c_n z^n$ admits an expansion into a continued fraction of the form (2) $k_0\gamma_0+k_0(1-\gamma_0\bar{\gamma}_0)z/\bar{\gamma}_0 z-1/k_1\gamma_1+k_1\cdot(1-\gamma_1\bar{\gamma}_1)z/\bar{\gamma}_1 z-1/k_2\gamma_2+\cdots$, where the γ_p are certain rational functions of the c_p, and the k_p are constants chosen so that $|\gamma_p|\neq 1$. Conversely, to every continued fraction (2) there corresponds a power series (1). The power series expansions for the $2p$th and $(2p+1)$th approximants of (2) agree with (1) up to and including the term involving z^p and z^{p-1} respectively. A simple algorithm is shown for the expansion of the power series (1) into the continued fraction (2). Conditions under which the continued fraction (2) converges uniformly to the function represented by the power series (1) are found. Also conditions on the γ_p and k_p are given for which (2) converges uniformly in certain specific regions. (Received July 23, 1948.)

462. R. E. Fullerton: *A characteristisation of L spaces.*

By using methods similar to those of Clarkson in characterizing spaces of continuous functions it is shown that necessary and sufficient conditions that a Banach space X be a space of integrable functions are: (1) The surface of the unit sphere contains a maximal closed convex set F which contains at least one point not on its boundary such that if C is the set $E\{\lambda x|x\in F, \lambda\geq 0\}$ then C has the property that for any points u, $v\in X$ there exists a $z\in X$ such that $(u+C)\cap(v+C)=z+C$. (2) If $\|x\|=\|y\|=1$ and $-[(x-C)\cap(y-C)]=C$ then all point of the segments joining x and y and

x and $-y$ lie entirely on the surface of the unit sphere, (3) The unit sphere is the closed convex set determined by F and $-F$. In such a space X the family of maximal convex sets on the surface of the unit sphere may be made into a Boolean algebra in such a way that X is equivalent to the space of all completely additive functions defined over the algebra which vanish at $-F$. If F is a convex set determined by a denumerable set of independent points then X is equivalent to the space l. (Received July 24, 1948.)

463t. P. R. Garabedian: *A problem of Robinson.*

Let D be a domain of the z-plane bounded by analytic curves $C_1, \cdots, C_n$, $n \geqq 2$. Let z_0 be a point of D, and let Ω be the class of analytic functions $F(z)$ in D with a possible simple pole at $z = z_0$ and with boundary values which are in modulus not greater than 1. Denote by $\sigma(\zeta)$ the maximum of $|F(\zeta)|$ for $F \in \Omega$, and denote by A the set of points $\zeta \in D$ such that $\sigma(\zeta) = 1$. Then the region $D - A$ is simply-connected. This result has been proved by Robinson (Duke Math. J. vol. 10 (1943) pp. 341–354) for the case $n = 2$. The method of the present paper is based on a continuity argument and the uniqueness of the extremal function $F_0 \in \Omega$ such that $F_0(\zeta) = \sigma(\zeta)$. It is further shown that for $n \geqq 3$ the set A has, except in special cases, interior points. If the class Ω is extended to include functions with $m \geqq 1$ possible simple poles, further results can be obtained by a method of contour integration. If the poles of $F(z)$ are located at the critical points of the Green's function of D, $G(z; \zeta)$, then $|F(\zeta)| \leqq 1$. (Received May 25, 1948.)

464t. J. J. Gergen and F. G. Dressel: *Mapping by p-regular functions.*

Let $p(x, y)$ be positive in the domain R. If $u(x, y) \in C'$, and if $(pu_x)_x + (pu_y)_y = 0$, in R, then u is p-harmonic in R. If $u, v \in C'$, and if $pu_x = v_y$, $pu_y = -v_x$, in R, then $F(z) = u + iv$, $z = x + iy$, is p-regular in R. This paper is concerned with some of the properties of functions, p-harmonic or p-regular in the interior S of a circle, under continuity and analytic conditions on p. The fundamental problem considered is that of extending the classical theorem on the conformal mapping of S onto a domain bounded by a simple, closed, rectifiable curve. An extension of this theorem is obtained for p positive and analytic in a domain containing the closure of S, and sup. bound $(|\nabla p|/p)$ on S sufficiently small. The analysis is based on the methods of Douglas and Courant. (Received August 2, 1948.)

465. L. M. Graves: *A mapping theorem.*

Let $G(x)$ be a function of class C' near the origin $x = 0$ in a Banach space X, with values in a Banach space Y, and suppose for convenience that $G(0) = 0$. It is well known that the equation $y = G(x)$ gives a one-to-one correspondence between neighborhoods of $x = 0$ and $y = 0$ whenever the differential relation $dy = dG(0; dx)$ does so. The present paper is concerned with the case when the condition of one-to-one-ness is omitted in hypothesis and conclusion. The method of successive approximations is still applicable in the proof. The result makes it possible to derive a multiplier rule for minima of abstract functions without the use of projections. (Received July 26, 1948.)

466t. P. R. Halmos: *A nonhomogeneous ergodic theorem.*

If X is a non-atomic measure space with measure μ, such that $\mu(X) = 1$, and T is a metrically transitive measure preserving transformation of X onto itself, then there exists a function f in $L_2(X)$ such that $\int f d\mu = 0$ and such that $\sum_{n=1}^{\infty} (1/n)f(T^n x)$ is *not* convergent in the mean of order two. This result shows that not only is Izumi's nonhomogeneous ergodic theorem vacuous, but even that there are no general hypotheses on T which imply its conclusion; to ensure the convergence of such series, special hypotheses on f are necessary. (Received June 8, 1948.)

467*t*. Israel Halperin: *Hamel's basis and nonmeasurable sets.*

Let a set of real numbers E be called s-nonmeasurable if $m_*(E) = 0$ and $m^*(EI) = m(I)$ for every measurable set I. Suppose that $f(x+y) = f(x) + f(y)$ for all x, y, that $f(x)$ is not identically 0 and that $f(x)$ has only a countable number of distinct values $c_1, c_2, \cdots, c_n, \cdots$. Let E_n be the set of x for which $f(x) = c_n$. Then the E_n are all s-nonmeasurable. Let $a_1, a_2, \cdots, a_\alpha, \cdots$ be a Hamel's basis so that every real x has a unique representation $\sum_\alpha r_\alpha a_\alpha$, with all r_α rational and only a finite number of them different from zero. Then the x for which $r_1 = 0$ form an s-nonmeasurable set. More generally, any restrictions, which are not self-contradictory and are not vacuous, and which involve a fixed finite or countable set of the r_α, will define an s-nonmeasurable set. This gives a construction to subdivide the set of real numbers (or any interval) into a countable family of mutually exclusive, congruent (under translation) s-nonmeasurable sets. (Received August 3, 1948.)

468*t*. Israel Halperin: *Non-finite solutions of the equation $f(x+y) = f(x) + f(y)$.*

The complete family of solutions as given by G. Hamel, in Math. Ann. vol. 60 (1905) p. 461, seems to include many functions which have infinite values for some x. However the indicated non-finite solutions are not unambiguous. It is now shown that, even with the widest possible allowance for non-finite solutions, the only ones are the trivial ones (1) $f(x) = (+\infty)$ for all x with the convention $(+\infty) + (+\infty) = (+\infty)$, and (2) $f(x) = (-\infty)$ for all x with the convention $(-\infty) + (-\infty) = (-\infty)$. (Received August 3, 1948.)

469*t*. W. L. Hart: *Line integrals independent of the path in real Hilbert space.*

Let H be the set of all $x = (x_1, x_2, \cdots)$ in real Hilbert space for which $\|x\| \leq r$. The paper considers $\mathfrak{J} = \int_C \sum_{i=1}^{\infty} k_i(x) dx_i$, where C is a continuous arc $x = x(t)$ in H for $0 \leq t \leq 1$, and consists of a finite number of regular pieces (on which there exists a continuous derivative $x'(t)$ with $\|x'(t)\| \neq 0$). It is proved that $\mathfrak{J}$ is independent of the path C in case (1) $k_i(x)$ is continuous in H and $\sum_{i=1}^{\infty} k_i^2(x)$ converges uniformly on every compact subset of H; (2) there exists $\partial k_i/\partial x_j$, continuous in H; (3) $\partial k_i/\partial x_j = \partial k_j/\partial x_i$. There is a corresponding theorem on the solution of the system $\partial f(x)/\partial x_i = k_i(x)$ $(i = 1, 2, \cdots)$. In form, (3) is not new. The novelty of the paper consists of the elimination of an assumption that $\sum_{i,j} (\partial k_i/\partial x_j)^2$ converges. This condition is involved if the independence of the path is proved as a special case of results phrased for much more general situations in Banach spaces by Kerner (Ann. of Math. (2) vol. 34 (1933) p. 546) and Michal (Acta Math. vol. 68 (1937) p. 71). (Received July 27, 1948.)

470. J. G. Herriot: *The polarization of a lens.* Preliminary report.

Consider a conductor placed in an electrostatic field uniform at infinity and having a given direction there. The intensity of the disturbance produced by the conductor may be measured by a quantity called the polarization in the given direction. (A more precise definition of this concept will be given in a forthcoming paper by M. M. Schiffer and G. Szegö. See Bull. Amer. Math. Soc. Abstract 54-7-354.) The mean polarization P_m is the average of the polarizations in any three mutually orthogonal directions. If V is the volume and C the electrostatic capacity, then for a sphere it is known that $P_m=2V=(8\pi/3)C^3$. It is the purpose of this paper to compare P_m with V and C for the lens which is a solid determined by the intersection of two spheres. For the bowl (limiting case of lens) it is shown that $P_m \geqq (8\pi/3)C^3$. For a lens with dielectric angle $\pi/2$ and for two tangent spheres (another limiting case of lens) it is shown that $P_m+V\geqq 4\pi C^3$. This implies that $P_m \geqq (8\pi/3)C^3 \geqq 2V$. In the first two cases the quantities were expressed in terms of elementary functions whereas in the third case the logarithmic derivative of the gamma function was involved. P_m was obtained from the above-mentioned paper. (Received July 23, 1948.)

471*t*. Einar Hille: *Remarks on a paper by Zeev Nehari.*

Zeev Nehari has proved in a paper to appear in Bull. Amer. Math. Soc. that $f(z)$ is univalent in the unit circle if its Schwarzian derivative satisfies the inequality $|\{f(z), z\}| \leqq 2[1-|z|^2]^{-2}$. In the present note it is shown that 2 is the best possible constant in the sense that for every $C>2$ there exists a function, holomorphic in the unit circle, which satisfies $|\{f(z), z\}| \leqq C[1-|z|^2]^{-2}$ and takes on certain values infinitely often in the unit circle. (Received August 28, 1948.)

472*t*. I. I. Hirschman: *The behaviour at infinity of certain convolution transforms*. Preliminary report.

Let $a_1, a_2, \cdots,$ be any sequence of real constants for which $\sum_1^\infty a_k^{-2}<\infty$. We define $E(s)=\prod_1^\infty(1-s/a_k)e^{s/a_k}$, $G(t)=(2\pi i)^{-1}\int_{-i\infty}^{i\infty}[E(s)]^{-1}e^{st}ds$, $\overline{E}(s)=\pi(1-s/|a_k|)e^{s/|a_k|}$, $\overline{G}(t)=(2\pi i)^{-1}\int_{-i\infty}^{i\infty}[\overline{E}(s)]^{-1}e^{st}ds$. $G(t)$ is said to belong to class Ia if there are both positive and negative a_k's and if $\sum_1^\infty|a_k|^{-1}=\infty$. $G(t)$ is said to belong to class II if there are only positive a_k's and if $\sum_1^\infty a_k^{-1}=\infty$. Let $\alpha(t)$ be a function of bounded variation in every finite interval and set (*) $f(x)=\int_{-\infty}^{\infty}G(x-t)d\alpha(t)$ where we suppose only that this integral converges conditionally for some one value of x. If $G(t)\in$ class II it is known that the transform (*) converges for $(\gamma_a<x<\infty)$ and diverges for $(-\infty<x<\gamma_a)$ where γ_a is a constant depending on $\alpha(t)$. The author proves that if, in this case, $f(x)=O[G(x-r)]$ as $(x\rightarrow\infty)$, then $\alpha(t)$ is constant for $(r<t<\infty)$. If $G(t)\in$ class Ia then the transform (*) necessarily converges for all x $(-\infty<x<\infty)$. It is shown, in this case, that if $f(x)=O[\overline{G}(x-r)]$ as $(x\rightarrow+\infty)$ for arbitrarily large negative values of r, then $f(x)=0$. (Received July 9, 1948.)

473*t*. I. I. Hirschman and D. V. Widder: *A representation theory for a general class of convolution transforms.*

Let $E(s)=e^{-cs^2}\prod_1^\infty[1-(s/a_k)]e^{s/a_k}$, where the constants a_k are real and such that $\sum_1^\infty 1/a_k^2<\infty$. The authors showed earlier (see Bull. Amer. Math. Soc. Abstract 54-5-193) that $1/E(s)$ is the bilateral Laplace transform of some function $G(t)$. The present paper obtains necessary and sufficient conditions for the representation of a function $f(x)$ as a convolution transform, $f(x)=\int_{-\infty}^{\infty}G(x-t)\,d\beta(t)$, where the function $\beta(t)$ is in a variety of special classes. In particular Bernstein's familiar representation of a completely monotonic function as a Laplace transform is generalized. (Received June 22, 1948.)

474. Witold Hurewicz: *Linearization problems.*

Let U be a neighborhood of the origin 0 in Euclidean E_n, and F a continuous mapping of U with $F(U) \in E_n$ and $F(0)=0$. When is F topologically equivalent to a linear transformation in the neighborhood of the origin, that is, when does there exist a topological mapping T defined over a certain n-hood of the origin such that TFT^{-1} is an affine mapping? In case F is represented by analytic functions of coördinates a sufficient condition can be given. In general the problem is unsolved. (Received July 27, 1948.)

475. M. A. Hyman: *On the identification principle for partial differential equations with singular coefficients.* Preliminary report.

A. Weinstein (Trans. Amer. Math. Soc. vol. 63 (1948) pp. 342–354) has considered the equation (I) $y\phi_{yy}+y\phi_{xx}+p\phi_y=0$ and proved the following identification principle for $p>0$: *Let* $\phi_1(x, y)$ *and* $\phi_2(x, y)$ *be two solutions of* (I) *even in* y *and analytic throughout a region* R *containing a segment* L *of the x-axis: then if* ϕ_1 *and* ϕ_2 *coincide on* L, *they coincide everywhere in* R. The author has extended Weinstein's result to cases where $p<0$ (but not one of the eigenvalues $p=-1, -2, -3, \cdots$) and obtained a necessary and sufficient condition that $\phi(x, y)$ be a solution of (I) analytic in R; from this condition it follows that ϕ must be even in y and uniquely determined by its values on L. The chief tools are expansions in powers of y. Using the same methods, the author has investigated the general equation (II) $y^n\phi_{yy}+A\phi_{xx}+B\phi_x+C\phi_y+D\phi=0$ where A, B, C, D are analytic functions of x, y in R and n is an integer not less than 1. He has found conditions on A; B, C, D such that an analytic solution $\phi(x, y)$ of (II) shall be even or shall be uniquely determined by its values on L. Extensions and applications of these results are now being studied. (Received July 28, 1948.)

476. W. G. Leavitt: *On matrices with elements holomorphic in an unbounded region and a generalization to Prüfer rings.*

Let $\mathfrak{C}$ be the set of all functions holomorphic in and on the boundary of every finite portion of a region R of the complex z-plane. Let A be any matrix with elements in $\mathfrak{C}$ and with characteristic equation each of whose roots is in $\mathfrak{C}$. It is shown that there exists a matrix $T(z)$ with elements in $\mathfrak{C}$ whose determinant $|T|$ is nonvanishing in R, such that $T^{-1}AT$ is in the normal form with zeros below the main diagonal. The theorem is proved by generalizing the proof (Duke Math. J. vol. 14 (1948)) for R a bounded region, requiring the additional theorem: there exists an integral function whose expansion is specified to a finite number of terms at each of a set of points having no finite accumulation point. An alternative proof is provided by showing that $\mathfrak{C}$ is a Prüfer ring, that is, a ring satisfying all postulates of a principal ideal ring with the exception of the infinite chain condition. It is found that the proof of Leavitt and Whaples (to appear in Bull. Amer. Math. Soc.) for principal ideal rings is also applicable to Prüfer rings, and thus to the ring $\mathfrak{C}$. (Received June 28, 1948.)

477t. Walter Leighton: *A substitute for the Picone formula.*

An idea first employed by Marston Morse is developed to provide a substitute and generalization for the Picone formula. It is shown that this generalization enables one to establish comparison theorems for solutions of self-adjoint differential systems to which the classical Sturm-Picone methods do not apply. (Received July 28, 1948.)

478*t*. G. R. MacLane: *Polynomials with zeros on a Jordan curve.*

Let D_1, D be bounded simply-connected domains in the z-plane, $D_1 \subseteq D$, and let the boundary of D be a rectifiable Jordan curve Γ. Let $f(z)$ be holomorphic and never zero in D_1. There exists a sequence of polynomials $P_n(z)$ with *all* zeros on Γ, such that $\lim_{n\to\infty} P_n(z) = f(z)$ uniformly in any closed subset of D_1. The zeros cannot be essentially more restricted since in general all the zeros of the sequence must be everywhere dense on Γ. In some cases results on the rapidity of convergence are obtained, for example: if Γ is analytic, $f(z)$ holomorphic and never zero in the closed domain $D+\Gamma$, then one can find $P_n(z)$ of degree n such that $|P_n(z) - f(z)| < c/nd^3$ where c is a constant and $d = \mathrm{dist}(z, \Gamma)$. Similar results are obtained for a piecewise analytic boundary. The results are extended to multiply-connected domains D, the approximating functions now being rational with exactly one pole (multiple) in each simply-connected domain exterior to D and with all zeros on the boundary of D. The approximating functions may be written explicitly in terms of an appropriate parametrization of the boundary. (Received June 24, 1948.)

479. Morris Marden: *On the polynomial solutions of the generalized Lamé differential equation.*

The generalized Lamé differential equation considered is $P(z)w'' + Q(z)w' + R(z)w = 0$ where $P(z)$, $Q(z)$ and $R(z)$ are real polynomials of degrees p, at most $p-1$ and at most $p-2$ respectively. $P(z)$ is assumed to have only simple zeros a_j $(j = 1, 2, \cdots, p)$ of which the first q $(q \leq p/2)$ lie in the upper half plane, the next q are the conjugate imaginaries of the first q and the remaining are real. In the expansion $Q(z)/P(z) = \sum_{j=1}^{p} [A_j e^{i\alpha_j}/(z - a_j)]$ it is assumed that all $A_j > 0$ and all $|\alpha_j| < \pi/2$. As is well known, choices of $R(z)$, to be denoted by $V(z)$, exist corresponding to which the differential equation has a polynomial solution $S(z)$. In a previous paper (M. Marden, Trans. Amer. Math. Soc. vol. 33 (1931) pp. 934–944) the zeros of the $V(z)$ and the $S(z)$ were shown to lie in the smallest convex region enclosing both the real a_j and the ellipses with foci at the conjugate imaginary pairs a_j and $\bar{a}_j$ and with eccentricities of $\cos \alpha_j$. In the present paper, k denotes the number of pairs of conjugate imaginary zeros of $S(z)$ and for each $a_j = b_j + ic_j$, $j = 1, 2, \cdots, q$, $E(a_j, k)$ denotes an ellipse with center at $e_j = b_j + c_j \tan \alpha_j$, with minor axis of $m_j = 2|c_j - e_j|$ parallel to the y-axis and with major axis of $k^{1/2} m_j$. It is shown that every non-real zero of $S(z)$ lies in at least one of the ellipses $E(a_j, k)$, that every non-real zero of the rth derivative of $S(z)$ lies in at least one of the ellipses $E(a_j, k+r)$ and that every non-real zero of the corresponding $V(z)$ lies in at least one of the ellipses $E(a_j, k+2)$. (Received July 19, 1948.)

480*t*. Mary K. Peabody: *Differential operators of infinite order.*

Let $G(\theta) = \sum_{k=0}^{\infty} g_k \theta^k$ where the coefficients are complex numbers generated by an entire function $G(w) = \sum_{k=0}^{\infty} g_k w^k$. In the most general case $\theta = p(z)d/dz + q(z)$; $p(z)$ and $q(z)$ being analytic functions and $\theta^k = \theta \cdot \theta^{k-1}$. Four modifications of θ are considered: $\theta_1 = d/dz + q(z)$, $\theta_2 = d/dz + P(z)$, $\theta_3 = zd/dz + C$, $\theta_4 = zd/dz + P(z)$, where $P(z)$ is a polynomial of degree n. Necessary and sufficient conditions (taking the form of restrictions upon the order and type of $G(w)$) are determined for the applicability of the operators $G(\theta_i)$ to various classes of functions such as holomorphic functions, entire functions, and entire functions of specified order and type. For instance: If ρ and σ are conjugate orders and α and β conjugate types, a necessary and sufficient condition that $G(\theta_2)$ apply to the class of entire functions $\mathcal{J}_{\rho,\alpha}$, ρ fixed $> n+1$, α fixed

($0<\alpha<\infty$), is that $G(w)\in \mathfrak{G}_{v,\gamma}$, where $\gamma<\beta$. Estimates of the order and type of the transforms $G(\theta_i)f(s)$ are made when such exist as entire functions. Many of the results of this paper are analogous to those of H. Muggli (Comment. Math. Helv. vol. 11 (1938) pp. 151–179) for $G(d/ds)$ and E. Hille (Duke Math. J. vol. 7 (1940) pp. 458–495) for $G(s^2-d^2/ds^2)$. (Received July 15, 1948.)

481. B. J. Pettis: *On rings and half-fields in measure theory.*

This paper deals with the existence and uniqueness of extensions of a modular real function ϕ defined on a subset E of a generalized Boolean algebra L, where $E\ni 0$ and $\phi(0)=0$, and with the preservation of modularity, lattice continuity, and bounded variation. Extensions to the smallest field and smallest σ-field containing E when E is a ring or a half-field are particularly considered. One by-product is the following. Suppose E is a ring of subsets of a topological space, ϕ is monotone increasing, and given x in E and $\epsilon>0$ there exist x', x'' in E, an open y, and a compact z such that $x'\geqq y\geqq x\geqq z\geqq x''$ and $\phi(x')-\epsilon\leqq\phi(x)\leqq\phi(x'')+\epsilon$; then ϕ has a unique completely additive extension to the Borel sets. When E equals the open sets this corrects a theorem of Caccioppoli (Rado, *Length and area*) and provides a somewhat shorter proof of the Riesz-Markoff theorem on the representation of linear functionals over continuous functions. (Received August 12, 1948.)

482t. B. J. Pettis: *Remarks on abstract Carathéodory measurability.*

Let D be a generalized Boolean algebra, R any space having a single associative binary operation $+$, and ϕ an arbitrary but fixed function on D to R. Call an element a in D *measurable* if $\phi(x)=\phi(x\cap a)+\phi(x-xa)$ holds for every x in D, and *c-measurable* if $\phi(x)=\phi(x\cap a)+\phi(x-xa)=\phi(x-xa)+\phi(x\cap a)$ for every x in D. Let M and C be the sets of measurable and c-measurable elements; then M is a sub-lattice of D and C is a field in D. Further results deal with relationships between M, C, and properties of ϕ, and include sufficient conditions that C be a σ-field. These results include certain standard elementary theorems concerning Carathéodory outer measure (real-valued) functions and are connected with, but independent of, certain results of M. F. Smiley concerning measurable elements in lattices. (Received July 26, 1948.)

483t. V. C. Poor: *The extension of certain theorems in analytic function theory to polygenic functions.*

The polygenic functions considered in a domain of the complex plane are to be single-valued continuous and regular. They are regular in the sense that they possess a differential at every point of the domain. The application of the theory of residues over an area developed in a previous paper permits the proof of the analogue to the Liouville theorem. The Rouché and the Hurwitz theorems are also proved for polygenic functions. This extension essentially justifies the definitions for residues of polygenic functions over an area. (Received May 14, 1948.)

484t. M. H. Protter: *A class of harmonic polynomials.*

Harmonic polynomials are considered which arise in the solution of the Cauchy problem for the Laplace equation in three independent variables. A generating function, recursion and differential expressions as well as explicit expressions for the polynomials are obtained. The relations between these polynomials and Bessel, Legendre, Lamé, and Mathieu functions are discussed. The polynomials are transformable to "wave polynomials" without the introduction of complex quantities; the general solution of the Cauchy problem for the wave equation may be represented as the limit of sequences of such polynomials. (Received July 24, 1948.)

485. Tibor Rado: *On convergence in area.*

Let $x^j = x^j(u, v)$, $j = 1, 2, 3$, $(u, v) \in Q: 0 \leqq u \leqq 1, 0 \leqq v \leqq 1$, be a representation of a continuous path-surface S in Euclidean 3-space x^1, x^2, x^3, and let $A(S)$ be the Lebesgue area of S. Let S^1, S^2, S^3 be the flat surfaces $S^1: x^1 = 0, x^2 = x^2(u, v), x^3 = x^3(u, v)$, $S^2: x^1 = x^1(u, v), x^2 = 0, x^3 = x^3(u, v)$, $S^3: x^1 = x^1(u, v), x^2 = x^2(u, v), x^3 = 0$ (the projections of S upon the coordinate planes). Let $S_n: x^1 = x_n^1(u, v), x^2 = x_n^2(u, v), x^3 = x_n^3(u, v)$ be a sequence of surfaces such that $x_n^j(u, v) \to x^j(u, v)$ uniformly in Q, and let S_n^j, $j = 1, 2, 3$, be the projections of S_n upon the coordinate planes. Suppose that $A(S) < \infty$ and $A(S_n) \to A(S)$. Theorem: Under the conditions just stated, we have the relations $A(S_n^j) \to A(S^j)$, $j = 1, 2, 3$. This theorem is an extension, to continuous path-surfaces in general parametric representation, of similar results established previously in the literature for arc-length, and for surfaces given in non-parametric form. (Received March 11, 1948.)

486. P. V. Reichelderfer: *Law of transformation for essential generalized jacobians.*

Let $\bar{x}_1, \bar{x}_2, \bar{x}_3$ be a right-hand rectangular coordinate system obtained in euclidean three-space by applying a rigid motion to the right-hand system x_1, x_2, x_3—thus $\bar{x}_j = c_j + c_{jk}x_k$, where the constants c_j, c_{jk} are real and the matrix (c_{jk}) is normal and orthogonal. Given a continuous surface of the type of a circular disc; choose any representation for it, and denote the jacobians (assuming these exist) of its projections on the $\bar{x}_2\bar{x}_3$, $\bar{x}_3\bar{x}_1$, $\bar{x}_1\bar{x}_2$ planes respectively by $\bar{J}_1, \bar{J}_2, \bar{J}_3$, the jacobians of its projections on the x_2x_3, x_3x_1, x_1x_2 planes respectively by J_1, J_2, J_3. When these are ordinary jacobians it is well known that $\bar{J}_j = c_{jk}J_k$. The purpose of this note is to prove that the same formula holds almost everywhere for the essential generalized jacobians introduced by T. Rado and P. Reichelderfer (*A theory of absolutely continuous transformations in the plane*, Trans. Amer. Math. Soc. vol. 49 (1941) pp. 258–307). In making this proof the writer uses an unpublished result by T. Rado on convergence in area for surfaces (see abstract 54-11-485). (Received March 15, 1948.)

487t. P. V. Reichelderfer: *On the semi-continuity of double integrals.*

The purpose of this note is to extend the results of T. Rado, *On the semi-continuity of double integrals in parametric form*, Trans. Amer. Math. Soc. vol. 51 (1942) pp. 336–361, by using essential generalized jacobians instead of ordinary jacobians. In particular, it is shown that the value of the integral I of an admissible integrand is independent of the choice of an essentially absolutely continuous representation for an oriented continuous surface oS; thus the integral may be denoted by $I(oS)$. If oSo is an oriented continuous surface having an essentially absolutely continuous representation in a neighborhood of whose elements the integrand is non-negative and at which the Weierstrass E-function is also non-negative, then $I(oS)$ is lower semi-continuous at oSo in the class of all oriented continuous surfaces having essentially absolutely continuous representations. (Received April 29, 1948.)

488t. T. D. Reynolds: *Analytic solutions of integral equations having nonanalytic kernels.*

By real variable methods the author proves the following theorem. Let R denote the rectangle $a<x<b$, $c<y<d$. Suppose that $K(x, y, u, v)$ has the following properties in R: (1) $K(x, y, u, v)$ is continuous with its partial derivatives $\partial^{m+n+p+q}/\partial x^m \partial y^n \partial u^p \partial v^q K(x, y, u, v)$, $m, n, p, q=0, 1, 2, \cdots$, for (x, y), (u, v) in R, $(x, y) \neq (u, v)$; (2) there exist constants λ, M such that $\left|\partial^{m+n}/\partial x^m \partial y^n K(x, y, u+\sigma(x-u), v+\sigma(y-v))\right| \leq M m! n! \lambda^{m+n}/(1-\sigma) \left[(x-u)^2+(y-v)^2\right]^{(m+n+1)/2}$ for $0 \leq \sigma < 1$, (x, y) (u, v) in R, $(x, y) \neq (u, v)$, $m, n=0, 1, 2, \cdots$. Let $V(x, y)$ be analytic in R. Let $Z(x, y)$ be continuous and bounded in R and satisfy in R the integral equation $Z(x, y) = V(x, y) + \iint_R K(x, y, u, v) Z(u, v) du dv$. Then $Z(x, y)$ is analytic in R. The above theorem, in conjunction with some of Seward's work (Amer. J. of Math. vol. 66 (1944) pp. 255–267), yields a real variable proof of Seward's theorem (Ibid. p. 255) on harmonic continuation in space of a harmonic function which assumes analytic values on an analytic surface. (Received August 9, 1948.)

489. E. H. Rothe: *Critical points and gradient fields in Hilbert space.*

The paper is concerned with the theory of nondegenerated critical points of a scalar $i(x)$ in a Hilbert space H. Especially a relation is derived between the "Morse numbers" of such critical points and the mapping degree of the mapping of H into itself induced by the gradient of $i(x)$. The theory is applied to a nonlinear integral equation, and an existence and uniqueness theorem generalizing an earlier one by M. Golomb is proved. (Received July 26, 1948.)

490. I. J. Schoenberg: *Variation-diminishing Stieltjes integral operators of the convolution type.*

Let $L(t)$ be a real function of bounded variation in $-\infty < x < \infty$. We say that the transformation (1) $g(x) = \int_{-\infty}^{\infty} f(x-t)\, dL(t)$ is ***variation-diminishing*** if and only if the inequality $V(g) \leq V(f)$ always holds for every continuous and bounded $f(x)$; here $V(f)$ and $V(g)$ represent the numbers of changes of sign of $f(x)$ and $g(x)$, respectively. Following an oral suggestion of A. Dvoretzky and I. I. Hirschman, the author's results, as announced in Proc. Nat. Acad. Sci. U. S. A. vol. 34 (1948) pp. 164–169, may be generalized as follows: The transformation (1) is variation-diminishing if and only if $L(t)$ has a Laplace-Stieltjes transform of the form (2) $\int_{-\infty}^{\infty} \exp(-st) dL(t) = C \cdot \exp (\gamma s^2 + \delta s) \prod_{\nu=1}^{\infty} \exp(\delta_\nu s)/(1+\delta_\nu s)$, where all constants are real, $\gamma \geq 0$, $\sum \delta_\nu^2$ converges, and where (2) is valid in a certain strip containing the imaginary s-axis. (Received July 29, 1948.)

491. W. T. Scott and R. H. Stark: *Infinite linear systems and continued fractions.*

The authors consider two systems of linear forms $L_p(X) = -a_{p-1}X_{p-1} + b_p X_p - a_p X_{p+1}$, $a_{-p} = 1$, $a_p \neq 0$, and $M_p(Y) = -c_{p-1} Y_{p-1} + d_p Y_p - c_p Y_{p+1}$, $p=0, 1, 2, \cdots$, where a_p, b_p, c_p, d_p, X_p, Y_p are complex numbers. By means of a Green's formula similar to that used by Hellinger and Wall (Ann. of Math. vol. 44 (1943) pp. 103–127) for one system of linear forms, comparison theorems relating solutions of the systems $L_p(X) = 0$ and $M_p(Y) = 0$ are obtained. For example, if there exist $\epsilon_p > 0$ for which $\sum [\epsilon_p^2 + |b_p - d_p|^2 \epsilon_p^{-2} + |a_p - c_p|^2 \epsilon_{p+1}^{-2} + |a_{p-1} - c_{p-1}|^2 \epsilon_{p-1}^{-2}] |X_p|^2$ converges for every solution X of $L_p(X) = 0$, then $\sum \epsilon_p^2 |Y_p|^2$ converges. This yields as special cases invariability theorems and other known results for J-fractions. A theory of continued fractions $-K_0^{\infty}(-a_{p-1}^2/b_p)$ is developed under conditions which insure that a related quadratic form is positive definite. When a_p and b_p are suitably chosen analytic functions of s for $I(s)>0$, Stieltjes integral representations for the equivalent functions of the continued fraction are obtained. (Received June 16, 1948.)

492. I. E. Segal: *Two-sided ideals in operator algebras.*

It is shown that a closed two-sided ideal in a C^*-algebra (= a uniformly closed self-adjoint algebra of operators on a Hilbert space) is the intersection of kernels of irreducible representations of the algebra. The quotient of the algebra modulo the ideal is isomorphic to a C^*-algebra whose state space is linearly and topologically equivalent to the collection of states of the original algebra which vanish on the ideal. The correspondence between ideals and closed convex sets of states which arises in this way has the property that there is an identity modulo the ideal (that is, the ideal is regular) if and only if the set is compact, and that the set corresponding to the intersection of two ideals is the closed convex sum of the corresponding sets. The intersection of a finite number of regular ideals in a C^*-algebra is regular. (Received July 20, 1948.)

493. I. M. Sheffer: *On the theory of sum-equations.* II.

This paper complements the work of an earlier one of the same title. The sum-equation system (1) $\sum_{j=0}^{\infty} a_{nj}x_{n+j}=c_n$ $(n=0, 1, 2, \cdots)$ is related to the problem of expanding an analytic function $f(t)$ in a series of form (2) $f(t)=\sum_{0}^{\infty} c_n t^n A_n(t)$ where (3) $A_n(t)=\sum_{j=0}^{\infty} a_{nj}t^j$. When system (1) is k-periodic (that is, $A_{nk+j}(t)=A_j(t)$), solutions for (1) and (2) can be expressed in "closed" form; and, under certain conditions, the nonperiodic case can be solved by a limiting process based on the periodic case. It is also shown to be possible to treat the nonperiodic case (under certain conditions) independently of the periodic case. (Received July 26, 1948.)

494. Andrew Sobczyk: *Generalized analytic functions and conformal manifolds.* Preliminary report.

Let E_n and E_m be differentiable manifolds, having real local coordinate systems, and satisfying an additional requirement as suggested below. Consider functions $y=f(x)$ on E_n to E_m, defined by $y_1=f^{(1)}(x_1, \cdots, x_n), \cdots, y_m=f^{(m)}(x_1, \cdots, x_n)$, in terms of the local coordinates, the $f^{(i)}$'s being assumed to be of class $C^{(1)}$. Generalizing a familiar property of regular functions of a complex variable, such a function will be called *orthomorphic* in case at each point of the x-domain in E_n the m gradient vectors are of equal length and mutually orthogonal. It is shown that the graph in $E_n \oplus E_m$ of an orthomorphic function is a special *conformal* manifold; that is, if local Euclidean coordinate systems are chosen in the natural way on the graph, the transformation between any two overlapping systems is of the form of a translation plus a constant times a linear orthogonal transformation. Examples of orthomorphic functions are studied. Numerous new properties of the functions, and analogues of classical theorems on analytic functions on Riemann surfaces, are obtained. (In case $n=m=3$, it is a known theorem of Liouville that the $C^{(3)}$ conformal group is generated by the motions and inversions. This drastically limits the extent of the class of orthomorphic functions.) (Received August 9, 1948.)

495*t*. D. B. Sumner: *The inversion of convolution transforms by means of on integral.*

The generalized Stieltjes transform $f(x)=\int_0^\infty \phi(t)dt/(x+t)^\rho$, $\rho>0$, has an inversion formula $-(2\pi i)^{-1}\int_C (s+t)^{\rho-1} f'(s)ds$, where C is the circle $|s|=t$ cut at the point $-t$, a generalization of the original result of Stieltjes for $\rho=1$. A similar type of inversion formula is obtained for the convolution transform $f(x)=\int_{-\infty}^{\infty}g(x-t)\phi(t)dt$, where it is assumed that the bilateral Laplace transform $\int_{-\infty}^{\infty}\exp(-su)g(u)du$ has simple poles at λ_n, $-\mu_n$, $0<\lambda_1<\lambda_2<\cdots$, $0<\mu_1<\mu_2<\cdots$, and $\lambda_n, \mu_n\rightarrow\infty$. The inversion formula utilizes the closure properties of the set of functions $\{\exp(i\omega\lambda_n), \exp(-i\omega\mu_n)\}$ over $(-\pi, \pi)$. It supplements recent discoveries by Widder and Hirschmann, who have treated the problem of inverting the convolution transform as the solution of a differential equation of infinite order with operator $e^{aD}D^{p}\prod_{1}^{\infty}(1-D/a_n)e^{D/a_n}$. (Received August 12, 1948.)

496. Otto Szasz: *Inequalities concerning ultraspherical polynomials and Bessel functions.*

Recently Turán and Szegö proved the following inequality for Legendre polynomials: $(P_n(x))^2-P_{n-1}(x)P_{n+1}(x)\geqq 0$, $-1\leqq x\leqq +1$; analogous inequalities were given by Szegö for ultraspherical and other orthogonal polynomials (cf. Szegö, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 401–405). In this paper sharper inequalities are given for Legendre polynomials and for a class of ultraspherical polynomials. The basic tool is the well known recurrence formula. The method is applicable to other sequences of functions, satisfying a recurrence formula. We prove in particular for Bessel functions the inequality: $(\mu+1)\{J_{\mu}^2(x)-J_{\mu-1}(x)J_{\mu+1}(x)\}\geqq J_{\mu}^2(x)$. (Received July 29, 1948.)

497. Otto Szasz and John Todd: *The convergence of Cauchy-Riemann sums to Cauchy-Riemann integrals.*

A set of conditions sufficient to ensure that $\lim h\sum_{\nu=1}^{\infty}f(\nu h)=\int_{0}^{\infty}f(x)dx$, as $h\rightarrow 0$, where the integral is taken in the Cauchy-Riemann sense, is derived. These are sufficient to cover such cases as these when $f(x)=\sin x/x$, $f(x)=J_0(x)$, where the integrals are not absolutely convergent. (Received July 24, 1948.)

498t. W. J. Thron: *Singularities of functions represented by continued fractions.* Preliminary report.

Let $0\leqq m\leqq d_n\leqq M\leqq\infty$. The function $f(z)$ defined by $1+d_0z+K(z/(1+d_nz))$ is easily seen to be holomorphic for all z not on the negative real axis. It is shown that $f(z)$ has at least two singular points (not poles) on the negative real axis. Denote these points by z_1 and z_2 $(z_1>z_2)$. Concerning the location of these points the following inequalities have been established: $-1/(2+M+2(1+M)^{1/2})\geqq z_1\geqq -1/(1+M)$; $-1/m\geqq z_2\geqq -1/(2+m-2(1+m)^{1/2})$. (Received July 24, 1948.)

499. C. J. Titus: *A topological characterization of a class of affine transformations.*

Let the symbols $\{x_\sigma\}_m$ and $\{y_\sigma\}_m$ denote real-valued sequences of period m, $m\geqq 2$; that is, $x_{\sigma+m}=x_\sigma$, $y_{\sigma+m}=y_\sigma$ $(\sigma=0, \pm 1, \pm 2, \cdots)$. Any pair of such sequences defines in the xy-plane the sequence of points (x_σ, y_σ) $(\sigma=1, 2, \cdots, m)$. If one connects these points by going from the σth to the $(\sigma+1)$st point $(\sigma=1, 2, \cdots, m)$, first by a horizontal line segment and then by a vertical line segment, a closed oriented polygon is formed. A closed oriented polygon is said to be of *non-negative circulation* (Loewner, Charles, *A topological characterization of a class of integral operators.* Ann. of Math. (1948)) if the order, with respect to the polygon, of every point not on the polygon is non-negative. Consider the transformation $y_\sigma=-\sum_{p=1}^{m}a_px_{\sigma-p+1}$, where the a_r are arbitrary real numbers. The transformation maps a periodic sequence $\{x_r\}_m$ into the periodic sequence $\{y_r\}_m$. These two sequences, as previously described, define a sequence of points which in turn determine a closed oriented polygon. The principal theorem can now be stated. *A necessary and sufficient condition that the transformation "generate" only polygons of non-negative circulation for all* $\{x_r\}_m$ *is that there exist non-negative* C_p, β_p *and* γ_p *such that* $a_{k+1}-a_k=\sum_{p=1}^{q} C_p(\beta_p)^{m-k+1}(\gamma_p)^{k-1}$ $(k=1, 2, \cdots, m-1)$, $q=[m/2]$. (Received July 21, 1948.)

500*t*. H. S. Wall: *Convergence of continued fractions in parabolic domains.*

It is shown that the determinate case holds for a positive definite continued fraction $K[-a_{p-1}/(b_p+z_p)]$, in which the $|b_p|$ are bounded, if, and only, if, the series $\sum|d_p|$ diverges, where $d_1=1$, $d_{p+1}=1/(d_pa_p^2)$, $p=1, 2, 3, \cdots$. This result is used to establish the following extension of the "parabola theorems." For each $p=1, 2, 3, \cdots$, let $|c_p|-R(c_p \operatorname{Exp} \theta_p)\leqq 2r\cos\phi_p\cos\phi_{p+1}(1-g_{p-1})g_p$, where $\theta_p=i(\phi_p+\phi_{p+1})$, $-(\pi/2)+c\leqq\phi_p\leqq(\pi/2)-c$, $0<c<\pi/2$, $0<r<1$, $0\leqq g_{p-1}\leqq 1$, r and c being independent of p. The continued fraction $K(c_{p-1}/1)$, $c_0=1$, converges if, and only if, the series $\sum|k_p|$ diverges, where $k_1=1$, $k_{p+1}=1/(c_pk_p)$. (Received May 24, 1948.)

501. S. E. Warchawski: *On conformal mapping of variable regions.*

The paper is a contribution to the study of the degree of variation of the mapping function of a simply connected region under a deformation of the boundary curve. A typical result is the following: Suppose that C_i $(i=1, 2)$ denote two simple closed curves with continuously turning tangent, which satisfy the hypotheses: (i) For some $\epsilon>0$, C_1 is in the "ϵ-neighborhood" of C_2 (that is, the region obtained by describing a circle of radius ϵ about every point of C_2), and C_2 in that of C_1; (ii) If $\alpha_i(P)$ denotes the tangent angle to C_i at P, then $|\alpha_1(P_1)-\alpha_2(P_2)|<m\epsilon$, provided $\overline{P_1P_2}<\epsilon$ (m is a constant); (iii) The ratio of any arc s of C_i to the corresponding chord c satisfies the inequality $s/c\leqq a$ ($=$constant); (iv) C_i contain the circle $|z|<r$ and are contained in $|z|<R$. (v) The curvature of one of the curves, say $K_1(s)$ of C_1, exists for almost all values of the arc length s, and $\int_{C_1}|K_1(s)|^p ds$ exists for some $p>1$. Then, if $f_i(z)$ maps $|z|<1$ onto C_i, $f_i(0)=0$, $f_i'(0)>0$, one has $\int_0^{2\pi}|f_1'(re^{i\theta})-f_2'(re^{i\theta})|^p d\theta\leqq M^p\epsilon^p$ for $0\leqq r<1$, where M depends on m, a, r, R and p *only*. In particular, this implies: $|f_1(z)-f_2(z)|\leqq(M/2)\epsilon$ for $|z|\leqq 1$. (Received July 27, 1948.)

502*t*. Antoni Zygmund: *On the existence of boundary values for regular functions of several complex variables.* I.

Let $f(z_1, z_2, \cdots, z_k)$ be a function regular for $|z_1|<1$, $|z_2|<1, \cdots, |z_k|<1$. Suppose that the integral $J=\int_0^{2\pi}\cdots\int_0^{2\pi}\log^+|f|(\log^+\log^+|f|)^{k-1}dx_1\cdots dx_k$ is bounded for $r_1, \cdots, r_k\to 1$, where f stands for $f(r_1e^{ix_1}, \cdots, r_ke^{ix_k})$. Then, for almost every point $(x_1, x_2, \cdots, x_k)$ in the cell $0\leqq x_1\leqq 2\pi, \cdots, 0\leqq x_k\leqq 2\pi$, the function $f(z_1, \cdots, z_k)$ tends to a finite limit $f(e^{ix_1}, \cdots, e^{ix_k})$ as the points $z_1, \cdots, z_k$ approach $e^{ix_1}, \cdots, e^{ix_k}$, each along a nontangential path. If in the integral J we drop the iterated logarithm, the boundedness of the resulting integral suffices for the existence, for almost every $(x_1, \cdots, x_k)$, of the *iterated* limit of $f(z_1, \cdots, z_k)$ as each of the variables $z_1, \cdots, z_k$ tends successively and nontangentially to the corresponding point e^{ix_j}. Let $\alpha>0$, and let $F(x_1, \cdots, x_k)=\sup_{r_1,\ldots,r_k}|f(r_1e^{ix_1}, \cdots, r_ke^{ix_k})|$. If the integral $J_\alpha=\int_0^{2\pi}\cdots\int_0^{2\pi}|f|^\alpha dx_1\cdots dx_k$ is bounded for all $r_j<1$ by a finite number M^α, then the function F is of the class L^α, and the integral of F^α over $0\leqq x_1\leqq 2\pi, \cdots,$

$0 \leq x_k \leq 2\pi$ does not exceed $C_\alpha M^\alpha$, where C_α depends on α only. In particular, if J_α is bounded, then $\lim \int_0^{2\pi} \cdots \int_0^{2\pi} \left| f(r_1 e^{ix_1}, \cdots, r_k e^{ix_k}) - f(e^{ix_1}, \cdots, e^{ix_k}) \right|^\alpha dx_1 \cdots dx_k = 0$, as $r_1, \cdots, r_k \to 1$. (Received June 25, 1948.)

APPLIED MATHEMATICS

503*t*. R. C. F. Bartels and O. Laporte: *Supersonic flow past a delta wing at angles of attack and yaw.*

The paper deals with certain solutions of the linearized potential equation for a supersonic flow which define conical flow fields. The method outlined in an earlier treatment is employed in the determination of the flow around a delta wing at angles of attack and yaw. The velocity field is expressed in terms of a function $F(\zeta)$ of the complex variable ζ which is analytic in a doubly connected region of the ζ-plane corresponding to the disturbed region of flow between the wing and its Mach cone. In this representation the components of velocity form the real parts respectively of the triplet of analytic functions defined by the integrals $\beta\int(1+\zeta^2)F(\zeta)d\zeta$, $i\beta\int(1-\zeta^2)F(\zeta)d\zeta$, $-2\int\zeta F(\zeta)d\zeta$, which resemble the formulas of Weierstrass for the solution of the minimal surface problem; here β is the cotangent of the Mach angle. The specification of $F(\zeta)$ is formulated as a boundary value problem for the doubly connected region of the ζ-plane in which the boundary conditions require that the velocity vanish along the Mach cone and be tangential to the surface of the wing. It is shown that the solution of this problem is not unique. There is, however, one and only one solution for which the normal force coefficient of the wing is finite. This solution is given in terms of elliptic functions. (Received July 27, 1948.)

504. Lipman Bers: *The Dirichlet problem for a partial differential equation of mixed type.* Preliminary report.

Let D be a plane domain situated in the upper half-plane whose boundary B lies partly on the line $y=0$. Let $K(y)$ be a sufficiently well-behaved function such that $K(0)=0$, $K(y)>0$. It is shown (by a method imitating Poincaré's "balayage") that in D the first boundary value problem for the equation $K(y)\psi_{xx}+\psi_{yy}=0$ always has a (unique) solution. Until now only the case $K(y)=y$ has been considered, by Tricomi, Hellerstedt and Frankl (see Frankl, Bull. Acad. Sci. USSR. vol. 10 (1946) pp. 166–182 and the literature quoted therein). These authors used rather complicated methods and imposed severe restrictions on the domain D. The author requires merely that for every sufficiently small $\epsilon>0$ the intersection of D with the half-plane $y>\epsilon$ be a connected Dirichlet domain. (Received July 27, 1948.)

505. A. S. Cahn: *The warehouse problem.*

John von Neumann, George B. Dantzig, T. C. Koopmans, and others are currently developing methods for solving the programming or management type problem in a rapidly changing "economy." This involves particularly the maximization of a linear form in variables subject to linear inequalities. It is shown that their methods are applicable to the problem of the optimum use of a storage warehouse for commodities subject to periodic price fluctuation. This example is important because the assumptions made in linearizing the economy are not seriously different from the realities of the situation. Also, the computation involved lies within the scope of present day computing machinery, so that the method could be immediately useful. The effects of various side conditions upon the size of the computation are discussed. (Received July 30, 1948.)

506t. G. B. Dantzig: *Programming in a linear structure.*

W. Leontief, Schlienger, Wald, von Neumann, and T. C. Koopmans have studied economic models of the type considered here. This paper differs essentially from those of the above authors in that it is concerned with the basic problem of programming in a rapidly changing "economy." The basic assumptions of the model lead to a fundamental set of linear equations expressing the conditions which must be satisfied by the various levels of activity, X_t, in the dynamic system. These variables are subject to the further restriction $X_t \geq 0$. The determination of the "best" choice of X_t is made to depend on the maximization (or minimization) of linear form in X_t (a typical example would be the minimization of the total budget over several time periods). This problem is equivalent to the maximization of a linear form whose variables are subject to linear inequalities. It is also closely related to the problem of determining a Min-Max of a bilinear form. In is proposed that computational techniques such as those developed by J. von Neumann and by the author be used in connection with large scale digital computers to implement the solution of programming problems. (Received July 28, 1948.)

507. A. H. Diamond: *Reproducing kernels for certain classes of biharmonic functions.*

The author considers the reproducing kernels for real biharmonic functions such that $\iint_D(\Delta u)^2dxdy$ and $u=0$ on the boundary of D. It is shown how to construct the kernel function for certain simply connected domains by orthonormalizing the system $\mathrm{Re}\{\bar{z}z^n\}$ and $\mathrm{Im}\{\bar{z}z^n\}$. The relationship between the kernel function and the Green's and Neumann's function is established and this is used to show how the kernel function can be used to solve the boundary value problem $\Delta\Delta u=0$ in D, $u=0$ and $\partial u/\partial\eta=f(s)$ on the boundary of D. (Received July 27, 1948.)

508. Wilfred Kaplan: *Dynamical systems with indeterminancy.*

In a previous paper (Bull. Amer. Math. Soc. Abstract 52-5-176) it was proposed that the differential equations $dx_i/dt=f_i(x_1\cdots x_n)$ of a physical system be modified to allow for errors in the f_i. One is thus led to consider as ϵ-solutions curves $x_i=x_i(t)$ whose tangent vectors are close to f_i in a certain sense. The order symbol $x'<x''$ is used to signify that x'' can be reached from x' along such an ϵ-solution with increasing time; the notations $\alpha(x)=E\{x'|x'<x\}$ and $\alpha(X)=\bigcup\alpha(x)$, $x\in X$ (X any subset of the phase space M) are used. It is shown that there exist certain disjoint open sets $X_1\cdots X_N$ (called "stable stationary states") in M with the properties: (1) $\alpha(X_j)$ is open; (2) $X_j\subset\alpha(X_j)$; (3) $\bigcup\alpha(X_j)=M$; (4) $X_j\cdot\alpha(X_k)=0$ for $j\neq k$. The nerve of the covering of M (assumed compact) by the $\alpha(X_j)$ is a complex K, each simplex of which corresponds to a class of points of M with equally uncertain future. This complex is thus a model for the qualitative structure of the system. In general, a sufficiently small change in the allowed error ϵ has no effect on K. It is shown that K can be found by an analysis on a finite subset of M. (Received July 21, 1948.)

509. M. Z. Krzywoblocki: *On the hodograph equation in adiabatic inviscid rotational flow.*

By a suitable choice of a function analogous to the velocity potential in a potential flow, the author succeeded in transferring the stream function equation into the hodograph plane. The final equation differs considerably from the known "Chaplygin equation" in the potential flow. (Received June 3, 1948.)

510*t*. J. P. LaSalle: *Relaxation oscillations.*

A case of relaxation oscillations which is simpler though not included in the general case of Levinson and Smith is studied. Sufficient conditions for the existence of stable periodic solutions (limit cycles) are found by the construction of regions each of which contains a single periodic solution. The regions approximate the location of the periodic solutions in the phase plane and provide bounds on the periods and amplitudes. The asymptotic case of relaxation oscillations is investigated by considering the behavior of the regions as a parameter approaches zero. (Received July 10, 1948.)

511*t*. C. C. Lin: *On the flow of compressible fluid through a group of regularly arranged airfoils.*

The complex potential function of the incompressible flow due to two complex sources in the ζ-plane at $\zeta = a_1$ and $\zeta = a_2$ outside the circular boundary $|\zeta| = 1$ is $F(\zeta) = A_1 \log (\zeta - a_1) + \bar{A}_1 \log (\zeta - 1/\bar{a}_1) + B_1 \log (\zeta - a_2) + \bar{B}_1 \log (\zeta - 1/\bar{a}_2)$, where A_1 and B_1 are complex constants, and $\operatorname{Re}(A_1) = -\operatorname{Re}(B_1)$. It is shown that the mapping $dz = g(\zeta)(\zeta - a_1)^{-1+1/n}(\zeta - a_2)^{-1-1/n}d\zeta - \bar{f}, = -4^{-1}F'^2(\zeta)\{g(\zeta)\}^{-1}(\zeta - a_1)^{1-1/n}(\zeta - a_2)^{1+1/n}d\zeta$ gives a compressible flow (with von Kármán-Tsien approximation) past a circular series of n identical blades in the z-plane, with velocity potential ϕ and stream function ψ given by $\phi + i\psi = F(\zeta)$, provided that $g(\zeta)$ satisfies the following general requirements: (a) It is regular in the region R, defined by $|\zeta| \geqq 1$. (b) It does not vanish in R except possibly at one point on the circle where $F'(\zeta) = 0$. (c) Along the circle $|\zeta| = 1$, $\oint dz = 0$. (d) Except in the neighborhood of $\zeta = a_1$, $|F'(\zeta)\{g(\zeta)\}^{-1}(\zeta - a_1)^{1-1/n} \cdot (\zeta - a_2)^{1+1/n}| < 2$ in R. In the limiting case $n \to \infty$, condition (d) is to be satisfied throughout R, and the above scheme defines the compressible flow past a straight series of infinite number of blades. (Received July 15, 1948.)

512. M. H. Martin: *A new approach to the theory of two-dimensional flows.*

When the pressure p and stream function ψ are chosen for independent variables in the two-dimensional steady flow of an inviscid fluid subject to no external forces, the stream lines and velocity vector are presented parametrically by $x = -\eta_p$, $y = \xi_p$, $u = \xi_\psi$, $v = \eta_\psi$, where $\xi = \xi(\psi, p)$, $\eta = \eta(\psi, p)$ satisfy the pair of partial differential equations $\xi_\psi^2 + \eta_\psi^2 = 2T$, $\xi_\psi \xi_{pp} + \eta_\psi \eta_{pp} = 0$, and $T = T(\psi, p)$, the specific kinetic energy, is an assigned function of ψ, p. Eliminating one of ξ, η from the pair, a Monge-Ampère partial differential equation results for the other and the Cauchy-Kowalewski existence theorem leads to a local existence and uniqueness theorem for flows possessing an assigned stream line and pressure distribution along it, together with a method for calculation of such flows. For irrotational flows $T_\psi = 0$ and the Monge-Ampère equation has an intermediate integral which yields a parametric representation for the well known Prandtl-Meyer flows. The theory is not restricted to irrotational flows. Applied to rotational flows, the Munk-Prim substitution principle is obtained and flows with one-dimensional hodographs are investigated. (Received July 26, 1948.)

513*t*. Leo Moser: *Linked rods and continued fractions.*

A set of n uniform rods $AB, BC, \cdots, MN$, each of mass m and length $2a$, are connected by smooth joints at $B, C, \cdots, M$, and lie in one straight line on a smooth horizontal table. A horizontal blow P is struck at N in a direction perpendicular to

BC. Let v_k and ω_k be the relative initial linear and angular velocities of the kth rod. Let p_k/q_k be the kth convergent to the continued fraction expansion of $3^{1/2}$. Recurrence formulae are found for the v's and ω's and by comparing these with the recurrence formulae for the p's and q's it is shown that $|v_k| = p_{2k-1}$, $|\omega_k| = q_{2k-1}$. (Received July 16, 1948.)

514. R. R. Reynolds: *Solution of boundary value problems for Laplace's equation in a multiply-connected domain.*

Methods developed in papers of Bergman and Schiffer (Duke Math. J. vol. 14 (1947) pp. 609–638) and Bergman (Quarterly of Applied Mathematics vol. 5 (1947) pp. 69–81) for obtaining the solution $u(x, y)$ of the Dirichlet problem for an m-ply connected domain B_m are applied to the case where B_m is the unit circle Γ_0 with $m-1$ circles Γ_r punched out. General formulas are given for determining a system of analytic functions $f_\nu(z) = \sum_{\mu=1}^{\nu} c_{\nu\mu} v_\mu(z)$ satisfying $\iint_B f_\mu(z)\overline{f_\nu(z)}\,dx\,dy = \delta_{\mu\nu}$ from a complete set $v_\mu(z)$. In particular, the $c_{\mu\nu}$ are calculated in a numerical example for B_3 and used to solve the boundary value problem in which $u(x, y)$ has given constant values u_r on $\Gamma_r (r=0, 1, 2)$. Finally, in connection with conformal mapping theorems, the coefficients $a_{\mu\nu}^{(N)}$ $(N=1, 2, \cdots, 15)$ of the approximating kernel $K_N(z, w) = \sum_{\nu=1}^{N} \sum_{\mu=1}^{N} a_{\mu\nu}^{(N)} v_\mu(z)\overline{v_\nu(w)}$ are computed for B_3. (Received September 3, 1948.)

515*t*. H. E. Salzer: *Coefficients for polar complex interpolation. I. Lagrangian coefficients.*

In a previous article by the author, namely ***Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs***, Journal of Mathematics and Physics vol. 25 (1945) pp.1 41–143, formulas for Lagrangian interpolation coefficients for complex analytic functions tabulated around a circular arc at points differing in phase by the constant amount θ are given as rather lengthy trigonometric functions of the angle θ. The present work supplements the preceding one by giving the numerical values, to eight decimals, of those functions of θ, for six cases quite likely to arise in practice, namely $\theta = 1°, 5°, 10°, 15°, 20°$ and $30°$, for the 3-, 4-, and 5-point Lagrangian interpolation formulas. (Received June 30, 1948.)

516*t*. H. E. Salzer: *Coefficients for polar complex interpolation. II. Auxiliary coefficients for Lagrangian interpolation.*

In the author's previous article, namely, ***Alternative formulas for direct interpolation of a complex function tabulated along equidistant circular arcs***, Journal of Mathematics and Physics vol. 27 (1947) pp. 56–61, formulas for certain auxiliary coefficients are given for direct Lagrangian interpolation for complex analytic functions tabulated around a circular arc at points differing in phase by the constant amount θ. Those formulas are (1) long trigonometric expressions in terms of θ, and (2) when evaluated for smaller values of θ, very many significant figures are lost. For these two reasons, it is a great help to have the formulas for the auxiliary coefficients evaluated for the angular intervals θ which are most likely to arise in practice. The present work supplements the above mentioned article by giving those complex coefficients for $\theta = 1°, 5°, 10°, 15°, 20°$ and $30°$, with an accuracy in the modulus of nine significant figures (eight figures guaranteed), for the 3-point through the 9-point Lagrangian interpolation formulas. (Received July 30, 1948.)

517*t*. E. A. Trabant: *Vibrations of cantilever beam as geodesics.*

Using the methods of A. D. Michael (*Studies in geodesics in vibrations of elastic beams*, Proc. Nat. Acad. Sci. U.S.A. vol. 31 (1945) pp. 38–43) the transverse vibrations of a cantilever beam which satisfy its fourth order partial differential equation with the usual boundary conditions are investigated. By a suitably defined arc-parameter s, the generalized Euler-Lagrange equation is obtained. Starting with the Euler-Lagrange equation the original differential equation is obtained. It is shown that the vibrations of the cantilever beam with total energy C can be represented as geodesics in a "Riemannian" space with element of arc-length given in terms of two arbitrary continuous functions which satisfy the boundary conditions. (Received July 26, 1948.)

518*t*. C. A. Truesdell: *On Cauchy's vorticity formula.*

The following kinematic formula for the vorticity $\mathbf{w}$ of any continuous motion of any continuum is derived: $\mathbf{w}/\rho=[\mathbf{w}_0/\rho_0+(1/\rho_0)\int_{t_0}^{t}\text{curl}\,(\text{grad}\,\mathbf{r}\cdot\mathbf{a})dt]\cdot\text{grad}\,\mathbf{r}$. Here ρ is the density, t_0 is the time at which $\mathbf{w}=\mathbf{w}_0$ and $\rho=\rho_0$, $\mathbf{r}$ is the Eulerian position vector, $\mathbf{a}$ is the acceleration, and all differential operations are taken with respect to the Lagrangian variables. In a circulation-preserving motion the integral, which represents the effect of diffusion of vorticity, vanishes, and there remains only the convective expression derived by Cauchy. The general formula is applied to a compressible viscous fluid. (Received August 3, 1948.)

GEOMETRY

519. C. C. Hsiung: *Invariants of intersection of certain pairs of curves in n-dimensional space.*

The purpose of this paper is to derive and characterize metrically, as well as projectively, some projective invariants associated with an ordinary point O of intersection of two curves C, C' in n-dimensional space S_n ($n\geqq 3$) for each of the following two cases: (a) C, C' have at O distinct osculating linear spaces S_k, S_k' ($k=1,\cdots,n-1$) of k dimensions. (b) C, C' have at O distinct tangents but $S_2\equiv S_2'$, $\cdots$, $S_r\equiv S_r'$, where r is any fixed integer satisfying $2\leqq r\leqq n-1$. (Received July 27, 1948.)

520*t*. H. T. Muhly and Oscar Zariski: *Hilbert's characteristic function and the arithmetic genus of an algebraic variety.*

The properties of the generalized characteristic function of Hilbert associated with a doubly homogeneous polynomial ideal are studied in the case where the homogeneous ideal is determined by a pair (U, V) of projective models of a field Σ of algebraic functions. This study yields the following results. (1) The virtual arithmetic genus $p_a(W)$ of a sectionally normal model W of Σ is a relative birational invariant, whereby a sectionally normal model is meant a normal model which is such that its sections by almost all linear subspaces of its ambient space are also normal. (2) If Σ is of dimension two or three over the ground field k, then the virtual arithmetic genu- of a nonsingular model of Σ is an absolute birational invariant. (3) If Σ is of dimens sion two over k and if U and V are normal models of Σ such that $U<V$, then $p_a(U)\geqq p_a(V)$. The theorem of Riemann-Roch for a large class of linear systems on an algebraic surface follows as a corollary to these results. (Received July 6, 1948.)

521*t*. Chenkuo Pa: *Some theorems on rectilinear congruences and transformations of surfaces.*

Let S be a nondevelopable analytic surface in ordinary space; let L be a rectilinear congruence with each of its generators passing through a corresponding point of S but not tangent to S, and let S' be a surface generated by the harmonic conjugate point of a point of S with respect to the foci of L on the corresponding generator through the point of S. The author proves the following results: 1. If the asymptotic net of S corresponds both to that of the focal sheets of L, then S is an R-surface and L is an R-congruence with its developables cutting S in an R-net. Conversely, for a given R-surface, there exist ∞^4 R-congruences having the above properties. 2. If the developables of L conjugate to S, and if the asymptotic nets of S and S' are in correspondence, then both S and S' are Jonas surfaces and the developables of L cut S and S' both in Jonas nets. Conversely, for a given Jonas surface S, there exist ∞^8 rectilinear conjugate congruences to S which satisfy the above conditions. (Received July 8, 1948.)

LOGIC AND FOUNDATIONS

522*t*. A. R. Schweitzer: *An outline of the history and the philosophy of the concept of orientation.* II. Preliminary report.

This paper continues a paper reported in Bull. Amer. Math. Soc. Abstract 44-7-304. From a psychological point of view this paper is concerned with "orientative thinking" classified as follows: 1. Speculative thinking, as represented, for example, by questions in the Book of Job. 2. Philosophical thinking, as represented, for instance, by the "elemental thinking" outlined by Albert Schweitzer (*Out of my life and thought*, New York, 1933, p. 260). 3. Postulational thinking, as represented by the hypotheses of mathematical physics and the assumptions of geometry and mathematical analysis. On another basis of classification the subject matter of this paper refers to man's position in: A. the universe (religion, philosophy); B. the world (sociology, social psychology, ethics); C. nature (science, mathematics). The term "orientation" is conceived so as to include organization (arrangement, interrelationship) of parts in a whole. (Received July 27, 1948.)

523*t*. A. R. Schweitzer: *On a relation of mathematics to philosophy.*

The author aims to interpret Kant's *Kritik der reinen Vernunft* (1781) as a means of gradual transition from mathematics to metaphysics. The principal references to Kant's treatises (apart from the above) are: (1) *Von dem ersten Grunde des Unterschieds der Gegenden im Raum* (1768), (2) *Prolegomena zu einer jeden kunftigen Metaphysik die als Wissenschaft wird auftreten konnen* (1783), (3) *Was heisst: Sich im Denken orientiren?* (1786). The transition is effected by classifying mathematics into: I. a science of extension: space (geometry) and time (arithmetic; in particular modular arithmetic), II. a science of discovery (heuristic mathematics) and abstraction from nature (applied mathematics; mathematical physics; analysis), III. a science of a priori principles; that is, axiomatic or postulational mathematics. These three types of mathematics are respectively associated with Kant's three universes of discourse, namely: I. sensibility, II. understanding, III. reason. The author discusses some of Kant's views concerning mathematical conceptions which seem unnecessarily restricted in the light of subsequent developments. In this connection reference is made to the monograph of Wilhelm Reinecke, *Die Grundlagen der Geometrie nach Kant*, Kantstudien vol. 8 (1903) pp. 345–395. (Received July 27, 1948.)

STATISTICS AND PROBABILITY

524. Herman Chernoff: *Asymptotic studentization in testing of hypotheses.*

If H is a hypothesis for which $t \leq c_1(\theta)$ would be a good test if the value of the nuisance parameter θ were known and $\hat{\theta}$ is an estimate of θ, then the following method of asymptotic studentization (obtaining critical regions of almost constant size) was suggested by Wald. Consider $t \leq \phi(\hat{\theta})$ where $\phi(\hat{\theta}) = c_1(\hat{\theta}) + \cdots + c_s(\hat{\theta})$ and $\Pr\{t \leq c_1(\theta)\} = \alpha$, $\Pr\{t - c_1(\hat{\theta}) \leq c_2(\theta)\} = \alpha, \cdots$, $\Pr\{t - c_1(\hat{\theta}) - \cdots - c_r(\hat{\theta}) \leq c_{r+1}(\theta)\} = \alpha$. It is shown that under reasonable conditions this test and various modifications, designed for those cases where the $c_r(\theta)$ are difficult to obtain exactly, have the asymptotic property that $\Pr\{t \leq \phi(\hat{\theta}) = \alpha + O(N^{-r/2})$ where N is the size of the sample involved or an analogous variable. This property can be extended to the case where θ is a k-dimensional variable. (Received May 24, 1948.)

525. K. L. Chung: *An estimate concerning the Kolmogoroff limit distribution.*

This paper sharpens Kolmogoroff's result (1933) on the maximum deviation of the empirical distribution based on n samples from the theoretical distribution $F(x)$, assumed to be continuous. Let $nF_n(x)$ be the number of sample values not exceeding x and $d_n = \sup_x |n(F_n(x) - F(x)|$; Kolmogoroff proved that for a fixed λ, $\Pr(d_n \leq \lambda n^{1/2})$ tends to the limit distribution $\Phi(\lambda) = \sum_{j=-\infty}^{\infty} (-1)^j \exp(-2j^2\lambda^2)$ uniformly in λ as $n \to \infty$. It is shown in this paper that the result holds even if $\lambda = \lambda(n)$ varies with n but lies within the range $(A_0 \lg n)^{-1}$ to $(A_0 \lg n)$ where A_0 is any positive constant; moreover the difference $\Pr(d_n \leq \lambda(n)n^{1/2}) - \Phi(\lambda(n))$ is of the order of magnitude of nearly $n^{-1/10}$. This estimate is obtained as consequence of a general theorem on "lattice distributions," after the problem is reduced (by Kolmogoroff) to one about the addition of independent random variables following a certain lattice distribution. From the asymptotic distribution follows a new law of the iterated logarithm, that is, $\Pr(\lim\sup d_n 2^{1/2}(n \lg n) = 1) = 1$, and also more refined statements. (Received August 30, 1948.)

526t. William Feller: *Fluctuation theory of recurrent events.*

Consider a sequence of independent or dependent trials but suppose that each has a discrete sample space. This paper studies recurrent patterns $\mathcal{E}$ which can be roughly characterized by the property that after every occurrence of $\mathcal{E}$ the process starts from scratch, the conditional probabilities coinciding with the original absolute probabilities. Typical examples are success runs, returns to equilibrium, zeros of sums of independent variables, passages through a state in a Markov chain. New methods are developed unifying and simplifying previous theories and applying to larger classes of recurrent events. It is shown in an elementary way that the probability that $\mathcal{E}$ occurs at the nth trial either has a limit or is asymptotically periodic. This theorem has many consequences. For example, the ergodic properties of discrete Markov chains follow in a few lines, and the difference between finite and infinite chains disappears. Several theorems of the renewal type are proved. Weak and strong limit theorems for the number N_n of occurrences of $\mathcal{E}$ in n trials are derived shedding new light on stable distributions. (Received July 29, 1948.)

527t. William Feller: *Infinitely divisible distributions.*

A simple derivation of P. Levy's formula is given starting from the following definition: a distribution function $F(x)$ is infinitely divisible if for every n it is possible to find finitely many distributions $F_{k,n}(x)$ such that $F(x)=F_{1,n}(x)*\cdots*F_{k_n,n}(x)$ and that $F_{k,n}(x)$ tends to the unitary distribution uniformly in n. This definition is more general than the one used by P. Levy and Khintchine. The equivalence of the two definitions was proved by Khintchine by deep methods. The new approach renders the equivalence obvious. Furthermore, a new characterization of infinitely divisible distributions is given; it is equivalent to Gnedenko's characterization but requires no special analytical tools. (Received July 29, 1948.)

528. E. L. Lehmann and Henry Scheffé: *Completeness, similar regions, and unbiased estimation.* Preliminary report.

A family $\mathfrak{M}$ of measures M on a space X of points x is defined to be *complete* if $\int_X f(x)dM=0$ for every M in $\mathfrak{M}$ implies $f(x)=0$ except on a set A for which $M(A)=0$ for every M in $\mathfrak{M}$. For a given family of measures the question of completeness may be regarded as the question of unicity of a related functional transform. Classical unicity results are applicable to many families of probability distributions that have been studied by statisticians. The notion of completeness throws light on the problem of similar regions and the problem of unbiased estimation. The concept of a *maximal* sufficient statistic—roughly, a sufficient statistic that is a function of all other sufficient statistics—is developed. A constructive method of finding such is given, which seems to apply to all examples ordinarily considered in statistical theory. A relation between completeness and maximality is found. (Received July 23, 1948.)

529. Herman Rubin (National Research Fellow): *Some results on the asymptotic distribution of maximum- and quasi-maximum-likelihood estimates.*

The author investigates the asymptotic normality of maximum- and quasi-maximum-likelihood estimates of parameters of systems of linear stochastic difference equations. The principal tool is the extension of the central limit theorem to dependent variables previously obtained by the author (Bull. Amer. Math. Soc. Abstract 54-7-280). The results obtained are analogous to those in the case in which no differences are present. Some extensions are also made to systems of stochastic difference equations linear in the coefficients but not necessarily in the variables. If the complete system of stochastic difference equations is linear in the jointly dependent variables, asymptotic efficiency is demonstrated for maximum-likelihood estimates. (Received July 26, 1948.)

530. J. E. Walsh: *Some nonparametric tests of whether the largest observations of a set are too large.* Preliminary report.

Let $x(1),\cdots,x(n)$ represent the values of n observations arranged in increasing order of magnitude. By hypothesis these observations have the properties: (1) They are independent and form continuous symmetrical populations; (2) For large n the variances of the tail order statistics are either very large or very small compared with the variances of the central order statistics; (3) For large n the tail order statistics are approximately independent of the central order statistics; (4) Each observation is from a population whose median is either θ or ϕ, where $x(n-r+1),\cdots,x(n)$

are from populations with median θ while the central and smaller order statistics are from populations with median ϕ. The test is: *Accept* $\phi<\theta$ *if* $\min\left[x(n-i_k)+x(j_k);\ 1\leqq k\leqq s\leqq r\right]>2x(t_\alpha)$, where $i_s<i_{s+1}$, $j_s<j_{s+1}$, $i_r=r-1$, and t_α is defined by $\Pr\left[x(t_\alpha)<\phi\mid\theta=\phi\right]=\alpha$. Here $\alpha=\Pr\left\{\min\left[x(n-i_k)+x(j_k);\ 1\leqq k\leqq s\leqq r\right]>2\phi\mid\theta=\phi\right\}$. For large n the significance level of the test is approximately α while the significance level does not exceed 2α for any value of n. Suitable values of α can be obtained for $r\geqq 4$. As $\theta-\phi\rightarrow-\infty$ the power function tends to zero, while the power function tends to unity as $\theta-\phi\rightarrow\infty$. For $\theta-\phi<0$ the power function is monotonically increasing. (Received July 29, 1948.)

TOPOLOGY

531. Richard Arens: *Approximation in, and representation of, linear algebras.*

A BQ^*-algebra is a Banach algebra (real scalars) with a linear involutory anti-isomorphism (*) in which $\|f\|^2\leqq\|ff^*+gg^*\|$ for any f and g, and ff^* lies in the center for all f. The quaternions Q form the simplest noncommutative example. The main representation theorem for any BQ^*-algebra A is this: there is a compact Hausdorff space X on which the orthogonal group G acts leaving a point x_0 of X fixed such that A is equivalent to the ring of all continuous functions on X with values in Q for which $f(x_0)=0$ and $f(Tx)=T_Q(f(x))$, T_Q being the automorphism of Q corresponding to T of G. Indispensable for the proof is this generalization of the Stone-Weierstrass theorem: Let X be a compact Hausdorff space, and let C be the ring of continuous $n\times n$ complex-entried matrix valued functions on X. Let A be a closed linear subset of C such that if A contains f then it contains f^* and the trace of $f+f^*$. Then a necessary and sufficient condition that A include a subset B of C is that for each x and y of X and each b of B there is an a in A such that $a(x)=b(x)$, $a(y)=b(y)$. (Received July 7, 1948.)

532*t*. R. R. Bernard: *Probability in dynamical transformation groups*. Preliminary report.

Let T be a multiplicative abelian topological group which is separable, metric, locally compact and has a compact generating system. Let μ be the Haar measure on T. Let X be a metric space and let T act as a transformation group on X. Denote the image of $x\in X$ under $t\in T$ by xt. Let A be a subset of X, V a subset of T and let $x\in X$. Denote the set of all $t\in V$ such that $xt\in A$ by $T(x, A, V)$. Let s, t^* be arbitrary elements of T. Let W be any open generator of T which contains the identity and such that $K=\overline{W}s$ is compact. If $\mu T(x, A, K^nt^*)$ exists and if the limit as $n\rightarrow\infty$ of $\mu T(x, A, K^nt^*)/\mu(K^nt^*)$ exists and is independent of the choice of K and t^* this limit is termed the *probability that* xt *is in* A. Denote it by $P(x, A)$. It is shown that if T is connected the limit is necessarily independent of t^*. If T is the additive group of integers or reals this definition is equivalent to the classical definition. Every point of a minimal center of attraction, $A(x)$, of a point $x\in X$ is regionally recurrent with respect to $A(x)$. If X is compact, the minimal center of attraction, A, of X is a closed invariant set and if $x\in X$ and N is any open set containing A then $P(x, A)$ exists and equals one. The set of central orbits contains A. (For definitions cf. Gottschalk and Hedlund, Bull. Amer. Math. Soc. Abstract 54-3-169). (Received June 7, 1948.)

533*t*. Felix Bernstein: *The four color problem in a point lattice*. III.

In the second paper of this series it has been shown that configurations requiring five colors exist. The proof however necessitates the disjunction between a great number of cases. Here a general topological method is presented which permits complete classification into four color and five color configurations for a well defined class of configurations. The essential tool of this method is the concept of a "polygraph of order n" as an ordinary graph to whose elements (vertices and sides) numbers from a given finite set $S(0, 1, 2, \cdots, n)$ are affixed. The sequences of numbers which can be represented by a continuous way in the polygraph of the class correspond to the configurations of the class which can be colored with four colors. Thus necessary and sufficient conditions are derived. While the classical four color problem is confined to a finite number of two-dimensional regions, the problems of coloring solved here embrace as well the case of the finite as that of the infinite configurations of a certain class, and can also be generalized to the higher-dimensional space. In outlook as well as in methods they are more closely akin to the interests of the field of general topology. (Received July 1, 1948.)

534. R. H. Bing: *Partitioning a set.*

A set M can be partitioned if for each positive number ϵ there exists a finite collection of mutually exclusive open subsets of M such that the sum of the elements of this collection is dense in M and each element of this collection is of diameter less than ϵ and has property S. It is shown that any plane set with property S can be partitioned. Also, any compact locally connected continuum can be partitioned if it is not locally separated by any arc. As an application of the notion of partitioning, it is shown that a compact locally connected continuum can be assigned a convex metric if it can be partitioned. (Received July 27, 1948.)

535*t*. R. P. Dilworth: *The lattice of continuous functions.*

The set $C(S)$ of all bounded continuous real functions over a topological space S forms a lattice under the containing relation $f(x) \leq g(x)$ all $x \in S$. $C(S)$ is, in general, not a complete lattice, that is, a bounded set of continuous functions need not have a least upper bound among the upper bounds in $C(S)$. In this paper we construct a set of real functions over S which forms a complete lattice under the above containing relation and which is isomorphic to the minimal completion of $C(S)$ by means of normal subsets. By this means it is proved that *the completion of* $C(S)$ *by normal subsets is distributive* (*indeed, completely distributive*!). (Received July 9, 1948.)

536*t*. E. E. Floyd: *A non-homogeneous minimal set.*

There is constructed a compact subset X of the plane and a homeomorphism T of X onto itself such that X is minimal with respect to T and such that X is of dimension 1 at some points and of dimension 0 at the others. This is a solution of the problem of constructing a compact minimal set which is not homogeneous. (Received June 14, 1948.)

537. A. M. Gleason: *A structure theorem for spaces with a compact Lie group of transformations.*

Let R be a Hausdorff space in which Tietze's extension theorem holds. Let G be a compact Lie group operating on R. If $x \in R$ denote by G_x the set of all elements of G leaving x fixed. Suppose that $p \in R$ is such a point that, for all q near p, G_q is a conjugate of G_p. Then there is a closed neighborhood of $G(p)$ (the orbit of p) which is the topological direct product of $G(p)$ and a closed set C in R. The principal tool is the construction of a matrix valued function K on the space R such that $K(p)$ is the identity matrix and $K(g(x)) = H(g)K(x)$ for all $g \in G$ and $x \in R$, where H is a faithful matrix representation of G. A corollary of particular interest arises if R is itself a topological group and G is a compact Lie subgroup. In this case there is a neighborhood of G in R homeomorphic to the direct product of G and a neighborhood of G in the factor space R/G. (Received July 24, 1948.)

538*t*. D. W. Hall: *A note on metrization.*

This note presents a new metrization theorem, the proof of which is very short and completely elementary. Known metrization theorems, the original proofs of which were much more difficult, are shown to follow from this theorem without difficulty. (Received July 26, 1948.)

539*t*. Marshall Hall: *A topology for free groups and related groups.*

Countable free groups are among the groups G possessing the following two properties: (1) G has a countable infinity of elements; (2) For each element $g \neq 1$ of G there is a subgroup U of finite index in G such that $g \notin U$. A topology may be defined for such groups in which a basis for open sets consists of cosets of subgroups U of finite index in G. This topology may be realized by a linear metric mapping the elements of G onto a linear set. The closure of this set is a Cantor set, C, and the points of C correspond to the completion $\widehat{G}$ of G which is a compact group. It is shown that C may be constructed in such a way that the Haar measure on $\widehat{G}$ is equal to the corresponding Lebesgue measure on C. A subgroup H of G is proved to be the intersection of the U's which contain it if and only if H is closed in the topology. Throughout this paper the central interest is attached to the particular properties of this topology and not to the previously known theorems on embedding in compact groups. (Received July 12, 1948.)

540. P. R. Halmos: *On the representation of σ-complete Boolean algebras.*

The purpose of this note is to give a simple proof of the following generalization of a theorem of Loomis. Every Boolean σ-algebra is σ-isomorphic to the class of all Baire sets, modulo Baire sets of the first category, in a totally disconnected, compact Hausdorff space. (Received July 14, 1948.)

541. Edwin Hewitt: *A general fixed point theorem.*

Let L be a locally convex linear topological space over the real numbers and let A be any closed convex subset of L. Then any continuous mapping ϕ of A into itself such that $\phi(A)$ has compact closure admits a fixed point. The proof employs methods of Tychonoff and a generalization of a theorem due to Mazur. (Received July 27, 1948.)

542. E. E. Moise: *Grille decomposition and convexification theorems for compact metric locally connected continua.*

The principal theorems of this paper are as follows: (I) Let S be a compact metric locally connected continuum. Then there is a sequence $G_1, G_2, \cdots$ of finite collections of mutually exclusive connected, uniformly locally connected open sets, such that for each i, (1) each point of S belongs to the closure of some element of G_i, (2) G_{i+1} is a refinement of G_i, and (3) each element of G_i has diameter less than $1/i$. (II) Let S be as in (I). Then S can be given a convex metric preserving the original topology. The convexification problem, of which (II) is the solution, was proposed by Menger in 1928 (Math. Ann. vol. 100, p. 75). (Received July 27, 1948.)

543*t*. G. D. Mostow: *A new proof of E. Cartan's theorem on the topology of semi-simple groups.*

A new proof is given for the theorem of Cartan: a semi-simple Lie group is topologically the direct product of a compact subgroup and a Euclidean space. The notion of a symmetric Riemannian space which plays an essential role in Cartan's proof is eliminated. (Received July 19, 1948.)

544*t*. B. J. Pettis: *On points of uniform convergence in topological spaces.*

Let S be a topological space (no separation axioms) and T a uniform space with ϕ' the cardinal number of the uniform neighborhoods $\{V_\alpha\}$ in T. Let $L=[\lambda]$ be a directed set having cardinal number ϕ''. For each λ in L let $f_\lambda(x)$ be a function on S to T and suppose $\lim_\lambda f_\lambda(x)=f(x)$ exists in the Moore-Smith sense for each x. Call p in S a *point of uniform convergence* if given any α there exists a neighborhood U of p and some λ in L such that x in U and $\mu>\lambda$ imply $f_\mu(x)$ is in $V_\alpha(f(x))$. The following then holds: if each f_λ is continuous the points in S not of uniform convergence form an L_ϕ set (Dunford, Ann. of Math. vol. 41 (1940)) for any $\phi \geqq \phi', \phi''$. Furthermore, every point of uniform convergence is a continuity point of $f(x)$. For countable L and metric S and T these results were given by Kuratowski (Fund. Math. vol. 5 (1924)) and earlier and more restrictedly by Banach, Osgood, and Baire. For complete metric S and metric T these results yield short direct proofs of Montgomery's theorem on the double continuity of xy in metrized groups, the Banach-Mazur generalized ergodic theorem, and the Nikodym-Saks theorems on the convergence of completely additive set functions. (Received July 26, 1948.)

545. E. H. Spanier: *The Mayer homology theory.*

W. Mayer has defined new homology groups based on a boundary operator whose pth power (p a prime) is zero, instead of the usual one whose square is zero. Mayer established the topological invariance of these groups but left unsettled the question of their relation with the classical homology groups. In this paper it is shown that the Mayer groups do not lead to new topological invariants but lead instead to interesting alternative definitions of the classical homology groups. This result is obtained by applying the axiomatic characterization of homology theory of Eilenberg and Steenrod to a suitably relabeled collection of Mayer groups. (Received July 20, 1948.)

546. G. W. Whitehead: *A generalisation of the Hopf invariant.* II.

Let f be a mapping of the n-sphere S^n into S^r representing an element α of the nth homotopy group $\pi_n(S^r)$ of S^r, and suppose that the Freudenthal suspension Ef of f is inessential. To each nullhomotopy h of Ef there is associated (if $n<3r-2$) an element $\beta\in\pi_{n+2}(S^{2r+1})$. It is shown that $E^2H(\alpha)=\beta+(-1)^r\beta$, where $H(\alpha)$ is the generalized Hopf invariant of α (Bull. Amer. Math. Soc. Abstract 52-5-215). Conversely, if $\beta\in\pi_n(S^{2r-1})$ with r even and $n<3r-3$, then there exists $\alpha\in\pi_n(S_r)$ such that $E\alpha=0$ and $H(\alpha)=2\beta$. This construction is used to answer in the negative a question proposed by Hopf (Fund. Math. vol. 25 (1935) pp. 427–440): if $r=4k+2>2$ there is no element of $\pi_{r-1}(S^r)$ with Hopf invariant 1. Moreover, S^{4k+1} does not admit a continuous multiplication with two-sided identity if $k>0$. (Received July 12, 1948.)

547t. G. T. Whyburn: *Continuous decompositions.*

Motivated by the fact that the decomposition of a region effected by an analytic function is l.s.c. but not necessarily u.s.c. in the open set sense, a study is made of decompositions G of a topological space A into disjoint closed sets which are (i) u.s.c. in the limit sense ($X\cdot \lim\inf X_i \neq 0$ implies $X\supset \lim\sup X_i$) and (ii) l.s.c. in the open set sense (sum of all elements intersecting an open set is open). The hyperspace A' of such a decomposition is a topological space and the natural mapping $\phi(A)=A'$ is continuous and interior (open). Accordingly, perfect separability as well as regularity plus local compactness go over from A to A'. For locally connected spaces A the condition (iii) for each region R in A' each component of $\phi^{-1}(R)$ maps onto R under ϕ, seems appropriate to add to (i) and (ii) to define a continuous decomposition. If A is locally compact, locally connected, separable and metric, G will be both u.s.c. and l.s.c. in the open set sense if and only if the elements of G are compact and G satisfies (i), (ii) and (iii). Further, the decompositions effected by important classes of analytic functions satisfy (i), (ii) and (iii). (Received May 11, 1948.)

548t. G. T. Whyburn: *Continuous decompositions and developability.*

Extending a concept of Stoïlow, we define a light interior mapping $f(A)=B$ on a locally connected generalized continuum A to be *developable* provided A is the sum of a strictly monotone increasing sequence of conditionally compact regions $[R_n]$, $\overline{R}_n\subset R_{n+1}$, such that $f(\overline{R}_n)\cdot f[F(R_{n+1})]=0$ and $f|\overline{R}_n$ is interior. It is shown that for f to be developable it is necessary and sufficient that non-compactness be invariant under f for closed generalized contimua in A or, equivalently, that each component of the f inverse of a continuum in B be compact. This condition in turn implies that the decomposition generated by f is continuous in a natural and significant sense. Further, the condition is satisfied by well defined classes of analytic functions. (Received May 11, 1948.)

549t. G. T. Whyburn: *Interior mappings on locally compact spaces.*

An interior (open) mapping $f(A)=B$ on a locally connected generalized continuum is said to generate a continuous decomposition of A provided (*) for any region R in B, each component of $f^{-1}(R)$ maps onto R under f. This condition is implied by a similar local conditions at points of B and also by the condition (†) for any continuum K in B, each component of $f^{-1}(K)$ is compact. In fact, a localized (†) implies (*); but (*) does not imply (†) as is shown by the mapping $w=e^z$. If (*) holds, the decomposition generated in each component of the inverse of a region in B is itself continuous. If A is a 2-manifold, f is light and (*) holds, then either $f^{-1}(y)$ is infinite for every $y\in B$ or f has a finite degree k, uniform for all $y\in B$. If f is light, for any locally connected generalized continuum Y in B whose interior rel. B is dense in Y, $f^{-1}(Y)$ is locally connected; further, any $a\in A$ is interior to an arbitrarily small locally connected continuum in A on which f is interior. (Received May 11, 1948.)

550t. G. T. Whyburn: *Sequence approximations to interior mappings.*

Given a sequence of continuous transformations $f_n(x)$ of a locally compact separable metric space A onto open subsets of a similar space B which converges uniformly on each compact subset of A to a mapping $f(x)$. Necessary and sufficient conditions are first established in order that $f(x)$ be an interior (or open) mapping. The sequence is said to be uniformly approximately interior on a subset A_0 of A provided that for any $\epsilon>0$, a δ and an N exist such that the f_n image of the ϵ-neighborhood of any x in A_0 contains the δ-neighborhood of $f_n(x)$ for all $n>N$. It is shown that if A is locally connected and the mappings f_n are interior and f is light, the sequence is uniformly approximately interior and f is interior. (Received April 5, 1948.)

551. R. L. Wilder: *Monotone mappings of manifolds.*

Monotone mappings of 2-manifolds have been extensively studied. The present paper attempts to begin a general study of r-monotone mappings (G. T. Whyburn, Amer. J. Math. vol. 57 (1935) p. 904) of n-manifolds. Simple examples show that 2-monotone images of the 3-sphere may no longer be manifolds; they will, however, be generalized manifolds. Accordingly the n-gm is adopted as the basic configuration, and it is shown that an $(n-1)$-monotone image of an orientable n-gcm is again an orientable n-gcm with the same homology characters. It is also shown that (1) an $(n-1)$-monotone image of an lc* compactum is lc*; (2) nasc that a locally compact space S be lc* is that if M is a compact subset of S such that $p^r(M)=0$ for some $r\leq n$, then for any open set U containing M there exists an open set V such that $M\subset V\subset U$ and all r-cycles on V bound on U; (3) if S is an orientable n-gm and M a compact subset of S such that $p^r(M)=p^{n-r-1}(M)=0$ for some r, $1\leq r\leq n-2$, then for any open set U containing M there exists an open set V such that $M\subset V\subset U$ and all compact r-cycles in $V-M$ bound in $U-M$. (Received July 26, 1948.)

552. G. S. Young: *On regular convergence of sequences of 2-manifolds.* II.

Two new types of regular convergence are defined, W-r-regular, and S-r-regular. Let $\{M_n\}$ be a sequence of two-manifolds in a locally compact metric space converging to a set M. If $\{M_n\}$ converges $W-1$-regularly to M, and the sequence $\{B_n; B_n$ the boundary of $M_n\}$ converges $W-0$-regularly to a subset of M, then M is a 2-manifold. If W-regularity is replaced by S-regularity, then M is homeomorphic to almost all M_n. (Received July 26, 1948.)

R. H. Bruck,
Associate Secretary

BOOK REVIEWS

Lezioni sulla teoria delle funzioni di una variabile complessa. By G. Sansone. Padova, Cedam, 1947. Vol. 1, 8+359 pp. L. 750. Vol. 2, 11+564 pp. L. 950.

These two volumes on the theory of analytic functions are based on lectures given by the author at the University of Florence. As can be expected from the size of this treatise a much wider variety of topics is covered than usual. A brief summary of the contents of the various chapters gives an indication of the material included in each of the two volumes.

Volume 1: Chap. 1, Power series and elementary functions; Chap. 2, Integral theorems of Cauchy, Laurent series; Chap. 3, Factorization theorem of Weierstrass; Chap. 4, Entire functions and the theorem of Picard; Chap. 5, Euler-Maclaurin and Lagrange series, asymptotic series and methods of summability.

Volume 2: Chap. 7, Dirichlet series, Riemann zeta function, hypergeometric series; Chap. 8, Conformal mapping, Riemann's theorem; Chap. 9, Harmonic functions, Dirichlet and Neumann's problems; Chap. 10, Hyperbolic metric and applications to conformal mapping; Chap. 11, Elliptic functions; Chap. 12, Fuchsian functions.

The broadness of the theory of analytic functions has naturally forced the author to restrict himself in his choice of topics and any choice would have invited a criticism. It seems, however, unfortunate that the theory of multivalued functions and Riemann surfaces were left out entirely. Both in his choice of topics and in their treatment the author has followed classical lines. In the opinion of the reviewer the injection at some places of a more modern point of view could have increased the value and interest of the book. The point set topology of the two-dimensional plane is not given as full a consideration as might be desired and in at least two cases the proofs are not complete on this point. But as an over-all judgment the author gives a clear, careful and thorough exposition of the theory. The many references to original papers and later contributions which are included in the text should be particularly valuable to the reader.

F. BOHNENBLUST

A treatise on set topology. Part I. By R. Vaidyanathaswamy. Madras, Mahadevan, 1947. 6+304 pp. Rs. 16–4.

This book begins with three introductory chapters on sets; topics treated here are algebras of subsets of a set, rings and fields of sets and algebras of partial order. Fundamental ideas concerning sets such as mapping, lattice, Boolean algebra, ideal and measure are included.

With these preliminaries out of the way, the author introduces topological spaces. The next three chapters deal with general topological spaces and some important special classes of spaces. Subjects discussed include Kuratowski's closure function and other topological concepts defined in terms of it, the neighborhood postulates due essentially to Hausdorff and the lattice of closed sets and the lattice of open sets. Concepts such as strengths of topologies, postulates of countability, separation axioms and bicompact and compact spaces are studied.

With the aid of the important properties of the spaces developed in the previous chapters, the book now journeys into a very essential part of the subject, the theory of mappings. The next chapter investigates various kinds of mappings: closed, open and continuous, including homeomorphic. Kolmogorov's theory of resolution spaces, chain resolution, Urysohn's lemma on normality, and Urysohn's extension theorem are treated. This chapter also has an excellent exposition of the important completely regular (Tychonoff) spaces.

The remaining four chapters deal with further structural properties of spaces, sets in spaces, mappings and convergence concepts. The derived set and related subjects are studied. Special topological products and classical theorems on topological products are handled. Convergence in metric spaces is treated; here analyses are made of compactness, completeness and topological completeness. The final chapter is on convergence topology; convergence is considered as a primitive concept given by a system of axioms which is very slightly more specialized than that of Fréchet's *L*-spaces.

This book has grown out of lectures delivered by the author at the University of Madras for several successive years. For the most part the book is self-contained; the portions demanding extra knowledge involve some elements of analysis, including an acquaintance with measure theory and the theory of transfinite numbers. It seems to the reviewer that a short section on transfinite numbers might be advantageously included so as to present at least the theorems of that subject needed in this text.

Throughout the book, the author emphasizes partial ordering. The author states "partial ordering is an inherent feature of the topological situation." In connection with partial ordering, the author raises two questions (cf. the Preface and p. 134, 1.20). They both admit affirmative answers. For a solution of the first, see the reviewer's Bull. Amer. Math. Soc. Abstract 54-5-212.

Since the book appears to be primarily intended for beginners, it seems that the author should mention and emphasize the methodology and consequences of the axiomatic approach. For instance, mention might profitably be made of the importance of function spaces in modern developments. Indeed, some not-so-bright novice might even be tempted to ask what this is all about. Aside from this general consideration, it seems that certain special aspects of set topology also fail to receive adequate emphasis; for example, the distinction between absolute (intrinsic) and relative concepts is not stressed and the metrization theorem of Urysohn appears as an exercise. Although α and μ ideals are introduced, and μ ideals of the sets in a topological space are studied, the author does not appear to be aware of the further development in the form of H. Cartan's important theory of filters (that is, proper α ideals). Certain classical landmarks fail to have the proper signposts. For instance, epithets such as Tychonoff space, Heine-Borel-Lebesque property, Urysohn's lemma on normality, and first and second axioms of countability do not stand over the corresponding places. The book also lacks an appendix, which can be profitably used especially in a book of this kind. However, some of these criticisms, particularly the last one, are probably premature, since the author plans to publish the second part of this treatise in 1948.

A few misprints and minor errors are observed. Most of the misprints can be easily corrected and are not likely to cause much confusion. Some of the slips are:

The "if" part of ex. 19, p. 58 is valid but not correctly proved; the equalities $H=\text{Int } X$ and $\text{Bd } F\cap H'=\text{Bd } X$ are both false.

On p. 111, 1.27 to 1.28 "$S\cap\overline{U}_s$ is the closure $\cdots$ of $\cdots$ $S\cap U_s$" should read "$S\cap\overline{U}_s$ *contains* the closure $\cdots$ of $\cdots$ $S\cap U_s$". This error occurs in several places.

The statement "the extension of f_0 will then give the extension of f" in the last line of p. 158 is not correct unless ∞ is also admitted as a value. However, Theorem 45.1 (Urysohn's extension theorem) is true without considering ∞ as a value; the unbounded case can be disposed of by a simple argument.

Ex. 9(2) of p. 164 is incorrect. The word "is" in 1.30 should be "contains." Consider, for example, the subspace $(0, 1/2, 1/3, 1/4, 1/5, \cdots)$ of the real number space.

The book is generally accurately and clearly written. A beginner who has had the prerequisites mentioned previously should find little difficulty in reading it. Every chapter of the book contains a large collection of exercises. These examples can serve well to illustrate the concepts and methods which occur; the more difficult ones among them might, in addition, help the reader to delve more deeply into the subject matter. Solutions of more difficult exercises are given or indicated in the text.

HING TONG

Integration in finite terms. By J. F. Ritt. New York, Columbia University Press, 1948. 7+100 pp. $2.50.

Advances in mathematical analysis have been intimately bound to the development of the notion of function. It was only in the last century that, both in the real and complex domain, the function concept was explicitly and completely elaborated. With this achievement it was possible to place on a sound foundation the calculus of earlier times. Having progressed so far, certain questions concerning the properties of functions of classical analysis lost their logical import. But their historical significance remains untarnished as generation after generation of young mathematicians is trained through the medium of the calculus.

The functions of classical analysis are the elementary functions: that is, those which can be constructed from the variables $x, y, \cdots$ by a finite number of algebraic operations and the taking of logarithms and exponentials. For example, $\cos y^{1/2}+\log [x^3+\arctan (x \log y)]$ is elementary. The young student quickly discovers that the closure of this set of functions under the operations of analysis is not an obvious fact, if it be a fact at all. Indeed, his instructor assures him that certain functions cannot be integrated in finite terms, that is, their integrals cannot be given an elementary representation. One hazards the guess that in a substantial number of the good courses in calculus offered in this and other countries, this is the one subject about which the instructor may not have first-hand knowledge. Rather, he imparts to his young charges, frequently with embarrassment, information which is based on hearsay.

Professor Ritt has now written a short book in which the reader will find all the material on this subject which should be the property of the complete mathematician. The theory exposed is one of considerable charm and of classical importance. A method is developed which may be used in attempting the solution of arbitrary problems on the representation of functions in an elementary manner. This method is then applied to certain specific questions where it yields complete results. It does not provide easy answers to the great variety of questions which one could propose. In spite of this fact, it possesses a certain degree of finality. It is truly extraordinary that no book has been written previously on this subject except in Russian.[1]

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[1] The book by G. H. Hardy, *The integration of functions of a single variable*, Cambridge Tracts, 1905, is not one in which the Liouville theory is expounded. The author

One may well propose that no other work need be printed for the next fifty years since our knowledge will hardly alter essentially in that period.

The material is so arranged as to be accessible to a good graduate student. The reader must be equipped with sound knowledge of the theory of functions of a complex variable including the elements of analytic continuation, Riemann surfaces of algebraic functions, and power series expansions of algebraic functions. The exposition is endowed with great clarity. The author bases his style on the staccato sentence. This makes the climb steady, the footing secure.

The founder of the theory here considered was Joseph Liouville. Over one hundred years ago he classified the elementary functions and discovered the important principles with which questions concerning them can be resolved. He determined the form which the integral of an algebraic function must have if it is elementary. He showed that the elliptic integrals of the first and second kind are not elementary. He investigated the integral of $e^{g(x)}\cdot y(x)$ where $g(x)$ and $y(x)$ are algebraic; this leads to the proof of the fact that e^{x^2} does not have an elementary integral. Introducing quadratures as an allowable operation, he showed that the Riccati equation $y'+y^2=x^n$ can be integrated in *finite* terms only if (the "if" is due to Daniel Bernouilli) $n=-2$ or $n=-4p/(1+2p)$ with p an integer. Thus, the Bessel equation with parameter ν has a finite solution if and only if 2ν is an odd integer. All these results are to be found in this monograph.

Besides Liouville, the leading other contributors to this theory have been the author himself and the Russian mathematician, D. Mordukhai-Boltovskoi. Results of these men here presented are concerned with the solution of problems by elementary methods in *implicit* terms. For example, the author reproduces his proof that if $y(x)$ is elementary and $w(x)$ is an integral of $y(x)$ then the existence of an elementary relation of two variables, $F(x, w)=0$, implies that w is itself an elementary function of x. A recent contribution of Ostrowski is also included: The question of solving problems in finite terms is essentially based on the notion of field extension. One begins with a differential field $\mathcal{F}$. If $\mathcal{F}$ is not sufficiently large for our purposes, successive adjunctions of the exponentials or logarithms of functions in $\mathcal{F}$ yield a field extension $\mathcal{F}'$. In $\mathcal{F}'$ one may look for functions which satisfy our desiderata. With these ideas one is in a position to extend Liouville's result about the integral of an algebraic function to a theorem concerning the integral of a function algebraic over a given differential field $\mathcal{F}$.

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does briefly mention certain of Liouville's results. He gives no proofs but refers the reader to original sources.

A frequently recurring difficulty permeates phases of our undergraduate teaching. Even our better students are puzzled and annoyed by a situation which could easily be improved. We are at fault in that we do not formally classify the distinct domains (types of function or types of numbers) in which we operate. Thus we state successively that $1/\log x$ cannot be integrated; that the differential equation $y'-1/\log x=0$ can be solved whereas others more complicated cannot; finally that all differential equations can be solved by series. Likewise, in the equation $y+\log y=x$ it is impossible to solve for y; yet the equation defines y as a function of x and hence we may compute dy/dx with the help of standard theorems on differentiation. The worst paradox of all is that since $x^2+1=0$ has no root, we must use a special symbol to represent it. It would seem that our students would thrive better if we gave less attention to the introduction of ϵ and δ and devoted a little time to the discussion of these questions. One ventures the prediction that future texts in the calculus will carry out such a program explicitly. The publication of *Integration in finite terms* may accelerate this very desirable end by making the Liouville theory current coin in mathematical circles.

E. R. LORCH

Matrix and tensor calculus with applications to mechanics, elasticity and aeronautics. By A. D. Michal. (Galcit Aeronautical series.) New York, Wiley; London, Chapman and Hall, 1947. 13+132 pp. $3.00.

The author states in the preface that the purpose of his book is "to give the reader a working knowledge of matrix calculus and tensor calculus, which he may apply to his own field." To accomplish that much in 130 pages is a difficult problem. Professor Michal attempts to solve it by omitting many proofs and by restricting severely the material presented. Thus several basic notions (for example, characteristic vectors of a matrix) are not mentioned. These omissions are partly compensated for by numerous *Notes* collected at the end of the volume and by an extensive bibliography. On the other hand, this book contains information on some non-standard topics. These include the theory of "multiple-point tensor fields" (originated by the author two decades ago), and a tensor treatment of the boundary layer theory (due to Lin).

The book consists of two largely independent parts, one dealing with matrices, the other with tensors. Each part begins with the fundamental definitions and theorems. Further mathematical concepts are introduced in connection with concrete applications which range over various fields of mechanics. While no problem is pursued very far the author succeeds in convincing the reader of the advantage resulting from attacking a problem in mechanics by appropriate mathematical tools.

Matrix theory occupies about one third of the book. A brief account of matrix algebra (including the Cayley-Hamilton theorem) is followed by a discussion of power series in matrices and of the calculus of matrix-valued functions. Applications are given to the theory of small oscillations around a point of stable equilibrium, to the theory of aircraft flutter and to elastic deformation theory.

Tensor calculus is developed primarily for the Euclidean three-space. The discussion centers around curvilinear coordinates, the metric tensor and covariant differentiation. As a first application the fundamental equations of mathematical physics are written in general curvilinear coordinates. A brief chapter on fluid dynamics is followed by a comparatively extensive treatment of tensor methods in the theory of elasticity. A distinctive feature of these chapters is the author's determination not to limit himself to infinitesimal deformations. The last two chapters are devoted to tensor calculus in Riemannian spaces with applications to classical mechanics and boundary layer theory.

The volume is based on a series of lectures given to a group of research engineers. It could be well used in a graduate or senior undergraduate course for engineering students. The exercises are not numerous but are selected skillfully. A few awkward expressions and slips of the pen (for instance on p. 32, l. 27, p. 34, l. 11, p. 41, l. 10, p. 88, l. 20) do not detract materially from the value of this enthusiastically written and useful book.

LIPMAN BERS

Theory and application of Mathieu functions. By N. W. McLachlan. New York, Oxford University Press, 1947. 9+401 pp. $12.50.

Part I of this book contains a comprehensive treatment of analytical and numerical methods which have been successfully used to obtain solutions of the various forms of Mathieu's differential equation, satisfying various conditions, and for complex as well as real values of the independent variable. It discusses also the theoretical background underlying these methods. A large part, estimated by the author as one third, of this material is new, filling gaps by extending over the whole field methods which had proved useful in part of it; the book will therefore be very useful to a reader who wants to make a new application without having to extend the theory.

The subjects covered include the integral equations and relations satisfied by the solutions; the distribution of their zeros; the periodic properties of the solutions and their expansions in Fourier series and the corresponding expansions of solutions that are not periodic; and the expansions in series of Bessel functions and of products of Bessel functions and the asymptotic expansions for large argument values. Powerful methods are given for computing the characteristic numbers and the coefficients in the expansions.

The applications fall naturally into two classes. Parametric resonance in dynamical systems where Mathieu's equation is to be considered the simplest special form of Hill's equation—also treated in Part I; and the solution of two-dimensional problems by separation of variables in elliptic coordinates, described for the wave equation also in Part I. The representative applications in Part II of the first class include treatments of amplitude distortion, frequency modulation, and sub-harmonics; of the second class the vibrations of elliptical membranes, plates, and lakes; eddy currents and thermal diffusion in elliptic cylinders; the propagation of electromagnetic waves in elliptical wave guides; and the diffraction of waves by elliptical cylinders. These, and the other applications given, should enable the reader to apply the theory to new cases as they arise.

The only large part of the theory not given is that connected with the treatment of the equation in terms of the singular points of its algebraic form.

The book is beautifully printed and a pleasure to read; it fulfills a long felt want and will take its place alongside Watson's *Bessel functions* as a necessary part of the equipment of an applied mathematician.

L. H. THOMAS

Algèbre et analyse linéaires. By A. Lichnerowicz. Paris, Masson, 1947. 316 pp. 800 fr.

This book is an introduction to linear mathematics for physicists based on lectures given by the author at Strasbourg. It is designed to fill in part the gap left in French scientific literature by the absence of works in French similar to the German treatises of Courant and Hilbert and of Frank and Mises.

The book has two parts, one on linear algebra and one on linear analysis, and each part is divided into four chapters. The first three chapters of Part I and the last three chapters of Part II present the standard elementary material on linear operators and bases in vector spaces with applications to the theory of expansions in orthogonal systems of functions and to the theory of integral equations. The middle two chapters present the algebra and analysis leading up to and including the formulation and proof of the generalized Stokes theorem for multiple integrals.

Keeping in mind his audience of physicists, the author treats his material with somewhat less generality and completeness than he might otherwise consider desirable. Occasionally he seems to go a little too far in this direction. For example one is astonished to find on page 266 a discussion of the spectrum of an operator in Hilbert space with no mention of anything but the point spectrum. Although he accepts the abstract approach to the extent of writing down the axioms for a vector space, he works as a rule with concrete vector spaces and in general uses coordinates and concrete representations more than an uncompromising abstractionist would consider necessary or desirable. On the other hand he usually, if not always, observes mathematical standards of rigor and at times assumes a degree of mathematical sophistication on the part of the reader uncommon among physicists, at least among those educated in the United States.

A more detailed outline of the contents of the book follows. Chapter I of Part I presents the fundamental facts about vector spaces, determinants and the solution of systems of linear equations. Chapter II contains a short treatment of inner products, the Schwartz inequality, the Bessel inequality, Parseval's equation and related matters for finite-dimensional vector spaces. Chapter III deals with linear transformations, the algebra of matrices, bilinear forms, characteristic values and the diagonalization of matrices. Chapter IV treats tensor products of vector spaces, the related algebra of tensors and the Grassman algebra of exterior forms. Chapter I of Part II begins with a treatment of exterior differential forms, their differentials and their integrals over m spreads in n-space. The familiar notions of gradient, divergence and curl in vector analysis are exhibited as examples of the differentials of differential forms and the chapter concludes with a proof of the generalized Stokes theorem. The principal topics presented in Chapter II include orthogonality for functions, convergence in the mean, completeness of systems of functions, the Riesz-Fischer theorem, the Weierstrass approximation theorem and its use in proving the completeness of systems of orthogonal polynomials and the trigonometric functions, properties of orthogonal polynomials, and the convergence of Fourier series and integrals. Chapter III deals with the elementary facts about linear operators and infinite matrices associated with vector spaces of functions. Chapter IV contains an exposition of the theory of the Fredholm integral equation.

George W. Mackey

NOTES

The National Council of Teachers of Mathematics will hold its ninth Christmas Conference at Ohio State University, Columbus, Ohio, on Wednesday and Thursday, December 29 and 30, 1948.

The School of Mathematics of the Institute for Advanced Study will allocate a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1949–1950. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, N. J., and are returnable by February 1, 1949.

Professor Elie Cartan of the University of Paris and Professor Marston Morse of the Institute for Advanced Study have been awarded honorary degrees by the University of Pisa.

Professor Charles Blanc of the University of Lausanne and Professor Albert Pfluger of the Swiss Federal Institute of Technology have been elected president and vice president respectively of the Swiss Mathematical Society.

Certificates of Merit have been awarded by the Army and Navy to Professors Marston Morse and Oswald Veblen of the Institute for Advanced Study, to Professor S. S. Wilks of Princeton University, and to Professor A. H. Taub of the University of Illinois.

It has been announced by the U. S. Army that the following mathematicians will constitute the Mathematics Advisory Committee charged with the consideration of questions involving the application of mathematics to Army problems: Dr. Hendrik Bode, Bell Telephone Laboratories; Professor Richard Courant, New York University; Professor John von Neumann, Institute for Advanced Study; Professor J. B. Rosser, Cornell University; Professor H. P. Robertson, California Institute of Technology; Professor J. J. Stoker, New York University. Professor E. R. Lorch of Columbia University will be scientific advisor to the Research and Development Group of the General Staff until February, 1949; he will serve as executive secretary of the committee.

Professor I. S. Carroll of Syracuse University has retired with the title emeritus.

Professor Emeritus Lennie P. Copeland of Wellesley College has received an honorary degree from the University of Maine.

Associate Professor H. E. Jordan of the University of Kansas has retired with the title emeritus.

Professor A. F. Kovarik of Yale University has retired with the title emeritus.

Professor Susan M. Rambo of Smith College has retired with the title emeritus.

Dr. P. R. Garabedian and Dr. Daniel Zelinsky have been awarded National Research Fellowships. Dr. E. J. Akutowicz and Dr. D. Feldman have been awarded Atomic Energy Commission postdoctoral fellowships.

Mr. Alberto Dias Coimbra has been appointed to a professorship at the Lyceum of Evora, Portugal.

Professor Vaclav Hlavaty of Charles University in Prague has been appointed to a visiting professorship at Indiana University.

Assistant Professor H. T. Hsu of Great China University, Shanghai, China, has been promoted to an associate professorship. He is at present at Yale University.

Professor L. K. Hua of Tsing Hua University, Peiping, China, has been appointed to a visiting professorship at the University of Illinois.

Dr. M. J. Pinl has been appointed to a professorship at the University of Cologne.

Dr. Gino Turrin of the University of La Plata has been appointed to a professorship at the University of Cuyo, San Luis, Argentina.

Professor Y. C. Wong of the National Sun Yat-sen University, Canton, China, has been appointed to a professorship at the University of Hongkong.

Mr. F. C. Andrews has been appointed an associate at the University of Washington.

Assistant Professor H. G. Apostle of the University of Chicago has been appointed to an associate professorship at Grinnell College, Grinnell, Iowa.

Dr. Alfred Basch has been appointed to a professorship at the Technical University of Vienna.

Professor A. H. Black of Lebanon Valley College, Annville, Pennsylvania, has been appointed to an associate professorship at Southern Illinois University.

Miss Thelma E. Bradford of Claflin University, Orangeburg, South Carolina, has been appointed to an assistant professorship at Delaware State College.

Miss Mildred J. Brannon of the University of Wisconsin has been appointed to an assistant professorship at Michigan State Normal College.

Associate Professor D. M. Brown of Central Michigan College has been appointed supervisor, Applied Mathematics Group, Aeronautical Reserve Center, University of Michigan.

Dr. L. J. Burton has been appointed to an assistant professorship at Bryn Mawr College.

Mr. P. F. Byrd has been appointed to an assistant professorship at Fisk University.

Dr. Lee Byrne of Purdue University has been appointed lecturer at the Arizona State College.

Dr. K. L. Chung of Princeton University has been appointed to an assistant professorship at Cornell University.

Mr. L. A. Colquitt of Ohio State University has been appointed to an assistant professorship at Texas Christian University.

Mr. W. C. Davis, formerly with the Commonwealth and Southern Corporation, Birmingham, Alabama, has accepted a position as engineer with the Debardeleben Coal Corporation, Birmingham, Alabama.

Mr. A. S. Day of Yale University has been appointed to an assistant professorship at Wesleyan University.

Vice President C. H. Dix of the United Geophysical Company, Pasadena, California, has been appointed to an associate professorship at the California Institute of Technology.

Mr. B. B. Dressler of Ohio State University has been appointed to an assistant professorship at Kent State University.

Assistant Professor A. B. Farnell of the University of Colorado has been appointed lecturer at Princeton University.

Associate Professor D. A. Flanders of New York University has been appointed senior mathematician at the Argonne National Laboratory, Chicago, Illinois.

Assistant Professor G. E. Forsythe of the University of California at Los Angeles has accepted a position as mathematician at the Institute for Numerical Analysis, National Bureau of Standards, University of California at Los Angeles.

Associate Professor G. N. Garrison of Lehigh University has been appointed to an assistant professorship at The City College, New York City.

Assistant Professor Landis Gephart of the University of Dayton has accepted a position as mathematician with the U. S. Air Force.

Mr. I. E. Glover of the University of Rochester has been appointed to a professorship at Langston University, Langston, Ohio.

Dr. M. J. Gottlieb of the Institute for Advanced Study has been appointed to an assistant professorship at the Newark College of Rutgers University.

Mr. Arthur Grad of Stanford University has accepted a position as mathematician at the Office of Naval Research, Washington, D. C.

Mr. A. G. Hansen of Purdue University has accepted a position as mathematician with the National Advisory Committee for Aeronautics, Cleveland, Ohio.

Mr. J. M. Heath of Yale University has accepted a teaching position at the Hopkins Grammar School, New Haven, Connecticut.

Professor W. R. Hutcherson of Berea College has been appointed to a professorship at Northwestern State College, Natchitoches, Louisiana.

Dr. C. M. Jensen of the University of Minnesota has been appointed to an assistant professorship at Augustana College, Rock Island, Illinois.

Professor B. W. Jones of Cornell University has been appointed to a professorship at the University of Colorado.

Dr. S. N. Karp of Brown University has been appointed senior research scientist at the Institute for Mathematics and Mechanics, New York University.

Mr. H. E. Kinerk of Seattle College has accepted the position of designing engineer at the Western Gear Works, Seattle, Washington.

Assistant Professor Cecilia Krieger of the University of Toronto has been appointed visiting lecturer at the University of British Columbia.

Mr. W. V. Leamon formerly with Western Union Telegraph Company has accepted a position as engineer with The Kellex Corporation, New York City.

Associate Professor M. M. Lemme of Illinois Wesleyan University has been appointed to an associate professorship at San Diego State College.

Assistant Professor J. N. B. Livingood of Rutgers University has accepted a position as aeronautical research scientist with the National Advisory Committee for Aeronautics, Cleveland, Ohio.

Professor Eugene Lukacs is on leave from Our Lady of Cincinnati College and has accepted a position as statistician at the United States Naval Ordnance Test Station, Inyokern, China Lake, California.

Mr. J. L. Lund has accepted a position at the Woodlawn High School, Woodlawn, California.

Assistant Professor P. E. Meadows of Washington and Lee University has been appointed to an associate professorship at Carroll College, Waukesha, Wisconsin.

Mr. B. W. Miller has accepted a position as staff engineer in electronics with Tele-Tronics, Incorporated, Miami Beach, Florida.

Professor A. M. Mood of Iowa State College of Agriculture and Mechanic Arts has accepted the position of research statistician with Rand Corporation, Santa Monica, California.

Miss Dorothy J. Morrow of the University of North Carolina has been appointed biometrician by the Civil Aeronautics Administration, Washington, D. C.

Professor M. E. Mosely of Arkansas Mechanical and Normal College has been appointed to a professorship at Jackson College, Jackson, Mississippi.

Professor G. A. Newton of Trinity University, San Antonio, Texas, has retired.

Assistant Professor P. B. Norman of Pratt Institute, Brooklyn, New York, has been appointed to an assistant professorship at Wagner College, Staten Island, New York.

Dr. O. G. Owens of the Institute for Mathematics and Mechanics, New York University, has been appointed to an assistant professorship at the University of Wisconsin.

Professor Annie M. Pegram of Greensboro College, Greensboro, North Carolina, has retired.

Assistant Professor Emily C. Pixley of Wayne University has been appointed to an assistant professorship at the University of Detroit.

Mr. Saul Rosen of the University of Delaware has been appointed lecturer at the University of California at Los Angeles.

Miss Sally I. Rothstein has accepted a teaching position at Taft High School in New York City.

Dr. Alice T. Schafer has been appointed to an assistant professorship at Swarthmore College.

Associate Professor D. R. Shreve of the University of Tulsa has accepted a position with the Carter Oil Co.

Dr. S. S. Shu of the Institute for Advanced Study has been appointed to an assistant professorship at the Illinois Institute of Technology.

Mr. W. H. Spragens of the University of Cincinnati has been appointed to an associate professorship at Florida State University.

Dr. R. H. Stark of Northwestern University has accepted a position as mathematician at the Los Alamos Scientific Laboratory.

Assistant Professor Feodor Theilheimer of Trinity College, Hartford, Connecticut, has been appointed mathematician at the Naval Ordnance Laboratory, Washington, D. C.

Dr. G. L. Tiller of the University of Kentucky has been appointed to an assistant professorship at the Utica College, Syracuse University.

Mr. H. I. Treiber of Princeton University has been appointed mathematician at the Watson Laboratories, Red Bank, New Jersey.

Mr. J. E. Warren has been appointed supervisor of the U. S. Government Section of Teleregister Laboratories, New York City.

Professor Kenichi Watanabe of Wabash College, Crawfordsville, Indiana, has been appointed physicist in the optics division of Naval Research Laboratories.

Mr. C. W. Williams has been appointed to an assistant professorship at Washington and Lee University.

Dr. A. W. Wundheiler, formerly senior physicist in the Bureau of Ordnance, Department of the Navy, has been appointed to a research professorship in mechanics at the Illinois Institute of Technology.

Professor H. L. Zeiders of Midland College, Fremont, Nebraska, has been appointed to a professorship at the University of Toledo.

Mr. J. A. Zilber of Harvard University has been appointed lecturer at Columbia University.

The following promotions are announced:

A. H. Hald, University of Copenhagen, to a professorship in statistics.

R. A. Rankin, Cambridge University, to a lecturer.

A. A. Aucoin, University of Houston, to a professorship.

E. W. Barankin, University of California, to an assistant professorship.

B. G. Capo, University of Puerto Rico, to an assistant director for research.

F. L. Celauro, Newark College of Engineering, to a professorship.

R. H. Cole, University of Western Ontario, to an associate professorship.

H. R. Cooley, New York University, to a professorship.

C. M. Cramlet, University of Washington, to a professorship.

A. R. Craw, United States Naval Academy, to an assistant professorship.

E. A. Davis, University of Nevada, to an assistant professorship.

Edward Fleisher, Brooklyn College, to a professorship.

Mariano Garcia, University of Puerto Rico, to a professorship.

E. C. Gras, United States Naval Academy, to an assistant professorship.

B. K. Hovey, University of Pittsburgh, to an associate professorship of electrical engineering.

J. A. Hratz, St. Ambrose College, Davenport, Iowa, to an assistant professorship.

S. P. Hughart, University of Chicago, to an assistant professorship.

Louise S. Hunter, Virginia State College, Petersburg, Virginia, to an associate professorship.

Charlotte D. Jeffrey, Psychometric Laboratory, University of Chicago, to a position as psychometrician.

S. C. Kleene, University of Wisconsin, to a professorship.

G. J. Mann, United States Naval Academy, to an assistant professorship.

M. G. Moore, Bradley University, Peoria, Illinois, to a professorship.

S. F. Neustadter, University of California, to a lectureship.

J. W. Odle, Naval Ordnance Test Station, Inyokern, China Lake, California, to head of the mathematics division.

I. E. Perlin, Georgia School of Technology, to an associate professorship.

Harry Pollard, Cornell University, to an associate professorship.

Walter Prenowitz, Brooklyn College, to an associate professorship.

V. N. Robinson, United States Naval Academy, to an associate professorship.

C. W. Seekins, United States Naval Academy, to an associate professorship.

H. K. Sohl, United States Naval Academy, to an assistant professorship.

W. J. Strange, United States Naval Academy, to an assistant professorship.

G. R. Strohl, United States Naval Academy, to an assistant professorship.

L. W. Swanson, Coe College, Cedar Rapids, Iowa, to a professorship.

J. L. Vanderslice, University of Maryland, to an associate professorship.

J. H. White, United States Naval Academy, to an assistant professorship.

D. R. Whitney, Ohio State University, to an assistant professorship.

The following appointments to instructorships are announced: Adelphi College, Garden City, New York: Mrs. A. M. Vogel; Amherst College, Philosophy Department: Mr. W. D. Stahlman; California Institute of Technology: Dr. Samuel Karlin; University of California: Dr. T. M. Apostol; University of California at Davis: Dr. H. A. Arnold; University of California at Los Angeles: Dr. Robert Steinberg, Dr. E. G. Straus; Catholic University of America: Dr. Paolo Nesbeda; City College, New York City: Dr. Abraham Schwartz; Clark School, Hanover, New Hampshire; Mr. F. D. Bosworth; University of Delaware: Mr. Morris Newman; Illinois Institute of Technology; Mr. R. H. Edwards; University of Illinois: Dr. F. E. Hohn, Dr. Irma M. Reiner; Indiana University, Extension Center, South Bend, Indiana: Mrs. J. G. Buck; Iowa State College of Agriculture and Mechanic Arts: Mr. R. F. Deniston; Johns Hopkins University: Mr. M. L. Juncosa, Dr. Sylvan Wallach: University of Michigan: Mr. R. K. Ritt; University of Nebraska: Miss F. Marion Clarke: University of Nevada: Mr. M. R. Demers; Northwestern University: Mr. E. W. Banhagel, Mr. W. M. Boothby; Ohio State University: Dr. O. W. Rechard; Ohio University: Mr. S. S. Anderson (Psychology Department), Mr. P. C. Stanger; University of Pennsylvania: Mr. Russell Remage, Mr. Herman Zabronsky; Princeton University: Mr. W. M. Gilbert, Dr. R. C. Lyndon; Purdue University: Dr. Harold Schniad; University of Rochester: Dr. W. J. Klimczak; University of the South, Sewanee, Tennessee: Mr. B. K. Dickerson; Tulane University of Louisiana: Mr. R. W. Schmied; University of Washington: Dr. D. B. Dekker; Wellesley College: Dr. Ilse L. Novak; University of Wisconsin: Dr. C. C. Hsiung, Mr. J. B. Kelly, University of Wisconsin in Milwaukee: Mr. E. G. Schuld.

Assistant Professor W. B. Campbell of the Philadelphia Textile Institute died August 12, 1948. He had been a member of the Society for twenty-two years.

Dr. Walther Mayer of the Institute for Advanced Study died September 12, 1948, at the age of sixty-one years.

Professor W. A. Wilson of Yale University died September 21, 1948 at the age of sixty-three years. He had been a member of the Society for thirty-eight years.

NEW PUBLICATIONS

BOURBAKI, N. *Algèbre linéaire.* Paris, Hermann, 1947. 132 pp.

CASOLLA, A. *Teoria fisica e matematica della musica.* I: *Meccanica del suono.* Naples, Severini, 1946.

CHURCHMAN, C. W. *Theory of experimental inference.* New York, Macmillan, 1948. 11+292 pp. $4.25.

DIEUDONNÉ, J. *Sur les groupes classiques.* Paris, Hermann, 1948. 82 pp.

FERRAR, W. L. *Higher algebra.* London, Oxford University Press, 1948. 320 pp. 17s 6d.

GOODSTEIN, R. L. *A text-book of mathematical analysis: the uniform calculus and its applications.* Oxford, Clarendon, 1948. 12+475 pp. $9.00.

LE LIONNAIS, F. *Les grands courants de la pensée mathématique.* Paris, 1948. 500 pp. 840 fr.

McCOY, N. H. *Rings and ideals.* (Carus mathematical monographs, no. 8.) Buffalo, Mathematical Association of America, 1948. 12+216 pp.

MAURIN, J. *Géométrie descriptive à quatre dimensions.* I: *Figures du premier degré. Définitions, positions, intersections.* Paris, Gauthier-Villars, 1948.

MURNAGHAN, F. D. *Introduction to applied mathematics.* New York, Wiley, 1948. 389 pp. $5.00.

ORE, O. *Number theory and its history.* New York, McGraw-Hill, 1948. 10+370 pp. $4.50.

PICONE, M., and TORTORICI, P. *Trattato di matematiche generali.* Vol. I. Rome, 1947. 16+589 pp. 1950 lires.

RELTON, F. E. *Applied differential equations.* London, Blackie, 1948. 8+264 pp. 20s.

RUTHERFORD, D. E. *Substitutional analysis.* Edinburgh, University Press, 1948. 11+103 pp. 25s.

SCHAAF, W. L., Ed. by. *Mathematics: Our great heritage. Essays on the nature and cultural significance of mathematics.* New York, Harper, 1948. xi+291 pp. $3.50.

SVOBODA, A. *Computing mechanisms and linkages.* Ed. by H. M. James. (Massachusetts Institute of Technology, Radiation Laboratory Series, No. 27.) New York, McGraw-Hill, 1948. 12+359 pp.

Tables of the Bessel functions of the first kind of orders twenty-eight through thirty-nine. (Annals of the Computation Laboratory of Harvard University, Vol. 10.) Cambridge, Harvard University Press, 1948. 12+694 pp. $10.00.

TORTORICI, P. See PICONE, M.

TRICOMI, F. *Equazioni differenziali.* Turin, Einaudi, 1948. 312 pp. 1800 L.

USPENSKY, J. V. *Theory of equations.* New York, McGraw-Hill, 1948. 8+353 pp. $4.50.

WALL, H. S. *Analytic theory of continued fractions.* New York, Van Nostrand, 1948. 13+433 pp. $6.50.

WEIL, A. *Sur les courbes algébriques et les variétés qui s'en déduisent.* Paris, Hermann, 1948. 85 pp.

BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

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1948

NOVEMBER, 1948

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK

1947

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[1] Succeeded A. C. Schaeffer, resigned July, 1947.

[2] Succeeded Warren Weaver, resigned November, 1947.

1948

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[1] Succeeded R. H. Bruck, resigned September, 1948.

[2] Succeeded G. W. Mullins, resigned January, 1948.

REPRESENTATIVES AND COMMITTEES OF THE SOCIETY

Representatives of the Society in the Division of Mathematical and Physical Sciences of the National Research Council:

1946–1947—A. B. Coble, Saunders MacLane, John von Neumann, M. H. Stone, Oswald Veblen, Warren Weaver.
1947–1948—R. V. Churchill, M. R. Hestenes, Saunders MacLane, John von Neumann, M. H. Stone, Oswald Veblen.
1948–1949—R. V. Churchill, M. R. Hestenes, T. H. Hildebrandt, M. H. Ingraham, John von Neumann, M. H. Stone.
1949–1950—R. V. Churchill, M. R. Hestenes, T. H. Hildebrandt, M. H. Ingraham, Marston Morse, Oscar Zariski.

Representatives on the Council of the American Association for the Advancement of Science:

1947—Solomon Lefschetz, G. T. Whyburn.
1948—R. P. Agnew, W. M. Whyburn.
1949—R. P. Agnew, W. M. Whyburn.

Representatives on the Editorial Board of the Annals of Mathematics:

K. O. Friedrichs, Norman Levinson, R. L. Wilder.

Representatives on the Editorial Board of the Duke Mathematical Journal:

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William Feller.

Colloquium Lecturers:

1946—Hassler Whitney.
1947—Oscar Zariski.
1948—Richard Brauer.
1949—G. A. Hedlund.

Committee to Select Gibbs Lecturers for 1948 and 1949:

Gabor Szegő (Chairman), W. T. Martin, J. J. Stoker.

Gibbs Lecturers:

1923—M. I. Pupin.
1924—Robert Henderson.
1925—James Pierpont.
1926—H. B. Williams.
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1929—Irving Fisher.
1930—E. B. Wilson.
1931—P. W. Bridgman.
1932—R. C. Tolman.
1934—Albert Einstein.
1935—Vannevar Bush.
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1937—C. A. Kraus.
1939—Theodore von Kármán.
1941—Sewall Wright.
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1944—John von Neumann.
1945—J. C. Slater.
1946—Subrahmanyan Chandrasekhar.
1947—P. M. Morse.
1948—Hermann Weyl.

Committee on Society Visiting Lectureship:

Hassler Whitney (Chairman), R. G. D. Richardson, Norbert Wiener.

Visiting Lecturers:

1927–1928—Constantin Carathéodory, University of Munich.
1928–1929—Hermann Weyl, Zurich Technical School.
1929–1930—Enrico Bompiani, University of Rome.
1930–1931—Wilhelm Blaschke, University of Hamburg.
1931–1932—R. L. Moore, University of Texas.
1936–1937—Thirjkkannapuram Vijayaraghavan, University of Dacca.
1948–1949—Casimir Kuratowski, University of Warsaw.

Auditors for 1948:

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Committee on Publication of Moore Manuscript:

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Committee on Role of Society in Mathematical Publication:

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Committee on Places of Meetings:

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Policy Committee for Mathematics:

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INTERNATIONAL CONGRESS OF MATHEMATICIANS

CAMBRIDGE, MASS., AUGUST 30-SEPTEMBER 6, 1950

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E. H. Moore, 1901–1902.
T. S. Fiske, 1903–1904.
W. F. Osgood, 1905–1906.
H. S. White, 1907–1908.
Maxime Bôcher, 1909–1910.
H. B. Fine, 1911–1912.
E. B. Van Vleck, 1913–1914.
E. W. Brown, 1915–1916.
L. E. Dickson, 1917–1918.
Frank Morley, 1919–1920.
G. A. Bliss, 1921–1922.
Oswald Veblen, 1923–1924.
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Virgil Snyder, 1927–1928.
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L. P. Eisenhart, 1931–1932.
A. B. Coble, 1933–1934.
Solomon Lefschetz, 1935–1936.
R. L. Moore, 1937–1938.
G. C. Evans, 1939–1940.
Marston Morse, 1941–1942.
M. H. Stone, 1943–1944.
T. H. Hildebrandt, 1945–1946.

ENDOWMENT FUND

In 1923 an Endowment Fund was collected to meet the greater demands on the Society's publication program caused by the ever increasing number of important mathematical memoirs. Of this fund, which now amounts to some $71,000, a considerable proportion was contributed by members of the Society. Under the terms of the will of the late Robert Henderson, for many years a Trustee of the Society, the Society receives approximately $4,000 yearly. Upon the death of the other legatees, the Society will receive the entire principal of the estate for its Endowment Fund.

SPECIAL FUNDS

The Bôcher Memorial Prize.

This prize was founded in memory of Professor Maxime Bôcher. It is awarded every five years for a notable research memoir in analysis which has appeared during the preceding five years in a recognized journal published in the United States or Canada; the recipient must be a member of the Society, and not more than fifty years old at the time of publication of his memoir.

First (Preliminary) Award, 1923: To G. D. Birkhoff, for his memoir *Dynamical systems with two degrees of freedom.*

Second Award, 1924: To E. T. Bell, for his memoir *Arithmetical paraphrases,* and to Solomon Lefschetz, for his memoir *On certain numerical invariants with applications to abelian varieties.*

Third Award, 1928: To J. W. Alexander, for his memoir *Combinatorial analysis situs.*

Fourth Award, 1933: To Marston Morse, for his memoir *The foundations of a theory of the calculus of variations in the large in m-space,* and to Norbert Wiener, for his memoir *Tauberian theorems.*

Fifth Award, 1938: To John von Neumann, for his memoir *Almost periodic functions and groups.*

Sixth Award, 1943: To Jesse Douglas, for his memoirs *Green's functions and the problem of Plateau, The most general form of the problem of Plateau,* and *Solution of the inverse problem of the calculus of variations.*

The Frank Nelson Cole Prize in Algebra.
The Frank Nelson Cole Prize in the Theory of Numbers.

These prizes were founded in honor of Professor Frank Nelson Cole, on the occasion of his retirement as Secretary of the American Mathematical Society and Editor of the Bulletin, after twenty-five years of service; the fund was later doubled by his son, Charles A. Cole. They are awarded at five-year intervals for contributions to algebra and the theory of numbers, respectively, under restrictions similar to those for the Bôcher prize.

First Award, 1928: To L. E. Dickson, for his book *Algebra und ihre Zahlentheorie,* Zurich, 1927.

Second Award, 1931: To H. S. Vandiver, for his several papers on Fermat's last theorem published in the Transactions of the American Mathematical Society and the Annals of Mathematics during the last five years, with special reference to a paper entitled *On Fermat's last theorem,* which appeared in volume 31 of the Transactions.

Third Award, 1939: To A. A. Albert, for his papers on the construction of Riemann matrices, published in volumes 35 and 36 of the Annals of Mathematics.

Fourth Award, 1941: To Claude Chevalley, for his paper entitled *La théorie du corps de classes,* published in volume 41 of the Annals of Mathematics.

Fifth Award, 1944: To Oscar Zariski, for four papers on algebraic varieties, published in volumes 61 and 62 of the American Journal of Mathematics and volumes 40 and 41 of the Annals of Mathematics.

Sixth Award, 1946: To H. B. Mann for his paper entitled *A proof of the fundamental theorem on the density of sums of sets of positive integers,* published in Volume 43 of the Annals of Mathematics.

The Eliakim Hastings Moore Fund.

This fund was founded in 1922 in honor of Professor Eliakim Hastings Moore, on the occasion of the twenty-fifth anniversary meeting of the Chicago Section of the American Mathematical Society. The income from the fund is to be used, at the discretion of the Council of the Society, for the publication of important mathematical books or memoirs, or the award of prizes.

The Marion Reilly Fund.

Dean Marion Reilly willed to the American Mathematical Society a portion of her estate to be used for the advancement of research in pure mathematics. The principal of this fund is $23,500.

The Ernest William Brown Fund.

From the estate of Professor Ernest William Brown, a fund of $1,000 is available, the interest on which, at the discretion of the Council, can be used for the furtherance of such mathematical interests as (a) the publication of important mathematical books, memoirs, and periodicals, and (b) lectures to be delivered on special occasions by invited guests of the Society.

The Robert Henderson Fund.

Upon his retirement from the Board of Trustees on December 31, 1940, Dr. Robert Henderson presented to the Society an unrestricted gift of $1,000 which the Trustees have set aside as the Robert Henderson Fund.

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October 1, 1948

The following list includes the name of every person who is a member of the Society as of October 1, 1948.

So far as information is available the listings are made up as follows: (a) Surname. A dagger before a surname indicates life membership. (b) Initials, or the given name when there is but one, except that the first given name of every woman member is given in full. Parentheses enclosing any part of a name indicate that that part is not generally used in correspondence. (c) Official title or position. (d) Name and address of institution or organization with which item (c) is associated. (e) The address to be used for mail if this is different from (d). The name of the city and postal district number, where necessary, are not repeated if they are the same as in (d). (f) Following the regular listing, information re military service or temporary appointments is given.

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Caywood, T. E.
Chernoff, H.
Cole, K. S.
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Everett, H. S.
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Hughart, S. P.
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Jeffrey, C. D.
Kaplansky, I.
Karush, W.
Lane, E. P.
Leech, J. S.
Logsdon, M. I.
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MacLane, S.
Merriell, D. M.
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Mist, H. F.
Newlander, A.
Northrop, E. P.
Opatowski, I.
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Ostrow, E. H.
Painter, N. H.
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Thomas, C. J.
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Northwestern Univ.
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Reiner, I. M.
Rochmes, N.
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Sinclair, A.
Snell, L. J.
Taub, A. H.
Townsend, E. J.
Trjitzinsky, W. J.
Turner, V. D.
Turquette, A. R.
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McCoy, N. H.
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Coll. of the Holy Cross
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Ann Arbor
Grennan, E. B.
Leisenring, K. B.
Roettinger-Kaplan, I.
Univ. of Michigan
Bartels, R. C. F.
Bradshaw, J. W.
Brauer, R.
Brown, D. M.
Browne, M. L.
Burroughs, W. K.
Cassen, P. N.
Churchill, R. V.
Coburn, N.
Coe, C. J.
Copeland, A. H.
Craig, C. C.
Dickinson, D. J.
Dolph, C. L.
Duncan, D. G.
Dushnik, B.
Dwyer, P. S.
Eisenstadt, B. J.
Field, P.
Fischer, C. H.
Ford, W. B.
Fowler, K. A.
Hahn, S. W.
Harary, F.
Hay, G. E.
Hildebrandt, T. H.
Hopkins, L. A.
Jerison, M.
Jones, P. S.
Kaplan, W.
Kincaid, W. M.
Kuller, R. G.
Lewis, D. J.
Love, C. E.
Moise, E. E.
Morkovin, M. V.
Myers, S. B.
Nesbitt, C. J.
Nyswander, J. A.
Perry, C. L.
Piranian, G.
Poor, V. C.
Rabson, G.
Rainich, G. Y.
Rainville, E. D.
Reade, M. O.
Ritt, R. K.
Rothe, E. H.
Running, T. R.
Samelson, H.
Sangren, W. C.
Scott, W. R.
Shapiro, A. S.
Spencer, G. L.
Stanton, R. G.
Thrall, R. M.
Tornheim, L.
Utz, W. R.
Wechsler, M. T.
Wend, D. V. V.
Wiegmann, N. A.
Wilder, R. L.
Woodbury, M. A.
Wright, J. B.
Young, G. S.
Bay City
Ewing, M. M.
Berrien Springs
Specht, E. J.
Detroit
Fleiger, A. G.
Pixley, E. C.
Lawrence Inst. of Tech.
Bagby, L. C.
Univ. of Detroit
McCarthy, E. D.
Mehlenbacher, L. E.
Smith, W. F.
Thompson, L. O.
Wayne Univ.
Borgman, W. M.
Cohn, H.
Coral, M.
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Dunford, M. J.
Epstein, B.
Folley, K. W.
Harrison, G.
Jacobson, A. W.
Kaplan, S.
Morrow, D. C.
Nelson, A. L.
Patterson, J. M.
Pixley, H. H.
Southard, T. H.
East Lansing
Plant, L. C.
Michigan State Coll.
Bell, J. H.
Black, H. L.
Blyth, M. I.
Carr, R. E.
Frame, J. S.
Groenewoud, C.
Grove, V. G.
Herzog, F.
Hill, J. D.
Katz, L.
Kelly, L. M.
Lapidus, L.
LeJunter, J. J.
McMillan, O. T.
Nordhaus, E. A.
Payne, M. H.
Powell, J. E.
Stelson, H. E.
Stewart, B. M.
Wells, C. P.
Zimmer, J. W.
Fenton
Tryon, G. M. V.
Ferndale
Jones, C. D.
Flint
Sitts, M. R.
Kalamazoo
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Marquette
Spooner, C. C.
Nazareth
Morrison, M. J.
Ypsilanti
Michigan State Normal Coll.
Brannon, M. J.
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Weaver, T.

MINNESOTA

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Collegeville
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Cothran, J. C.
Lindquist, C. B.
Minneapolis
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Univ. of Minnesota
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Barnett, H. H.

Bearman, J. E.
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Carlson, E.
Clamons, E. H.
Fischer, I. C.
Franck, A.
Fulks, W.
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Hall, N. A.
Hart, W. L.
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Kalisch, G. K.
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Munro, W. D.
Nering, E. D.
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Wang, C. Y.
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St. Olaf Coll.
Carlson, C. S.
Norlie, O. M.
St. Paul
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Ohnsorg, F. R.
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Sheridan, L. W.
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Macalester Coll.
Camp, E. J.
Winona
Seiler, L. D. L.

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Mississippi State Coll.
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Walker, B. M.
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Univ. of Mississippi
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St. Louis
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Lewis, M. T.
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Regan, F.
Rust, C. H.
Vezeau, W. A.
Washington Univ.
Downs, T. L.
Dunkel, O.
Haimo, F.
Harrison, C. E.
Kreezer, G. L.
Leighton, W.
Mathews, C. W.
Middlemiss, R. R.
Nehari, Z.
Rider, P. R.
Roberson, R. E.
Roever, W. H.
Schaerf, H. M.
Sholander, M.
Thron, W. J.
Van Schaack, G. B.
Varney, R. N.
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Warrensburg
Akers, L. W.

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Montana State Coll.
Hurst, J. W.
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Roscoe, M. K.
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Butte
Smith, A. J.
Garrison
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Montana State Univ.
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Lennes, N. J.
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Ostrom, T. G.

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Union Coll.
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Omaha
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Creighton Univ.
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Duchesne Coll.
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Municipal Univ. of Omaha
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Wayne
Boyce, J. W.

NEVADA

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Univ. of Nevada
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Wood, F.

NEW HAMPSHIRE

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Univ. of New Hampshire
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Slobin, H. L.
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Hanover
Bosworth, F. D.
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Dartmouth Coll.
Brown, B. H.
Doyle, T. C.
Durfee, W. H.
King, A. L.
Mathewson, L. C.
Perkins, F. W.
Robinson, R.
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Wilder, C. E.
Keene
Goodrich, M. T.

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Gutmann, P.
Clifton
Mulligan, J. H.
Convent
O'Neill, A. C.
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Gilbert, N. M.
Ft. Monmouth
Norwood, L.
Hoboken
Davis, H. N.
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Jersey City
Jersey City Junior Coll.
Ayres, H. C.
St. Peter's Coll.
Kruse, W. E.
Markow, C.
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Sternberg, W.
Lakewood
Wallick, E. E.
Lawrenceville
Kiernan, P. J.
Long Branch
Rosenthal, J. E.
Monmouth Junior Coll.
Higgins, C. H.
Wulf, L. M.
Madison
Battin, I. L.
Greenspan, B.
Maplewood
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Montclair
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Murray Hill
Bell Telephone Labs.
Bode, H. W.
Dietzold, R. L.
Farley, B. G.
Gray, M. C.
Hamming, R. W.
Ling, D. P.
McMillan, B.
Olmstead, P. S.
Shannon, C. E.
Shewhart, W. A.
Newark
Fenichel, S. A.
Mossesson, Z. I.
Neifeld, M. R.
Mutual Benefit Life Ins. Co.
Kirk, D. B.
Rhodes, E. E.
Thompson, J. S.
Newark Coll. of Eng.
Capecelatro, A.
Celauro, F. L.
Fithian, J. H.
Hoffmann, P. O.
Jaffe, W. J.
Molina, E. C.
Rutgers Univ.
Gottlieb, M. J.
McCarthy, J. J.
Sherak, B.
New Brunswick
Rutgers Univ.
Barlaz, J.
Bartholomay, A. F.
Bateman, F. D.
Bunyan, L. H.
Cohn, R. M.
Court, L. M.
Darling, D. A.
Dutka, J.
Fender, F. G.
Firestone, C. D.
Galuten, A.
Gokhale, V. D.
Goodman, A. W.
Grant, H. S.
Hazard, K. E.
LeLeiko, M.
Makarov, A. G.
Meder, A. E.
Morris, R.
Nelson, C. A.
Ott, E. R.
Phelps, C. R.
Robertson, M. S.
Saalfrank, C. W.
Starke, E. P.
Walter, R. M.
Zimmerberg, H. J.
Nutley
Assadourian, F.
Orange
Morris, C. R.
Paterson
Chiefa, S. R.
Princeton
Inst. for Advanced Study
Akutowicz, E. J.
Alexander, J. W.
Bateman, P. T.
Bigelow, J. H.
Bourgin, D. G.
Chandrasekharan, K.
Charney, J. G.
Chowla, S.
Dowker, Y. N.
Dvoretzky, A.
Gödel, K.
Goldstine, H. H.
Harish-Chandra
Kuiper, N. H.
Leipnik, R. B.
Mariani, J.
Masani, P. R.
Montgomery, D.
Morse, M.
Mostow, G. D.
von Neumann, J.
Olum, P.
Ryser, H. J.

Sherman, S.
Silverman, E.
Slutz, R. J.
Transue, W. R.
Veblen, O.
Weyl, H.
Zelinsky, D.
Princeton Univ.
Adams, E. P.
Artin, E.
Bargmann, V.
Blakers, A. L.
Blankinship, W. A.
Bochner, S.
Borgers, A. M. F.
Chevalley, C.
Church, A.
Cronin, J. S.
Crosby, D. R.
Davis, M.
Eisenhart, L. P.
Farnell, A. B.
Fox, R. H.
Freeman, M. F.
Gale, D.
Gilbert, S. A.
Gilbert, W. M.
Hawley, N. S.
Henkin, L. A.
Househam, K. O.
Langenhop, C. E.
Lefschetz, S.
Lyndon, R. C.
MacKenzie, R. E.
Massey, W. S.
Meier, P.
Mills, W. H.
Prim, R. C.
Rauch, L. L.
Steenrod, N. E.
Stephan, F. F.
Suppiger, E. W.
Teichmann, T.
Tucker, A. W.
Tukey, J. W.
Wightman, A. S.
Wigner, E. P.
Wilks, S. S.
Red Bank
Newman, P.
Treiber, H. I.
River Edge
Hoyt, R. S.
South Orange
Stanwick, C. A.
Seton Hall Coll.
Elston, F. G.
Glusman, S.
Herschdorfer, M
Summit
McMillan, A. W.
Teaneck
Meyer, H. K.
Teterboro
Enequist, L. N.
Trenton
Ferester, A. H.
Shuster, C. N.
Upper Montclair
Boggs, E. M.
Campbell, G. A.
New Jersey State Teachers Coll.
Davis, D. R.
Humphreys, T. R.
Mallory, V. S.
West Orange
Edison, T. M.
Mead, D. G.

NEW MEXICO

Albuquerque
New Mexico School of Mines
Cassity, C. R.
Schutzberger, H.
Univ. of New Mexico
Boldyreff, A. W.
Hendrickson, M. S.
Hildner, R. C.
LaPaz, L.
Las Vegas
Rodgers, T. G.
Los Alamos
Allen, W. A.
Carlson, B. G.
Everett, C. J.
Gittings, H. T.
Goldstein, L.
Goldstein, M.
Hammer, P. C.
Mark, J. C.
Stark, R. H
Ulam, S M.
Raton
Kloss, R. E.
Roswell
Roudebush, W. C.
Socorro
Beach, J. W.

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Thompson, W. R.
New York State Coll. for Teachers
Birchenough, H.
Lester, C. A.
Stokes, E. C.
Alfred
Butler, L. C.
Freund, J. E.
Annandale-on-Hudson
Tremblay, C. J.
Aurora
Wells Coll
Carroll-Rusk, E.
Hollcroft, M. P.
Hollcroft, T. R.
Peacock, T. E.
Baldwin
Grove, C. C.
Beacon
Sherman, J.
Bethpage
Grumman Aircraft Eng. Corp.
Gross, G. L.
Mack, C. E.
Tripp, R. H.
Bronxville
Manning, A. W.
Strodt R. C.
Brooklyn
Gardner, C. S.
Gerst, I.
Hopkins, M. W.
Kallish, A.
Kramer-Lassar, E. E.
Kravetz, S.
LaSala, L.
Maddaus, I.
Miller, J. L.
Salkind, C.
Salkind, W.
Wayne, A.
Wynne, W. M.
Zabb, N. J.
Brooklyn Coll.
Bishop, M. L.
Boeker, M. D.
Borofsky, S.
Bowden, M.
Boyer, C. B.
Fleisher, E.
Griffin, H.
Herbach, L. H.
Johnson, R. A.
Kennison, L. S.
Kieval, H. S.
Kormes, J. P.
Landers, A. W.
Luchins, E. H.
Maria, A. J.
Maria, M. H.
Moore, L. T.
Prenowitz, W.
Richardson, M.
Secrist, J. B.
Shapiro, E. I.
Singer, J.
Washnitzer, G.
Wolfe, J. M.
Woodbridge, M. Y.
Long Island Univ.
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Ritterman, M. B.
Polytech. Inst. of Brooklyn
Belsky, A.
Fenn, I. H.
Fialkow, A. D.
Foster, R. M.
Hutchinson, L. C.
Kahal, R.
Klamkin, M. S.
Lowe, R. B.
Morduchow, M.
Rubin, J.
Sachs, B.
Schacknow, A. B.
Terzuoli, A. J.
Weber, E.
Whitford, D. E.
Pratt Inst.
Beckman, F. S.
Cowles, W. H. H.
Helme, G. C.
Moore, A. H.
Thompson, J. E.
St. Francis Coll.
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Dice, W. F.
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St. John's Univ.
Cassidy, W. F.
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Sarno, A. H.
Tolle, L. F.

St. Joseph's Coll. for Women
Larkin, F. X.
Buffalo
Cramer, R. H.
Dixon, G. H.
Kavanagh, A. J.
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Walker, G. W.
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Wilkins, J. E.
D'Youville Coll.
Maloney, M. M.
Univ. of Buffalo
Baeumler, H. W.
Brendel, R. A.
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Gordon, R. D.
Montague, H. F.
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Welmers, I. W.
Canton
Bates, O. K.
Peters, R. M.
Cleveland
Morenus, E. M.
Clinton
Ferry, F. C.
Gere, B. H.
Cortland
Benedick, J. W.
Elmhurst
Eckweiler, H. J.
Elmira
Suffa, M. C.
Endicott
Kent, J. R. F.
Wright, F. M.
Far Rockaway
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Flushing
Ripton, T. B.
Queens Coll.
Archibald, R. G.
Brown, A. B.
Cohen, L. W.
Cope, T. F.
Critchlow, F. G.
Eaton, J. E.
Feld, J. M.
Hoffmann, B.
Laderman, L. M.
Raudenbush, H. W.
Sard, A.
Sullivan, M. M.
Williamson, J.
Zippin, L.
Garden City
Bowden, J.
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Geneva
Hobart Coll.
Beinert, R. L.
Durfee, W. H.
Hubbs, H. N.
Great Neck
Stewart, S. W.
Hamilton
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Wardwell, J. F.
Hempstead
Hofstra Coll.
Hove, E. M.
Ollmann, L. F.
Stabler, E. R.
Houghton
Luckey, R. R.
Ithaca
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Cornell Univ.
Agnew, R. P.
Carver, W. B.
Chung, K. L.
Donsker, M. D.
Feller, W.
Goldberg, S.
Hurwitz, W. A.
Kac, M.
Kuo, Y. H.
Laush, G.
Malti, M. G.
Morton, J. E.
Murdock, W. L.
Pollard, H.
Rosser, J. B.
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Snyder, V.
Tuckerman, B.
Walker, R. J.
Williams, C. S.
Wing, G. M.
Yood, B.
Lake Success
Flexner, W. W.
Grant, A. M. C.
Hauser, A. A.
Riley, M. H.
Loudonville
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Mt. Kisco
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Mt. Vernon
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Franken, P. A.
Grossman, E. A.
Grossman, G.
Halperin, M.
Hardin, R. I.
Holdman, R. E.
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Ingram, W. H.
Jablon, S.
Joffe, S. A.
Kagno, I. N.
Katz, W.
Keyser, S. Y.
Kimball, B. F.
Kiss, S. A.
Kormes, M.
Lewis, J. H.
Lipschutz, M. A.
Meyerson, H. B.
Miller, K. S.
Miloslavsky, G. C.
Mitchell, H. B.
Monsky, M.
Morrison, N.
Neuman, S.
Nodelman, H. M.
Parsons, C. H.
Perlstein, M.
Porter, E. A.
Preston, B.
Rempfer, R. W.
Roosevelt, G. E.
Rothstein, S. I.
Ruderman, H. D.
Salerno, J.
Sasuly, M.
Schwartz, J. T.
Senior, J. L.
Shook, R. C.
Siegel, K. M.
Simon, L. G.
Stachel, J. J.
Tanzola, J. J.
Vitale, R. L.
Weaver, W.
Wernick, W.
White, M. E.
Williams, E. C.
Williams, H. B.
Wolfe, H.
American Math. Society
Hull, E. M.
Kellar, M. E.
American Telephone and Telegraph Co.
Belcher, D. R.
Halbert, K. W.
Hodgkinson, W.
Bell Telephone Labs., Inc.
Fry, T. C.
Gilbert, E. N.
Hebbert, C. M.
Jones, P. C.
MacColl, L. A.
Mead, S. P.
Riordan, J.
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Vaage, E. F.
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Zobel, O. J.
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Barber, S. F.
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Cortell, R.
Dwork, J. S.
Fagerstrom, W. H.
Fischer, E.
Fischer, I.
Garrison, G. N.
Gill, B. P.
Hinman, H. H.
Hubert, W. G.
Hurwitz, S.
Linehan, P. H.
Linial, S.
Lorch, L.
Nathan, D. S.
Post, E. L.
Radin, A.
Robinson, S. L.
Schwartz, A.
Singer, D.
Updike, G. Z.
Weingarten, H.
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Zeig, J. J.
Coll. of Mt. St. Vincent
Conlan, M. L.

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Ascher, R. N.
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Bazer, J.
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Brown, R. H.
Cathcart, A.
Chen, K. T.
Chessin, P. L.
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Franklin Inst.
Miller, B. L.
Swann, W. F. G.
Swarthmore Coll.
Brinkmann, H. W.
Carruth, P. W.
Dresden, A.
Marriott, R. W.
Schafer, A. T.
Wasow, W. R.
Washington
Dorwart, H. L.
Shaub, H. C.
Wayne
Feather, E. H.
Snyder, A. K.

PUERTO RICO

Hato Rey
Bobonis, A.
Mayagüez
Univ. of Puerto Rico
García, M.
Morales, J. D.
Sánchez-Díaz, R.
Rio Piedras
Capo, B. G.
Univ. of Puerto Rico
Bocanegra-Saldaña
Colon, J. A.

RHODE ISLAND

Kingston
Stauffer, J. R. K.
Rhode Island State Coll.
Haggerty, G. B.
Pease, E. M. J.
Providence
Adams, R. B.
Sherman, W. L.
Brown Univ.
Adams, C. R.
Albert, R. G.
Archibald, R. C.
Arsove, M. G.
Bennett, A. A.
Biot, M. A.
Boas, R. P.
Buck, E. F.
Buck, R. C.

Carlen, M. E.
Carrier, G. F.
Chen, P. P.
Chiarulli, P.
Clarke, A. B.
Demers, M. R.
Diaz, J. B.
Ehlers, F. E.
Federer, H.
Freeman, J. C.
Freilich, G.
Gildea, R. A. J.
Gilman, R. E.
Greenberg, H. J.
Hammond, E. E.
Heins, M. H.
Isenberg, J. S.
Jónsson, B.
Keck, W.
Krumhansl, J. A.
van der Kulk, W.
Lindsay, R. B.
Manning, H. P.
Neugebauer, O. E.
Pease, D. K.
Penez, J.
Prager, W.
Richardson, R. G. D.
Schwartz, B. L.
Smiley, C. H.
Stewart, F. M.
Truell, R.
Weil, H.
Whitehead, G. W.
Whitehead, K. B.
Providence Coll.
Gallagher, E. H.
McKenney, J. L.
Rhode Island Coll. of Education
Mitchell, C. R.
Portsmouth
Taliaferro, R. C.
West Warwick
Brant, L.

SOUTH CAROLINA

Anderson
Paschal, R. C.
Charleston
Dye, L. A.
Sutton, C. S.
Clemson
Coker, E. C.
Vause, R. Z.
Clinton
Sherratt, W. A.
Columbia
Coleman, J. B.
Univ. of South Carolina
Hedberg, E. A.
Jackson, J. B.
Lee, T. H.
Lytle, R. A.
Novak, J. D.
Robinson, L. V.
Weber, W. W.
Williams, W. L.
Rock Hill
Pepper, R. I.
Spartanburg
Patten, W. E.

SOUTH DAKOTA

Brookings
South Dakota State Coll.
MacDougal, H. B.
Walder, O. E.
Wente, I. L.
Vermillion
Beaty, M. H.

TENNESSEE

Alcoa
Harris, L. J.
Knoxville
Univ. of Tennessee
Albert, G. E.
Blackstock, M. C.
Bradley, H. J.
Cooley, J. A.
Eaves, E. D.
Ficken, F. A.
Givens, W.
Harrold, O. G.
Johnson, R. B.
Lee, H. L.
Miller, D. D.
Rivers, M. E.
Rutledge, W. A.
Snyder, W. S.
Watson, M.
Williams, J. F.
Wilson, R. L.
Martin
Taylor, W. C.
Memphis
MacQueen, M. L.
Smith, R. F.
Nashville
Fisk Univ.
Byrd, P. F.
Vanderbilt Univ.
Boyce, M. G.
Bright, S. K.
Hyden, J. A.
Miser, W. L.
Shanks, E. B.
Oak Ridge
Chelius, L. G.
Coveyou, R. R.
Dismuke, N. M.
Garrett, G. A.
Householder, A. S.
Hurd, C. C.
Nelson, L.
Thalgott, F. W.
Sewanee
Bruton, G. S.
Dickerson, B. K.

TEXAS

Abilene
Abilene Christian Coll.
Mullings, M. E.
Hardin-Simmons Univ.
Burnam, J. E.
McMurry Coll.
Kean, H. P.
Arlington
Lassen, L. L.
Austin
Decherd, M. E.
Univ. of Texas
Batchelder, P. M.
Bell, E. L.
Clarkson, M. H.
Coleman, W. B.
Craig, H. V.
Dyer, S. E.
Estill, M. E.
Ettlinger, H. J.
Greenwood, R. E.
Hamstrom, M. E.
Hurt, J. M.
Jones, F. B.
Lane, R. E.
Lubben, R. G.
MacNerney, J. S.
Moore, R. L.
Porter, G. H.
Porter, M. B.
Titt, E. W.
Vandiver, H. S.
Wall, H. S.
Chapman Ranch
Newton, G. A.
College Station
Agric. and Mech. Coll. of Texas
Basye, R. E.
Beeman, W. E.
Daum, J. A.
Hurt, J. T.
Klipple, E. C.
Koss, W. E.
McCulley, W. S.
Moore, B. C.
Wells, N. W.

Commerce
Wright, C. B.
Dallas
McDermott, E.
Montgomery, P. V.
Sorrells, R. C.
Southern Methodist Univ.
Starr, D. W.
Denton
North Texas State Teachers Coll.
Brown, M. C.
Maple, C. G.
Texas State Coll. for Women
Ashburn, A.
Miller, H. C.
El Paso
González, J. M.
Ft. Worth
Colquitt, L. A.
Georgetown
Southwestern Univ.
Kester, W. W.
Whitmore, R. M.
Wilcox, M. E.
Houston
Eberman, J. W.
Pearson, E. H.
Rust, W. M.
Slotnick, M. M.
Weinstein, A. K.
Rice Inst.
Bray, H. E.
Brunk, H. D.
Calkin, J. W.
Dean, A. C.
Houston, W. V.
Lovett, E. O.
MacLane, G. R.
Reiter, H.
Taylor, H. E.
Ulrich, F. E.
White, T. J.
Univ. of Houston
Abernethy, J. J.
Aucoin, A. A.
Grover, B. B.
Hutchinson, J. D.
Newhouse, A.
Parson, R. H.
Rader, C. B.
Wright, M.
Huntsville
Querry, J. W.
Kingsville
Dorroh, J. L.
Oliphant, E. A.
Lubbock
Texas Tech. Coll.
Crouch, R. P.
Michie, J. N.
Roberts, V. B.
Parker, B.
Marlin
Falconer, B. L.
Mercedes
Shaw, W. F.
San Angelo
Bright, H. F.
San Antonio
Gough, M. de L.
Hill, M. L.
San Marcos
Tulloch, L. H.
Waco
Williams, W. J.

UTAH

Logan
Hunsaker, N. C.
Tingey, V. H.
Provo
Marshall, M.
Salt Lake City
Univ. of Utah
Biesele, F. C.
Chamberlin, R. E.
Hayes, J. J.
Henriques, A. S.
Horsfall, I. O.
Pehrson, E. W.
Smith, S. S.
Thorne, C. J.
Wylie, C. R.

VERMONT

Bennington
Ellis, G. A.
Bristol
Billard, R. G.
Burlington
Stibitz, G. R.
Swift, E.
Univ. of Vermont and State Agric. Coll.
Bullard, J. A.
Larrivee, J. A.
McClay, D. T.
Sherman, B.
Simond, R. G.
Wilson, A. L.
Middlebury
Ballou, D. H.
Bowker, J. G.
Montpelier
Miller, R. M.
Northfield
McConnon, M.

VIRGINIA

Arlington
Tompkins, C. B.
Blacksburg
Virginia Polytech. Inst.
Hatcher, T. W.
Lawrence, V. S.
McFadden, L.
Charlottesville
Robbins, R. B.
Univ. of Virginia
Bernard, R. R.
Botts, T. A.
Butler, J. B.
Floyd, E. E.
Jeffries, M. B.
Kasriel, R. H.
Klee, V. L.
Linfield, B. Z.
McShane, E. J.
Oglesby, E. J.
Pace, W. E.
Whyburn, G. T.
Whyburn, L. S.
Dahlgren
Naval Proving Ground
Barker, J. E.
Bramble, C. C.
Dresch, F. W.
Ross, C.
Emory
Young, N. F.
Falls Church
McBee, E. L.
Fredericksburg
Burns, G. P.
Carter, H. C.
Hampton
Andrews, T. B.
Hampton Institute
Eshleman, J. D.
Harrisonburg
Partlow, B. W.
Hopewell
Fazel, C. S.
Langley Field
Benscoter, S. U.
Lexington
Virginia Military Inst.
Byrne, W. E.
Knox, R. H.
Paxton, E. K.
Washington and Lee Univ.
Welch, F. P.
Williams, C. W.
Wine, R. L.
Lynchburg
Larew, G. A.
Wiggin, E. P.
Petersburg
Hunter, L. S.
McDaniel, R. R.
Richmond
Barr, W. J.
Staunton
Taylor, M. E.
Sweet Briar
Iwanowski, R. M.
Lee, M. A.
University of Richmond
Gaines, R. E.
Grable, E. S.
Wheeler, C. H.
Williamsburg
Coll. of William and Mary
Phalen, H. R.
Smith, R. E.
Stetson, J. M.

WASHINGTON

College Place
Schlotthauer, L. G.
East Sound
Melvin, M. A.
Olympia
Raymond, J.
Pullman
State Coll. of Washington
Butler, L. G.
Caton, W. B.
Hacker, S. G.
Knebelman, M. S.
Richland
Bennett, C. A.
Seattle
Kinerk, H. E.
Lanczós, C.
Seattle Pacific Coll.
Beegle, B. L.
Univ. of Washington
Andrews, F. C.
Avann, S. P.
Ball, R. W.
Ballantine, C. R.
Ballantine, J. P.
Beaumont, R. A.
Birnbaum, Z. W.
Bryan, W. S.
Carpenter, A. F.
Cramlet, G. M.
Dekker, D. B.
Haller, M. E.
Hewitt, E.
Hildebrand, J. L.
Jerbert, A. R.
Kingston, J. M.
McFarlan, L. H.
Owen, D. B.
Paulson, E.
Uhrich, G. E.
Winger, R. M.
Yagi, F.
Zuckerman, H. S.
Spokane
Ryan, D. R.
Walla Walla
Hutchings, W. L.

WEST VIRGINIA

Bethany
Bethany Coll.
Cramblet, W. H.
Moser, C. E.
Tye, B.
Huntington
Knifley, L. M.
Institute
Crawley, A. B.
Ponds, J. W.
Keyser
Iverson, P. F.
Morgantown
West Virginia Univ.
Crisler, E. H.
Cunningham, A. B.
Davis, H. A.
Pollock, H. S.
Reynolds, C. N.
Turner, B. M.
Vest, M. L.

WISCONSIN

Appleton
Berry, A. C.
Stewart, J. C.
Beloit
Conwell, H. H.
Huffer, R. C.
Green Bay
Zarling, L. B.
LaCrosse
Adkins, L. K.
Madison
Ericksen, W. S.
Univ. of Wisconsin
Adler, F. T.
Allen, F. E.
Andree, R. V.
Arnold, K. J.
Bing, R. H.
Bruck, R. H.
Campbell, H. E.
Cashwell, E. D.
Cohen, H. J.
Colvin, B. H.
Davies, R.
Dinneen, G. P.
Eberlein, W. F.
Emerson, M. P.
Engel, J. H.
Evans, H. P.
Fleming, S. G.
Fulkerson, D. R.
Fuller, L. E.
Fullerton, R. E.
Gaddum, J. W.
Goldhaber, J. K.
Hood, R. T.
Hsiung, C. C.
Ingraham, H. H.
Jamrich, J. X.
Kelly, J. B.
Kleene, S. C.
Langer, R. E.
Larson, C. W.
MacDuffee, C. C.
March, H. W.
Mark, A. M.
Mayor, J. R.
Mitchell, M.
Morrison, D. R.
Norton, D. A.
Owens, O. G.
Renno, J. G.
Rose, G. F.
Ross, V. H.
Schurrer, A. L.
Snowdon, S. C.
Sokolnikoff, E. S.
Specht, R. D.
Strehler, A. F.
Trump, P. L.
Wysocki, T.
Zemmer, J. L.
Marinette
Wagner, R. D.
Milwaukee
Beckwith, E. R.
Herr, D. L.
Krug, R. E.
Vass, J. I.
Marquette Univ.
Moeller, A. C.
Pettit, H. P.
Wilczewski, J.
Milwaukee-Downer Coll.
Weisbecker, F.
Mt. Mary Coll.
Vaudreuil, M. F.
State Teachers Coll.
Norris, R. E.
Univ. of Wisconsin
Perkinson, G. A.
Univ. of Wisconsin in Milwaukee
Bardell, R. H.
Kenney, J. F.
Marden, M.
Schuld, E. G.
Spitzbart, A.
Wolf, L. A.
Neenah
Nash, J. P.
Racine
May, A. E.
Ripon
Alexander, A. B.
Graham, J. W.
River Falls
Eide, M. C.
Sheboygan
Battig, L.
Waukesha
Meadows, P. E.
West DePere
DeCleene, L. A. V.
Williams Bay
Chandrasekhar, S.
Smith, B.

WYOMING

Casper
Bailey, A. W.
Lamont
Bellamy, B. C.
Laramie
Univ. of Wyoming
Brady, W. G.
Bristow, L.
Rechard, O. H.
Smith, W. N.
Varineau, V. J.

CANADA

ALBERTA

Edmonton

Univ of Alberta

Campbell, J. W.
Cook, A. J.
Fostvedt, T. M.
Jacka, R. C.
Phibbs, E.
Sheffield, E. T.
Sheldon, E. W.
Wyman, M.

BRITISH COLUMBIA

Vancouver

Univ. of British Columbia

Buchanan, D.
Derry, D.
Gage, W. H.
James, R. D.
Jennings, S. A.
Moyls, B. N.
Murdoch, D. C.
Nowlan, F. S.
Simons, W. H.

MANITOBA

Winnipeg

Univ. of Manitoba

Lawson, J. W.
McEwen, W. H.
Mendelsohn, N. S.
Moser, L.
Prendergast, H. B.

NEW BRUNSWICK

Frederickton

Edwards, L. P.

Sackville

Crawford, W. S. H.

St. Joseph's

Poitras, A.

NOVA SCOTIA

Halifax

Adshead, J. G.
Walmsley, C.

Wolfville

Haley, K. D. C.
Rothberger, F.

ONTARIO

Hamilton

McMaster Univ.

Bankier, J. D.
Britton, F. R.
Findlay, W.
Johns, A. E.
McCallion, W. J.

Kingston

Queen's Univ.

Ellis, H. W.
Halperin, I.
Jeffery, R. L.
Miller, N.

London

Univ. of Western Ontario

Cole, R. H.
Kingston, H. R.
Magee, G. R.
Moore, J. T.

Ottawa

Duprat, J. T.
MacPhail, M. S.

Toronto

DeLury, D. B.
Grant, A. A.
Wolfenden, H. H.

Univ. of Toronto

Beatty, S.
Burk, J. D.
Coxeter, H. S. M.
Fryer, K. D.
Gilchrist, L.
Krieger, C.
Lapidus, B.
Morris, T. F.
Pounder, I. R.
Robinson, G. de B.
Schild, A.
Waddell, M. E. G.
Webber, W. J.

QUEBEC

Montreal

Benjamin, I.
Nichols, W. A.

Loyola Coll.

O'Connor, R. E.

McGill Univ.

Duffie, J. A. H.
Rosenthall, E.
Shaw, A. N.
Williams, W. L. G.

Univ. of Montreal

Gauthier, A.
L'Abbé, M.
Stock, R.

SASKATCHEWAN

Saskatoon

Univ. of Saskatchewan

Crosby, W. J. R.
Ferns, H. H.
Kozakiewicz, W.
Miller, C. E.
Scherk, P.
Vigder, J. S.

AFRICA

EGYPT

Cairo

Dreyfus, D. M.

Fouad I Univ.

Dana, V.
Mostafa, H. H.

SOUTH AFRICA

Cape Town

Pauc, C. Y.
Young, L. C.

Johannesburg

Tannenbaum, B.

ASIA

CHINA

Canton

Wong, Y. C.

Changsha

Wu, S. C.

Hangchow

Chen, K. K.

Nanking

Chern, S. S.
Chiang, L. F.

Peiping

Lu, C. H. L.

Tsing Hua Univ.

Chien, W. Z.
Min, S. H.
Tuan, H. F.

Santung

Hsieh, L. F.

INDIA

Aligarh

Shah, S. M.

Annamalainagar

Rao, K. N.

Bangalore

Narayana Row, K. R.
Rao, B. S. M.

Bombay

Kosambi, D. D.
Levi, F. W.

Delhi

Seth, B. R.

Madras

Ananda-Rau, K.

New Delhi

Chand, P.

Poona

Paranjpye, R. P.

Waltair

Minakshisundaram, S.
Vijayaraghavan, T.

ISRAEL

Tiberias

Artzy, R.

Rehovoth

Pekeris, C. L.

JERUSALEM

Jabotinsky, E. T.
Preiser, T. H.

Hebrew Univ.

Amira, B. A.
Fraenkel, A. A.
Levitzki, J.
Motzkin, T. S.
Schiffer, M. M.

LEBANON

Beirut

Kennedy, E. S.

TURKEY

Istanbul

Taskin, H. K.
Uluçay, C.

Zonguldak

Demir, H.

AUSTRALASIA

AUSTRALIA

Brisbane
Raybould, E. H.

Hobart
Jaeger, J. C.

Melbourne
Univ. of Melbourne
Behrend, F. A.
Cherry, T. M.
Schwerdtfeger, H. W. E.

Perth
Weatherburn, C. E.

Sydney
Univ. of Sydney
Carslaw, H. S.
Room, T. G.
Thorne, H. H.

NEW ZEALAND

Auckland
Forder, H. G.

Christ Church
Gill, P. J.

Dunedin
Martyn, W. J.

CENTRAL AND SOUTH AMERICA

ARGENTINA

Cordoba
Perrine, C. D.

Buenos Aires
Military School of Rio Santiago
De Cesare, E. A.
Univ. of Buenos Aires
Baidaff, B. I.
Barral-Souto, J.
Blaquier, J.
Cotlar, M.
González-Domínguez, A.

LaPlata
Univ. of LaPlata
Cesco, R. P.
Duranona y Vedia, A.
Florio, J. S.
Machado, E. A.
Pérsico, H. A.
Ricabarra, R. A.
Sagastume Berra, A. E.
Trejo, C. A.

Olivos
Bula, C. A.

Rosario
Univ. of Rosario
Gaspar, F. L.
Levi, B.
Santaló, L. A.

San Luis
Turrin, G.

San Martín
Guglialmelli, J. P.

BRAZIL

Distrito Federal
de Carvalho, C. A.

Paraná
Lepecki, Z.

Recife
Freire, L. B.

Rio de Janeiro
Galper, M.
Military Coll.
Quintella, A. N. M.
National School of Eng.
Peixoto, M. C.
Peixoto, M. M.
Univ. of Brasil
Abdelhay, J.
Abdelhay, M. Y. de M. N.
Gomes, A. M.
Martynes, G. W. L. D.
Monteiro, A. A.
Nachbin, L.
de Oliveira Castro, F. M.
de Oliveira Junior, E. L.
Rocha, P. S.

Salvador
Santos, A. M. V.

São Paulo
de Toledo Piza, A. P.
Univ. of Brasil
Breves Filho, J. A.
Univ of São Paulo
Castrucci, B.
Catunda, O.
Dias, C. L. da S.
Farah, E.
Furquim de Almeida, F.
Gomide, E. F.
Lima Filho, G. S.

COLOMBIA

Bogota
Laserna, M.
Thullen, P.

COSTA RICA

San Jose
Trejos-Fernández, J. J.

CUBA

Havana
González, M. O.
Gutiérrez, L. G.

MEXICO

Mexico
Medina, A.
National Polytech. Inst.
Vallarta, M. S.
Univ. of Mexico
Barajas, A.
González-Baz, E.
Graef Fernandez, C.
Nápoles-Gándara, A.

Monterrey
Treviño García, J.

San Luis Potosi
Flodin, I. A.

PANAMA

Santiago
Evans, J. L.

PERU

Lima
de Losada y Puga, C.
Secada, C. F.

URUGUAY

Montevideo
Laguardia, R.
Massera, J. L.

VENEZUELA

Caracas
Goa, C. S.
Hodell, L. R.
Michalup, E. M.

EUROPE

AUSTRIA

Vienna
Basch, A. E.
Funk, P.

Graz
Jardetzky, W.

BELGIUM

Brussels
Debever, R. G.
Sips, R.

St. Nicolas-Waas
Van Bergen, F. E.

CZECHOSLOVAKIA

Brünn
Borůvka, O.

Prague
Janko, J. J.
Schoenbaum, E.

DENMARK

Copenhagen
Fog, D.
Følner, E.
Hansen, H. H.
Pihl, M.
Tech. Univ. of Denmark
Andersen, A. F.

Buch, K. R.
Fabricius-Bjerre, F. G.
Fenchel, W.
Lauritzen, S. E.
Nielsen, J.
Petersen, R.
Univ. of Copenhagen
Bohr, H. A.
Bundgaard, S. B. E.
Busk, T.
Hald, A. H.
Jessen, B.
Nørlund, N. E.
Rasch, G. W.

ENGLAND

Acomb
Melmore, S.
Alnwick
Collingwood, E. F.
Birmingham
Kober, H.
Bristol
Heilbronn, H. A.
Cambridge
Cambridge Univ.
Burkill, J. C.
Hall, P.
Ingham, A. E.
Mordell, L. J.
Rankin, R. A.
Smithies, F.
East Molesey
Taussky, O.
Todd, J.
Englefield Green
McCrea, W. H.
Hindhead
Western, A. E.
Leeds
Milne, W. P.
Univ. of Leeds
Brodetsky, S.
Cowling, T. G.
Perrott, I. B.
Ruse, H. S.
Liverpool
Fuchs, W. H. J.
London
Dalal, R. D.
Univ. of London
Booth, A. D.
Britten, K. H.
Davenport, H.
Pedoe, D.
Semple, J. G.
Simpson, H.
Vermes, P.
Manchester
Univ. of Manchester
Neumann, B. H.
Newman, M. H. A.
Stone, A. H.
Stone, D. M.
Oxford
Oxford Univ.
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RESTRICTED MEASURABILITY

TREVOR J. McMINN

1. Introduction. It is our aim to introduce into the general theory of measure a concept of restricted measurability[1] and to establish certain conditions under which it is equivalent to measurability in the familiar Carathéodory sense.

We follow the theory with two applications.

2. Notations and conventions. If C is a sequence of sets and n is an integer, we agree that

$$\bigcup_{j=0}^{-1} C_j, \qquad \bigcup_{j=0}^{n} C_j, \quad \text{and} \quad \bigcup_{j=0}^{\infty} C_j$$

are respectively 0, the union of terms of C numbered from 0 to n, and the union of all terms of C.

We agree that a family H of sets is *hereditary* if each subset of each member of the family is a member of the family and is *countably additive* if the union of members of each countable sub-family of H is a member of H. By σH we mean the union of all members of H.

If A and B are sets, we agree that $A \sim B$ is the set of points in A and not in B.

Throughout this paper we consider $\mathfrak{S}$ to be a fixed set with respect to which we make definitions.

3. Restricted measurability.

3.1. Definition. ϕ is a *measure* if and only if ϕ is such a function on

$$\underset{\beta}{E}(\beta \subset \mathfrak{S}) \quad \text{to} \quad \underset{t}{E}(0 \leqq t \leqq \infty)$$

that

$$\phi(A) \leqq \sum_{\beta \in H} \phi(\beta)$$

whenever H is such a countable family that

$$A \subset \sigma H \subset \mathfrak{S}.$$

Since empty sums are zero, we see that, if ϕ is a measure, then:

I. $\phi(0) = 0$;

II. $\phi(A) \leqq \phi(B)$ whenever $A \subset B \subset \mathfrak{S}$;

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[1] A. P. Morse has contributed much to the development of this theory.

III. $\phi(\sigma H) \leqq \sum_{\beta \in H} \phi(\beta)$ whenever H is such a countable family that $\sigma H \subset \mathfrak{S}$.

3.2. DEFINITION. A is ϕ *measurable* F if and only if ϕ is a measure and, for each member B of F, $\phi(TB) = \phi(TBA) + \phi(TB \sim A)$ whenever $T \subset \mathfrak{S}$.

It is rather evident that a set, ϕ measurable F, is ϕ *measurable* F' where F' is the family of subsets of members of F.

In accordance with Carathéodory, we agree that A is ϕ *measurable* if and only if A is ϕ measurable

$$\underset{\beta}{E}\,(\beta \subset \mathfrak{S}).$$

Following the next definition is the principal theorem of this paper.

3.3 DEFINITION. F is ϕ *convenient* if and only if ϕ is a measure, F is hereditary, and, corresponding to each T of finite ϕ measure, there exists such a sequence C that

$$\phi\left(T \sim \bigcup_{j=0}^{\infty} C_j\right) = 0,$$

and, for each integer n,

$$C_n \subset C_{n+1} \in F, \quad C_n \text{ is } \phi \text{ measurable } F.$$

3.4. THEOREM. *If F is ϕ convenient, then A is ϕ measurable whenever A is ϕ measurable F.*

PROOF. If $\phi(T) = \infty$, then, clearly, $\phi(T) = \phi(TA) + \phi(T \sim A)$.

Suppose $\phi(T) < \infty$. We secure such a sequence C that, for each integer n,

$$C_{-1} = 0, \quad \text{and} \quad \phi\left(T \sim \bigcup_{j=0}^{\infty} C_j\right) = 0. \tag{1}$$

$$C_n \subset C_{n+1} \in F,\ C_n \text{ is } \phi \text{ measurable } F.$$

We note that, for each integer j, $C_j \sim C_{j-1} \in F$.

Now, for each integer n and $T \subset \mathfrak{S}$, we have

$$\phi(TC_n) = \phi(TC_nC_{n-1}) + \phi(TC_n \sim C_{n-1}) = \phi(TC_{n-1}) + \phi(TC_n \sim C_{n-1}).$$

By induction we obtain, for each integer n,

$$\phi(T) \geqq \phi(TC_n) = \sum_{j=0}^{n} \phi(TC_j \sim C_{j-1}) = \sum_{j=0}^{n} \phi\{T(C_j \sim C_{j-1})A\}$$

$$+ \sum_{j=0}^{n} \phi\{T(C_j \sim C_{j-1}) \sim A\}.$$

Hence,

$$\begin{aligned}\phi(T) &\geqq \sum_{j=0}^{\infty}\phi\{T(C_j\sim C_{j-1})A\} + \sum_{j=0}^{\infty}\phi\{T(C_j\sim C_{j-1})\sim A\}\\ &\geqq \phi\left\{\bigcup_{j=0}^{\infty} T(C_j\sim C_{j-1})A\right\} + \phi\left\{\bigcup_{j=0}^{\infty} T(C_j\sim C_{j-1})\sim A\right\}\\ &= \phi\left\{TA\bigcup_{j=0}^{\infty} C_j\right\} + \phi\left\{(T\sim A)\bigcup_{j=0}^{\infty} C_j\right\}\\ &\quad + \phi\left\{TA\sim\bigcup_{j=0}^{\infty} C_j\right\} + \phi\left\{(T\sim A)\sim\bigcup_{j=0}^{\infty} C_j\right\}\\ &\geqq \phi(TA) + \phi(T\sim A) \geqq \phi(T),\end{aligned}$$

where the third step follows from (1).

The proof is complete.

3.5. REMARK. The next definition and two obvious theorems following it enable us by using extant measure theory[3] to avoid imitative direct methods for proofs that the complement of a set ϕ measurable F and the union and intersection of a countable family of sets ϕ measurable F are ϕ measurable F and consequently to establish 3.9. The notion of intersection measure is also useful in proving 4.2.

3.6. DEFINITION. sect (ϕ, B) is the function f on

$$\underset{\beta}{E}\,(\beta\subset\mathfrak{S})$$

such that $f(\alpha)=\phi(\alpha B)$ for $\alpha\subset\mathfrak{S}$.

3.7. THEOREM. *If ϕ is a measure, then, for each B,* sect (ϕ, B) *is a measure.*

3.8. THEOREM. *A is ϕ measurable F if and only if ϕ is a measure and, for each member B of F, A is* sect (ϕ, B) *measurable.*

3.9 THEOREM. *If H is a family of sets ϕ measurable F and B is the intersection of all countably additive, $\mathfrak{S}$ complemental families which contain H, then each member of B is ϕ measurable F.*

4. **Applications.** For Theorems 4.1 through 4.6 we assume that ρ metrizes $\mathfrak{S}$ and agree that

$$\operatorname{dist}(A, B) = \inf_{x\in A}\ \inf_{y\in B}\rho(x, y).$$

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[3] Measure theoretic results we use can be found in H. Hahn, *Theorie der reellen Funktionen*, vol. 1, Berlin, 1921, pp. 424–432.

Due to Carathéodory is the following well known theorem.

4.1. THEOREM. *If* ϕ *is a measure and* $\phi(A\cup B)=\phi(A)+\phi(B)$ *whenever* dist $(A, B)>0$, *then each open set is* ϕ *measurable.*

Stronger than 4.1 is 4.3 below.

4.2. THEOREM. *If* F *is hereditary and* $\phi(A\cup B)=\phi(A)+\phi(B)$ *whenever* A *and* B *are such members of* F *that* dist $(A, B)>0$, *then each open set is* ϕ *measurable* F.

PROOF. Suppose dist $(A', B')>0$. From the hereditariness of F we see that

$$\text{sect}\,(\phi, T)(A'\cup B') = \text{sect}\,(\phi, T)(A') + \text{sect}\,(\phi, T)(B')$$

whenever $T\in F$. By 4.1 each open set is ϕ measurable F.

4.3. THEOREM. *If* F *is* ϕ *convenient and* $\phi(A\cup B)=\phi(A)+\phi(B)$ *whenever* A *and* B *are such members of* F *that* dist $(A, B)>0$, *then each open set is* ϕ *measurable.*

PROOF. Use 3.4 and 4.2.

An application of the foregoing is the following theorem.

4.4. THEOREM. *If* ϕ *is a measure and* $\phi(A\cup B)=\phi(A)+\phi(B)$ *whenever* A *and* B *are such bounded sets that* dist $(A, B)>0$, *then each open set is* ϕ *measurable.*

PROOF. Let F be the family of bounded subsets of $\mathfrak{S}$. We establish the ϕ convenience of F by taking a sequence of open spheres of integer radii the union of whose terms is $\mathfrak{S}$ and inferring from 4.2 the ϕ measurability F of these spheres. Reference to 4.3 completes the proof.

4.5. REMARK. The above theorem, which was communicated to me by A. P. Morse, instigated the present investigation. Morse proved it directly using a method of alternate annular shells.

I know of no correspondingly direct proof of the following application of 4.3.

4.6. THEOREM. *If* ϕ *is a measure,* R *is a sequence of non-dense sets,*

$$F = \underset{\beta}{E}\left(\beta\subset\mathfrak{S} \text{ and } \beta\bigcup_{j=0}^{\infty} R_j \text{ is non-dense}\right),$$

$\phi(A\cup B)=\phi(A)+\phi(B)$ *whenever* A *and* B *are such members of* F *that* dist $(A, B)>0$, *then each open set is* ϕ *measurable.*

PROOF. Let C be such a sequence that, for each integer n,

$$C_n = \bigcup_{j=0}^{n} (\text{closure } R_j) \cup \{\mathfrak{S} \sim \bigcup_{j=0}^{n} (\text{closure } R_j)\}.$$

Clearly, for each integer n,

$$\bigcup_{j=0}^{\infty} C_j = \mathfrak{S}. \qquad C_n \subset C_{n+1} \in F,$$

After checking the hereditariness of F, we infer from 4.2 that each open set is ϕ measurable F. Hence, if we recall 3.5, C_n is ϕ measurable F for each integer n. Thus F is ϕ convenient. Reference to 4.3 completes the proof.

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ON THE DISTRIBUTION OF THE VALUES OF $|f(z)|$ IN THE UNIT CIRCLE

ROBERT BREUSCH

1. **Summary.** Let $f(z) = 1 + a_1 z + \cdots$ be analytic for $|z| \leqq 1$, $f(z) \not\equiv 1$. Then $|f(z)|$ will be greater than 1 at some points of the unit circle, and less than 1 at others. Calling $A(f)$ the area of the set of points within the unit circle, for which $|f(z)| \geqq 1$, let α and β be the two largest non-negative constants such that $\alpha \leqq A(f) \leqq \pi - \beta$, for every $f(z)$. It is shown that $\alpha = \beta = 0$; in other words, if ϵ is arbitrarily small positive, there are functions $f(z)$ such that $A(f) < \epsilon$, and others such that $A(f) > \pi - \epsilon$. The same is true, if $f(z)$ is restricted to polynomials $\prod_{\nu=1}^{n}(z - z_\nu)$ with $\prod_{\nu=1}^{n}|z_\nu| = 1$. These statements will be proved in §2. §3 contains a few additional results, given without proofs.

2. **Proofs.** The statements made in the summary are contained in the following theorem.

THEOREM. *Let P stand for the set of polynomials over the complex field of the form $f(z) = \prod_{\nu=1}^{n}(z - z_\nu)$, with $\prod_{\nu=1}^{n}|z_\nu| = 1$; let $A(f)$ denote the area of the set of points in the unit circle, for which $|f(z)| \geqq 1$; let ϵ be an arbitrarily small positive number. Then P contains polynomials $f_1(z)$ and $f_2(z)$ such that $A(f_1) > \pi - \epsilon$, and $A(f_2) < \epsilon$.*

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PROOF. Consider $f_1(z) = 1 + Nz + z^2$, where N is a real positive number greater than 3. Then in the unit circle, $|f(z)| \geqq |Nz| - 2$, which is greater than 1, if $|Nz| > 3$, $|z| > 3/N$. Thus

$$A(f) > \pi - \frac{9}{N^2}\pi,$$

and for $N > 3(\pi/\epsilon)^{1/2}$, this is greater than $\pi - \epsilon$. This proves the first part of the theorem.

To prove the second part, consider a function $F_0(z) = b_1 z + b_2 z^2 + \cdots$, analytic for $|z| \leqq 1$, $F(z) \not\equiv 0$. The real and the imaginary part of a function $F(z)$ will be designated by Re $F(z)$ and Im $F(z)$, respectively. Call $\bar{B}(F_0)$ the set of points in the unit circle for which Re $F_0(z) \geqq 0$, and $B(F_0)$ the area of $\bar{B}(F_0)$. Let n_1 be a positive real number.

The function $F_1(z) = \exp\,(n_1 F_0(z)) - 1 = n_1 b_1 z + \cdots$ is again analytic for $|z| \leqq 1$. Its real part is Re $F_1(z) = \exp\,(n_1 \mathrm{Re}\ F_0(z)) \cdot \cos\,(n_1 \mathrm{Im}\, F_0(z)) - 1$. Therefore Re $F_1(z)$ is positive or zero only where Re $F_0(z) \geqq 0$ *and* $\cos\,(n_1 \mathrm{Im}\ F_0(z)) \geqq \exp\,(-n_1 \mathrm{Re}\ F_0(z)) > 0$. It will be shown (see the lemma below) that for sufficiently large values of n_1, $\cos\,(n_1\ \mathrm{Im}\ F_0(z))$ is negative in a subregion of $\bar{B}(F_0)$ of area greater than $(1/3)B(F_0)$. Then $B(F_1) < (2/3)B(F_0) < (2/3)\pi$. Similarly a function $F_2(z) = \exp\,(n_2 F_1(z)) - 1$ may be formed such that $B(F_2) < (2/3)B(F_1) < (2/3)^2\pi$. Continuing in this way, for any arbitrarily chosen positive number ϵ, a function $F_t(z) = c_1 z + \cdots$ may be formed which is analytic for $|z| \leqq 1$, and for which $B(F_t) < (2/3)^t\pi < \epsilon/3$, for t large enough.

Then the function $f(z) = \exp\,(F_t(z)) = 1 + a_1 z + \cdots$ is analytic for $|z| \leqq 1$, and $|f(z)| = \exp\,(\mathrm{Re}\ F_t(z))$. Therefore $|f(z)| \geqq 1$ for the same points for which Re $F_t(z) \geqq 0$. If $A(f)$ stands again for the area of the set of points in the unit circle for which $|f(z)| \geqq 1$, $A(f) = B(F_t) < \epsilon/3$.

Now choose a positive number η so small that the set of points in the unit circle where $1 - 2\eta \leqq |f(z)| \leqq 1$ has an area not exceeding $\epsilon/3$. Finally choose a positive integer q so large that $|a_q z^q + a_{q+1} z^{q+1} + \cdots| < \eta$ for $|z| \leqq 1$ and that $|z^q| < \eta$ for $|z| \leqq 1 - \epsilon/(6\pi)$. Then in the unit circle $|1 + a_1 z + \cdots + a_{q-1} z^{q-1} + z^q| < 1 - 2\eta + \eta + \eta = 1$, except in the two regions where either $|f(z)| \geqq 1 - 2\eta$, or $|z| > 1 - \epsilon/(6\pi)$. The area of the first region is less than $\epsilon/3 + \epsilon/3 = 2\epsilon/3$, and the area of the second one is less than $2\pi \cdot \epsilon/(6\pi) = \epsilon/3$. Thus $A(f_2) < \epsilon$, with $f_2(z) = 1 + a_1 z + \cdots + a_{q-1} z^{q-1} + z^q$.

In the proof of the theorem, use has been made of the following lemma.

LEMMA. *If* $F(z)=b_1z+b_2z^2+\cdots\neq 0$ *is analytic for* $|z|\leq 1$, *if* $\bar{B}$ *is the set of points of the unit circle for which* Re $F(z)\geq 0$, *and* B *the area of* $\bar{B}$, *and if* n *is a sufficiently large positive number, then* cos (n Im $F(z)$) *is negative in a subset of* $\bar{B}$ *whose area is greater than* $B/3$.

PROOF OF THE LEMMA. For $z=re^{i\theta}$, Im $F(z)$ is a function $g_r(\theta)$ of r and θ. Choose a positive number m so small that the absolute value of $g_r'(\theta)=\partial/\partial\theta(\text{Im } F(z))$ will be not less than m except in a subset of $\bar{B}$ whose area is less than $B/12$. In other words, if $\bar{B}'$ is the set of points in the unit circle where Re $F(z)\geq 0$ *and* $|g_r'(\theta)|\geq m$, then B', the area of $\bar{B}'$, will be between $11B/12$ and B.

Call M the largest value of $|g_r'(\theta)|$ in $|z|\leq 1$, T the largest value of $|g_r''(\theta)|$ in $|z|\leq 1$, and S the largest number of intersections between the boundary of $\bar{B}'$ and any of the circles $r=\text{constant}\leq 1$. For every r, this number of intersections is not greater than the sum of the numbers of intersections between $r=\text{const.}$ and the three curves Re $F(z)=0$; $\partial/\partial\theta$ (Im $F(z)$) $=+m$; $\partial/\partial\theta$(Im $F(z)$) $=-m$. Thus a finite value of S certainly exists. Therefore we know that, for every r,

$$(1)\qquad m\leq|g_r'(\theta)|\leq M,\quad\text{and}\quad|g_r''(\theta)|\leq T,\text{ in }\bar{B}'.$$

(2) $\bar{B}'$ contains less than S separate intervals of the circumference of the circle $r=\text{constant}$.

The subscript r in $g_r(\theta)$ will be omitted from here on.

Let (σ,τ) $(\sigma\leq\theta\leq\tau)$ be one of the intervals mentioned in (2). Then $g'(\theta)$ is either not less than m throughout (σ,τ), or not greater than $-m$ throughout. Assume $g'(\theta)\geq m$ in (σ,τ). Choose a positive number n, and call $\theta_1,\theta_2,\cdots,\theta_s$ the values of θ in (σ,τ) for which cos $(ng(\theta))=0$ $(\sigma\leq\theta_1<\theta_2<\cdots<\theta_s\leq\tau)$. Assume for the present that $s\geq 2$. Then

$$(3)\qquad ng(\theta_{i+1})-ng(\theta_i)=\pi,\qquad\text{for } i=1,2,\cdots,s-1.$$

In the subinterval (θ_i,θ_{i+1}), $g(\theta)=g(\theta_i)+(\theta-\theta_i)g'(\bar{\theta})$, with $\theta_i\leq\bar{\theta}\leq\theta$. Since $g'(\bar{\theta})$ differs from $(g(\theta_{i+1})-g(\theta_i))/(\theta_{i+1}-\theta_i)$ absolutely by less than $(\theta_{i+1}-\theta_i)\cdot\max g''(\theta)$,

$$g(\theta)=g(\theta_i)+(\theta-\theta_i)\frac{g(\theta_{i+1})-g(\theta_i)}{\theta_{i+1}-\theta_i}+\kappa(\theta_{i+1}-\theta_i)^2T,\text{ with }|\kappa|<1,$$

or, using (3),

$$ng(\theta)=ng(\theta_i)+\frac{\pi(\theta-\theta_i)}{\theta_{i+1}-\theta_i}+n\kappa(\theta_{i+1}-\theta_i)^2T.$$

Since $\cos(ng(\theta_i))=0$, $ng(\theta_i)=2q\pi\mp\pi/2$ ($q=$integer). Here the upper or the lower sign applies according to whether $\cos(ng(\theta))$ is positive or negative throughout (θ_i,θ_{i+1}). Thus

$$\begin{aligned}\cos(ng(\theta)) &= \cos\left[\mp\frac{\pi}{2}+\frac{\pi(\theta-\theta_i)}{\theta_{i+1}-\theta_i}+\kappa n(\theta_{i+1}-\theta_i)^2T\right]\\ &= \pm\sin\left[\frac{\pi(\theta-\theta_i)}{\theta_{i+1}-\theta_i}+\kappa n(\theta_{i+1}-\theta_i)^2T\right]\\ &= \pm\sin\frac{\pi(\theta-\theta_i)}{\theta_{i+1}-\theta_i}+\kappa' n(\theta_{i+1}-\theta_i)^2T \text{ with } |\kappa'|\leqq|\kappa|<1.\end{aligned}$$

Therefore

$$\int_{\theta_i}^{\theta_{i+1}}\cos(ng(\theta))d\theta=\pm\frac{2}{\pi}(\theta_{i+1}-\theta_i)+\lambda_i n(\theta_{i+1}-\theta_i)^3T, \quad \text{with } |\lambda_i|<1. \tag{4}$$

For any θ-interval (γ,δ), let $I_{\gamma\delta}$ stand for the sum of the lengths of all those subintervals of (γ,δ), where $\cos(ng(\theta))$ is negative. Then $(\delta-\gamma)-I_{\gamma\delta}$ is equal to the sum of the lengths of the subintervals of (γ,δ), where $\cos(ng(\theta))$ is positive.

From (4) follows

$$\begin{aligned}\int_{\theta_1}^{\theta_s}\cos(ng(\theta))d\theta &= \frac{2}{\pi}\left[(\theta_s-\theta_1-I_{\theta_1\theta_s})-I_{\theta_1\theta_s}\right]\\ &\quad+\lambda(s-1)nT\cdot\max(\theta_{i+1}-\theta_i)^3 \quad \text{with } |\lambda|<1.\end{aligned}$$

According to (1), $m\leqq(g(\theta_{i+1})-g(\theta_i))/(\theta_{i+1}-\theta_i)\leqq M$. Therefore, and because of (3),

$$\begin{aligned}&\frac{\pi}{nM}\leqq\theta_{i+1}-\theta_i\leqq\frac{\pi}{nm}\\ &\qquad\text{for } i=1,2,\cdots,s-1, \text{ and } s-1\leqq\frac{\tau-\sigma}{\pi/nM}<2nM.\end{aligned} \tag{5}$$

Similarly, since $ng(\theta_1)-ng(\sigma)<\pi$, and $ng(\tau)-ng(\theta_s)<\pi$:

$$\theta_1-\sigma<\frac{\pi}{nm}, \quad \text{and} \quad \tau-\theta_s<\frac{\pi}{nm}. \tag{5'}$$

Thus $|\lambda(s-1)nT\max(\theta_{i+1}-\theta_i)^3|<2nMnT\pi^3/(nm)^3=2\pi^3MT/(nm^3)$, and

$$\int_{\theta_1}^{\theta_s} \cos (ng(\theta))d\theta = \frac{4}{\pi}\left[\frac{\theta_s - \theta_1}{2} - I_{\theta_1\theta_s}\right] + \lambda' \frac{2\pi^3MT}{nm^3},$$

with $|\lambda'| < 1$.

Therefore $(4/\pi)[(\tau-\sigma)/2 - I_{\sigma\tau}]$ differs from $\int_{\theta_1}^{\theta_s} \cos (ng(\theta))d\theta$ absolutely by less than $(2/\pi)(\theta_1 - \sigma + \tau - \theta_s) + 2\pi^3MT/(nm^3)$, or finally by (5′)

$$\left|\int_{\theta_1}^{\theta_s} \cos (ng(\theta))d\theta - \frac{4}{\pi}\left[\frac{\tau - \sigma}{2} - I_{\sigma\tau}\right]\right| < \frac{C_1}{n}, \tag{6}$$

where $C_1 = 4/m + 2\pi^3MT/m^3$ is independent of r and n.

On the other hand, if $s \geqq 3$, for $i = 2, 3, \cdots, s-1$:

$$\int_{\theta_{i-1}}^{\theta_{i+1}} \cos (ng(\theta))d\theta = \int_{g(\theta_{i-1})}^{g(\theta_{i+1})} \cos (ng) \frac{d\theta}{dg} dg,$$

$$\frac{d\theta}{dg} = \left(\frac{d\theta}{dg}\right)_{\theta_i} + (g(\theta) - g(\theta_i))\left(\frac{d^2\theta}{dg^2}\right)_{\bar\theta},$$

with $\bar\theta$ between θ_i and θ.

By (1), $|d^2\theta/dg^2| = |g''(\theta)/(g'(\theta))^3| \leqq T/m^3$, and by (3), for $\theta_{i-1} \leqq \theta \leqq \theta_{i+1}$, $|g(\theta) - g(\theta_i)| \leqq \pi/n$. Therefore

$$\int_{\theta_{i-1}}^{\theta_{i+1}} \cos (ng(\theta))d\theta = \left(\frac{d\theta}{dg}\right)_{\theta_i} \int_{g(\theta_{i-1})}^{g(\theta_{i+1})} \cos (ng)dg + \mu 2 \frac{\pi^2}{n^2} \frac{T}{m^3},$$

with $|\mu| < 1$.

Since $ng(\theta_{i-1})$ and $ng(\theta_{i+1})$ differ by 2π,

$$\int_{g(\theta_{i-1})}^{g(\theta_{i+1})} \cos (ng)dg = 0, \quad \text{and} \quad \left|\int_{\theta_{i-1}}^{\theta_{i+1}} \cos (ng(\theta))d\theta\right| < \frac{2\pi^2T}{n^2m^3}.$$

Thus

$$\begin{aligned} \left|\int_{\theta_1}^{\theta_s} \cos (ng(\theta))d\theta\right| &< (\theta_s - \theta_{s-1}) + \frac{s-1}{2} \frac{2\pi^2T}{n^2m^3} \\ &< \frac{\pi}{nm} + \frac{2\pi^2MT}{nm^3} = \frac{C_2}{n}, \end{aligned} \tag{7}$$

according to (5). Formula (7) is obviously also correct if $s = 2$. The term $\theta_s - \theta_{s-1}$ may be omitted if s is odd. The constant $C_2 = \pi/m + 2\pi^2MT/m^3$ is again independent of r and n.

Combining (6) and (7) we get

$$(8)\qquad \left|\frac{\tau-\sigma}{2}-I_{\sigma\tau}\right|<\frac{C}{n},\quad\text{with}\quad C=\frac{\pi}{4}(C_1+C_2).$$

In the derivation of (8) it has been assumed that $s\geqq 2$, that there are at least two values of θ in (σ,τ) for which $\cos(ng(\theta))=0$. If there is only one such value in (σ,τ), or none at all, then by (5′)

$$\left|\frac{\tau-\sigma}{2}-I_{\sigma\tau}\right|\leqq\frac{1}{2}(\tau-\sigma)<\frac{1}{2}2\frac{\pi}{mn}<\frac{C}{n},\quad\text{since}\quad\frac{\pi}{m}<\frac{\pi}{4}C_1<C.$$

If $g'(\theta)\leqq -m$ throughout (σ,τ), the same conclusion is reached in a similar manner.

Thus (8) is correct for any interval (σ,τ).

From (2) and (8) follows: for any value of r between 0 and 1, the sum of the lengths of *all* the θ-intervals within $\overline{B}'$ where $\cos(n \operatorname{Im} F(z))$ is negative differs from half the sum of the total lengths of the θ-intervals within $\overline{B}'$ by less than SC/n. Therefore the area of the subset of $\overline{B}'$ formed by the points for which $\cos(n \operatorname{Im} F(z))$ is negative differs from $B'/2$ by less than $(SC/n)\cdot\int_0^1 r\,dr=SC/(2n)$. Take n so large that $SC/(2n)<B/12$. Then, since $B'>11B/12$, $\cos(n \operatorname{Im} F(z))$ is negative in a subset of $\overline{B}'$ (and therefore of $\overline{B}$) whose area is greater than $11B/24-B/12>B/3$. This completes the proof of the lemma.

3. Some additional remarks. Let $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ be further restricted by the condition that $|z_\nu|=1$ for $\nu=1, 2, \cdots, n$. Let α and β again be the two largest possible numbers such that $\alpha\leqq A(f)\leqq\pi-\beta$ for every $f(z)$ of this form. In this case the values of α and β are still unknown. It can be shown that $\alpha=\beta$, and that $\alpha\leqq .43$. Dr. Erdős[1] quotes Mr. Eröd as possessing an unpublished proof that $\alpha>0$.

It is possible, however, to construct polynomials $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ with $\prod_{\nu=1}^{n}|z_\nu|=1$ such that $A(f)<\epsilon_1$, and $1-\epsilon_2\leqq|z_\nu|\leqq 1+\epsilon_2$ for $\nu=1, 2, \cdots, n$. where ϵ_1 and ϵ_2 are arbitrarily small positive numbers which are independent of each other.

If $F(z)=b_1z+\cdots$ is analytic for $|z|\leqq 1$ and such that the set $B(F)$ and its complement in the unit circle are both simply connected regions, then it can be proved that there exist positive numbers α such that for every $F(z)$ of this kind $\alpha\leqq B(F)\leqq\pi-\alpha$. The largest possible value of α in this case can be shown to be not less than .141 nor greater than .283.

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[1] Paul Erdős, *Note on some elementary properties of polynomials*, Bull. Amer. Math. Soc. vol. 46 (1940) p. 954.

THE COMPLETELY MONOTONIC CHARACTER OF THE MITTAG-LEFFLER FUNCTION $E_a(-x)$

HARRY POLLARD

The Mittag-Leffler function is defined by the equation

$$E_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(ka+1)}.$$

A considerable literature is devoted to a study of the analytic character of this function. (See, for example, vol. 29 of Acta Mathematica.) Recently W. Feller communicated to me his discovery—by the methods of probability theory—that if $0 \leqq a \leqq 1$ the function $E_a(-x)$ is completely monotonic for $x \geqq 0$. This means that it can be written in the form

$$E_a(-x) = \int_0^\infty e^{-xt} dF_a(t),$$

where $F_a(t)$ is nondecreasing and bounded. In this note we shall prove this fact directly and determine the function $F_a(t)$ explicitly.

Since $E_0(-x) = 1/(1+x)$, $E_1(-x) = e^{-x}$ there is nothing to be proved in these cases. We assume then that $0 < a < 1$. By a standard representation[1]

$$E_a(-x) = \frac{1}{2\pi i a} \int_L \frac{e^{t^{1/a}}}{t+x} dt, \tag{1}$$

where L consists of three parts as follows:

C_1: *the line* $y = -(\tan \psi)x$ *from* $x = +\infty$ *to* $x = \rho$, $\rho > 0$.

C_2: *an arc of circle* $|z| = \rho \sec \psi$, $-\psi \leqq \arg z \leqq \psi$.

C_3: *the reflection of* C_1 *in the* x-axis.

We suppose $\pi > \psi/a > \pi/2$, while ρ is arbitrary but fixed.

In (1) replace $(x+t)^{-1}$ by $\int_0^\infty e^{-(x+t)u} du$. The resulting double integral converges absolutely, so that one can interchange the order of integration to obtain

$$E_a(-x) = \frac{1}{2\pi i a} \int_0^\infty e^{-xu} du \int_L e^{t^{1/a}} e^{-tu} dt.$$

It remains to compute the function

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[1] L. Bierberbach, *Lehrbuch der Funktionentheorie*, vol. 2, 1931, p. 273.

$$F'_a(u) = \frac{1}{2\pi i a}\int_L e^{t^{1/a}} e^{-tu}\,dt, \tag{2}$$

and to prove it is non-negative when $u \geqq 0$. An integration by parts in (2) yields

$$F'_a(u) = \frac{1}{2\pi i a u}\int_L e^{-tu}\left(\frac{1}{a}\,t^{1/a-1}\right) e^{t^{1/a}}\,dt.$$

Now let $tu = s^a$. Then

$$F'(u) = \frac{u^{-1-1/a}}{a}\left\{\frac{1}{2\pi i}\int_{L'} e^{-s^a} e^{su^{1/a}}\,ds\right\}, \tag{3}$$

where L' is the image of L under the mapping.

Now consider the function

$$\phi_a(t) = \frac{1}{2\pi i}\int_{L'} e^{-s^a} e^{st}\,ds.$$

This is known to be the inverse Laplace transform of

$$e^{-s^a} = \int_0^\infty e^{-st}\phi_a(t)\,dt,$$

which is completely monotonic.[2] Hence

$$F'_a(u) = \frac{u^{-1-1/a}}{a}\,\phi_a(u^{-1/a}) \geqq 0.$$

From the explicit series[2] for $\phi_a(t)$ we find also that

$$F'_a(u) = \frac{1}{\pi a}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\sin \pi a k\,\Gamma(ak+1)u^{k-1}, \tag{4}$$

so that $F'_a(u)$ is an entire function.

It is of course possible to obtain (4) directly from (3). But a proof of its non-negative character without the intervention of the function e^{-s^a} eludes me.

It follows finally that $E_a(x)$ has no real zeros when $0 \leqq a \leqq 1$.

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[2] H. Pollard, *The representation of e^{-x^λ} as a Laplace integral*, Bull. Amer. Math. Soc. vol. 52 (1946) p. 908. The contour γ of that paper differs slightly from L', but is easily deformed into it.

AN INITIAL VALUE PROBLEM FOR HYPERBOLIC DIFFERENTIAL EQUATIONS

BENHAM M. INGERSOLL

We consider the normal form of the linear partial differential equation of hyperbolic type

$$(1)\qquad L[u] \equiv u_{xy} + au_x + bu_y + cu = d.$$

It is well known that if the coefficients of (1) satisfy certain continuity conditions, a unique solution $U(x,y)$ of (1) can be determined over any rectangle with sides parallel to the coordinate axes which lies entirely within the domain of continuity of the coefficients of (1) by prescribing the solution along any two adjacent sides of the rectangle.[1] No generality will be lost by assuming that the adjacent segments lie on the coordinate axes, so that a vertex of the rectangle is on the origin.

It will be shown here that for a certain sub-class of equations of this type a unique solution is obtainable by prescribing merely two partial derivatives of $u(x, y)$, one along each of two adjacent sides of the rectangle, that is, by prescribing $\partial^k u(x, y)/\partial x^k|_{x=0}$ and $\partial^m u(x, y)/\partial y^m|_{y=0}$ where k and m are any non-negative integers, instead of $u(x, 0)$ and $u(0, y)$. This result is obtained by reducing the new problem to the classic one $(k=m=0)$.

It should also be noted that the result to be proved complements, in a certain sense, results of Bergman[2] on elliptic differential equations. If $a=\sum_{m,n}a_{mn}x^my^n$, $b=\bar{a}$, $c=\sum_{m,n}c_{mn}x^my^n$, $c_{mn}=\bar{c}_{nm}$, are entire functions of two complex variables x and y where y is conjugate to x, and we write $x=X+iY$, $y=X-iY$; then (1) becomes an equation of elliptic type in the X, Y plane. It has been shown, by means of Bergman's operator method, that in order to determine the regularity domain of a solution $U=\sum_{m,n}U_{mn}x^my^n$, $U_{mn}=\bar{U}_{nm}$, from a given subsequence $\{U_{mk}\}$, k fixed, $m=0, 1, 2, \cdots$, it is sufficient to know the subsequences $\{a_{m\nu}\}$, $\{a_{\nu m}\}$, $\{c_{m\nu}\}$, $\nu=0, 1, 2, \cdots, k$, $m=0, 1, 2, 3, \cdots$, of the coefficients in the power series expansion of a and c in (1). If, by the same change of variables, the result of the present paper is formulated as a theorem for equations of elliptic type, it becomes

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[1] See Courant and Hilbert, *Mathematische Physik*, vol. 2, pp. 315–316.

[2] See S. Bergman, *Linear operators in the theory of partial differential equations*, Trans. Amer. Math. Soc. vol. 53 (1945) pp. 130–155, in particular p. 141.

apparent that our results belong to somewhat the same range of ideas.

We first treat a simple example which will indicate the method of proof in the general case.

We wish to determine a solution $u(x, y)$ of the partial differential equation

$$\text{(A)} \qquad u_{xy} + u_x + u_y + u = 0$$

which satisfies the initial conditions

$$\text{(B)} \qquad u_{yy}(x, 0) = x, \qquad u_{xx}(0, y) = y.$$

Differentiating (A) with respect to y, then setting $y=0$ and employing the first of the conditions (B), there results the system of ordinary linear differential equations for the functions $u(x, 0)$, $u_y(x, 0)$

$$\text{(C)} \qquad \begin{aligned} &u_y'(x, 0) + u'(x, 0) + u_y(x, 0) + u(x, 0) = 0, \\ &u_y'(x, 0) \qquad\qquad\quad + u_y(x, 0) \qquad\qquad = -x-1, \end{aligned}$$

where primes denote derivatives with respect to x.

The complete integral of the system (C) is given by

$$\text{(D)} \qquad u(x, 0) = K_1 e^{-x} + x, \qquad u_y(x, 0) = K_2 e^{-x} - x.$$

If each equation of the system (C) is differentiated once, upon adjoining to (C) the new equations and setting $x=0$ in the resulting system, we obtain by employing the initial conditions (B),

$$\text{(E)} \qquad \begin{aligned} &u_y'(0, 0) + u'(0, 0) + u_y(0, 0) + u(0, 0) &&= 0, \\ &u_y'(0, 0) \qquad\qquad\quad + u_y(0, 0) &&= -1, \\ &u_y'(0, 0) + u'(0, 0) &&= -1, \\ &u_y'(0, 0) &&= -2. \end{aligned}$$

Hence $u_y(0, 0)=1$, $u(0, 0)=0$; thus $K_1=0$, $K_2=1$, and $u(x, 0)=x$, $u_y(x, 0)=e^{-x}-x$.

By symmetry, $u(0, y)=y$, $u_x(0, y)=e^{-y}-y$.

The original problem has now been reduced to one of the classical type, namely, the determination of a solution $u(x, y)$ of (A) satisfying the boundary conditions

$$\text{(F)} \qquad u(x, 0) = x, \qquad u(0, y) = y.$$

Riemann's integration method yields

$$u(x, y) = 0 + \int_0^y e^{\eta-x-y}(1+\eta)d\eta + \int_0^x e^{\xi-x-y}(1+\xi)d\xi,$$

that is, $u(x, y) = ye^{-x} + xe^{-y}$.

The following work follows the above lines.

We shall use the notation

$$\frac{d^i f(x)}{dx^i} = f_i(x); \qquad \frac{\partial^{i+j} u(x, y)}{\partial x^i \partial y^j} = u_{ij}(x, y); \qquad \frac{\partial^{i+j} L[u]}{\partial x^i \partial y^j} = L_{ij}(u).$$

By $\alpha_{ij;\xi\beta} = \alpha_{ij;\xi\beta}(x, y)$ we shall mean the coefficient of $u_{\xi\beta}(x, y)$ in $L_{ij}[u]$. The letters k and m will be thought of as fixed integers throughout the paper. If in an equation containing x and y such as (1), we wish, for example, to change (x, y) to $(3, 4)$, we shall refer to it as the equation (1; 3, 4).

Form the system of equations

$$L_{ij}[u] = d_{ij} \qquad (i = 0, \cdots, k-1; j = 0, \cdots, m-1) \tag{2}$$

and the matrix of the $u_{\xi\beta}$ occurring in (2)

$$\begin{pmatrix} u_{00} & u_{01} \cdots u_{0m} \\ u_{10} & u_{11} \cdots u_{1m} \\ \cdot \; \cdot \; \cdot & \cdot \; \cdot \; \cdot \; \cdot \\ u_{k0} & u_{k1} \cdots u_{km} \end{pmatrix}. \tag{3}$$

In the equations (2) when the terms containing $u_{k\beta}$ and $u_{\xi m}$ $(\beta = 0, \cdots, m; \xi = 0, \cdots, k)$ are transposed to the right-hand side, there results a system of km equations in km functions u_{ij} $(i = 0, \cdots, k-1; j = 0, \cdots, m-1)$. We shall refer to these as the $(2)^*$ equations. Since the reordering of a system of equations cannot affect the absolute value of their determinant, we shall call

$$A(x, y) \tag{4}$$

the absolute value of the determinant of the system $(2)^*$ in the variables u_{ij}, $(i = 0, \cdots, k-1;\ j = 0, \cdots, m-1)$. The elements of $A(x, y)$ are a, b, c and their derivatives.

The subsets $L_{0j}[u] = d_{0j}$ $(j = 0, \cdots, m-1)$ and $L_{i0}[u] = d_{i0}$ $(i = 0, \cdots, k-1)$ of the system $(2)^*$ are fundamental in this paper, and we write them in the forms

$$\begin{aligned} \sum_t \alpha_{0j;1t} u_{1t} + \sum_t \alpha_{0j;0t} u_{0t} &= d_{0j} \\ &\qquad (0 \leqq j \leqq m-2;\ t = 0, \cdots, j+1) \\ &= d_{0j} - u_{1m} - b u_{0m} \\ &\qquad (j = m-1;\ t = 0, \cdots, m-1), \end{aligned} \tag{5}$$

$$
\begin{aligned}
\sum_{\xi} \alpha_{i0;\xi 1} u_{\xi 1} + \sum_{\xi} \alpha_{i0;\xi 0} u_{\xi 0} &= d_{i0} \\
&\qquad (0 \leqq i \leqq k-2;\ \xi = 0, \cdots, i+1) \\
&= d_{i0} - u_{k1} - a u_{k0} \\
&\qquad (i = k-1;\ \xi = 0, \cdots, k-1).
\end{aligned} \tag{6}
$$

The determinant $|\alpha_{0j;1t}|$ of the terms u_{1t} in (5) will be denoted by

$$B(x, y) = |\alpha_{0j;1t}| \qquad (j, t = 0, \cdots, m-1), \tag{7}$$

and the corresponding determinant of (6) by

$$C(x, y) = |\alpha_{i0;\xi 1}| \qquad (i, \xi = 0, \cdots, k-1). \tag{8}$$

We may now state our principal result.

THEOREM 1. *In the equation* $L[u]=d$ *let* a, b, c, d, *be of class* C^k in x and C^m in y *and let* $f(x)$, $g(y)$ *be preassigned functions of classes* C^{k+1} *and* C^{m+1} *respectively, with* $f_k(0)=g_m(0)$. *Let*

$$A(0, 0) \neq 0, \tag{9}$$

$$B(x, 0) \neq 0 \qquad (0 \leqq x < s_1), \tag{10}$$

$$C(0, y) \neq 0 \qquad (0 \leqq y < s_2). \tag{11}$$

Under these conditions there exists a unique solution $u(x, y)$ *of* $L[u]=d$ $(0 \leqq x \leqq s_1,\ 0 \leqq y < s_2)$, *such that*

$$u_{0m}(x, 0) = f(x) \qquad (0 \leqq x < s_1), \tag{12}$$

$$u_{k0}(0, y) = g(y) \qquad (0 \leqq y < s_2). \tag{13}$$

The following proof holds for $mk \neq 0$. The slight changes necessary for the case when m or k is zero will be indicated.

We shall first prove a series of lemmas. The first three of the lemmas will show that all the terms in the matrix (3; 0, 0) are determined by the assignment of its first row and last column, or its first column and first row, or its last row and last column.

LEMMA 1. *Under the conditions of Theorem* 1 *the assignment of the values of the first row and last column of* (3; 0, 0), *namely*

$$
\begin{aligned}
u_{0j}(0, 0) &= \overset{*}{u}_{0j} \qquad (j = 0, \cdots, m-1), \\
u_{im}(0, 0) &= f_i(0) \qquad (i = 0, \cdots, k)
\end{aligned} \tag{14}
$$

together with the equations (2; 0, 0) *uniquely determine all the terms in the matrix* (3; 0, 0).

In using the values (14) in (5; 0, 0), we note that only the terms in $u_{1i}(0, 0)$ $(i=0, \cdots, m-1)$ remain to be found, but since $B(0, 0) \neq 0$, we can solve uniquely for these values. On differentiating the equations (5) once with respect to x, the only way terms in $u_{2i}(x, y)$ $(i=0, \cdots, m-1)$ can occur is from differentiating the u_{1i} in the terms $\alpha_{0j;1i}u_{1i}$. This shows that the determinant $|\alpha_{1j;2i}| = B(x, y)$. Hence we may find the $u_{2i}(0, 0)$ $(i=0, \cdots, m-1)$ after having determined the $u_{1i}(0, 0)$. Owing to our continuity conditions on a, b, c, d, we may differentiate (5) at least $k-1$ times with respect to x. Hence the lemma is seen to hold.

Using the equations (6) as we did the equations (5), we may prove the following lemma.

LEMMA 2. *Under the conditions of Theorem* 1 *the assignment of the first column and last row of* (3; 0, 0), *namely*

$$\begin{aligned} u_{i0}(0, 0) &= u_{i0}^{*} && (i = 0, \cdots, k-1), \\ u_{kj}(0, 0) &= g_j(0) && (j = 0, \cdots, m) \end{aligned} \tag{15}$$

together with the equations (2; 0, 0) *uniquely determine all the terms in the matrix* (3; 0, 0).

LEMMA 3. *If the conditions of Theorem* 1 *hold the assignment of values to the last row and last column*

$$\begin{aligned} u_{kj}(0, 0) &= g_j(0) && (j = 0, \cdots, m-1), \\ u_{im}(0, 0) &= f_i(0) && (i = 0, \cdots, k-1), \\ u_{km}(0, 0) &= f_k(0) = g_m(0) && \end{aligned} \tag{16}$$

together with the system (2; 0, 0) *uniquely determine all members of the matrix* (3; 0, 0).

This follows from the facts that (16) determine all the right members of the system (2; 0, 0)* and the absolute value $A(0, 0)$ of the determinant of this system of algebraic equations is not zero.

Since the functions $f(x)$ and $g(y)$ are assumed to be known, we may assign the values in (16); the resulting values to the elements of the matrix (3; 0, 0) will be denoted by

$$u_{ij}(0, 0) = F_{ij}(f, g). \tag{17}$$

In the equations (5; x, 0) write

$$u_{0i}(x, 0) = z^{i}(x), \quad u_{1i}(x, 0) = \frac{d}{dx} z^{i}(x) \qquad (0 \leqq i \leqq m-1), \tag{18}$$

$$u_{0m}(x, 0) = f(x), \quad u_{1m}(x, 0) = f_1(x). \tag{18'}$$

This gives us the system of ordinary differential equations

$$\sum_t \alpha_{0j,1t}(x,0)\frac{dz^t}{dx} + \sum_t \alpha_{0j,0t}(x,0)z^t = d_{0j}(x,0)$$

$$(0 \leqq j \leqq m-2;\ t = 0, \cdots, j+1) \tag{5'}$$

$$= d_{0j}(x,0) - f_1(x) - b(x,0)f(x)$$

$$(j = m-1;\ t = 0, \cdots, m-1).$$

We assign as initial values to $z^0, \cdots, z^{m-1}$,

$$z^i(0) = F_{0i}(f, g) \qquad (i = 0, \cdots, m-1). \tag{19}$$

Since $|\alpha_{0j,1t}(x,0)| = B(x,0) \neq 0$ for $0 \leqq x < s_1$, and because of the continuity conditions satisfied by the coefficients of (1), the system (5′) is seen to have a unique solution[3] of class C^k. Since the system (2; 0, 0) can be formed as follows

$$\left\{\frac{d}{dx^i}\left[L_{0j}(u)\Big|_{y=0}\right]\right\}_{x=0} = \left\{\frac{d}{dx^i}\left[d_{0j}(x,y)\Big|_{y=0}\right]\right\}_{x=0},$$

we see on adding to the system (5′) all equations obtained by differentiating it 1, 2, $\cdots$, $k-1$ times with respect to x, and then putting $x=0$, we have the system (2; 0, 0). In this system (18′) assures

$$u_{im}(0,0) = f_i(0) \qquad (i = 0, \cdots, k). \tag{20}$$

Thus from (17), (19), (20), and Lemma 1 we have:

LEMMA 4. *Under the conditions of Theorem* 1 *the unique solution of equations* (5′) *with initial conditions* (19) *satisfies the conditions*

$$\left.\frac{d^i z^j(x)}{dx^i}\right|_{x=0} = F_{ij}(f, g) \qquad (i = 0, \cdots, k;\ j = 0, \cdots, m-1). \tag{21}$$

In the equations (6; 0, y) we make the replacements

$$u_{i0}(0,y) = w^i(y), \quad u_{i1}(0,y) = \frac{dw^i}{dy} \qquad (0 \leqq i \leqq k-1), \tag{22}$$

$$u_{k0}(0,y) = g(y), \quad u_{k1}(0,y) = g_1(y). \tag{22'}$$

Calling the resulting system of ordinary differential equations (6′), we assign initial values

$$w^i(0) = F_{i0}(f, g) \qquad (i = 0, \cdots, k-1). \tag{23}$$

We may treat (6′) as we treated (5′) and get the lemma.

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[3] See E. L. Ince, *Ordinary differential equations*, p. 72.

LEMMA 5. *Assuming conditions of Theorem* 1 *hold, the unique solution of system* (6′) *with initial conditions* (23) *satisfies the conditions*

$$\left.\frac{d^i w^j(y)}{dy}\right|_{y=0} = F_{ji}(f, g) \qquad (i = 0, \cdots, m;\ j = 0, \cdots, k-1). \tag{24}$$

We now return to equation (1) and solve it by the classical theory for the unique solution with initial conditions

$$u(x, 0) = z^0(x), \qquad 0 \leqq x < s_1, \tag{25}$$

$$u(0, y) = w^0(y), \qquad 0 \leqq y < s_2, \tag{26}$$

The solution of $L[u]=d$ for $0\leqq x<s_1$, $0\leqq y<s_2$ so obtained we shall denote by $U(x, y)$. Theorem 1 will be complete if we can show

$$U_{0m}(x, 0) = f(x), \qquad U_{k0}(0, y) = g(y), \tag{27}$$

and U is the only such solution of $L[u]=d$.

$U(x, y)$ is unique, for Lemma 3 implies that $f(x)$ and $g(y)$, where $f_k(0)=g_m(0)$, together with (1) determine uniquely (3: 0, 0). Accordingly the other lemmas guarantee the uniqueness of the $z^j(x)$ and $w^i(y)$ $(j=0, \cdots, m-1; i=0, \cdots, k-1)$, in particular that of $z^0(x)$ and $w^0(y)$, and thus that of U.

If from the equations (5′), say, we wish to consider a single one, for example, the one obtained by setting $j=7$, we shall designate it by (5′) $(j=7)$, and shall use corresponding notation for other equations picked from sets.

In equations (5′) make the replacements

$$z^{j+1} = p^{j+1}, \qquad \frac{dz^{j+1}}{dx} = \frac{dp^{j+1}}{dx}, \tag{28}$$

$$f(x) = p^m, \qquad f_1(x) = \frac{dp^m}{dx}. \tag{28′}$$

This yields m differential equations

$$\begin{aligned} &\alpha_{0j;1,j+1}(x, 0)\frac{dp^{j+1}}{dx} + \alpha_{0j;0,j+1}(x, 0)p^{j+1} \\ &\quad = d_{0j}(x, 0) - \sum_i \alpha_{0j;1i}(x, 0)\frac{dz^i(x)}{dx} - \sum_i \alpha_{0j;0i}(x, 0)z^i(x) \\ &\qquad (0 \leqq j \leqq m-1;\ i = 0, \cdots, j) \end{aligned} \tag{5′′}$$

[(5′′) $(j=m-1)$ need not be written separately since $\alpha_{0j,1,j+1}(x, 0)=1$ and $\alpha_{0j;0,j+1}(x, 0)=b(x, 0)$].

Now by (5; x, 0) ($j=0$) and (25), $U_{01}(x, 0)$ is a solution of (5'') ($j=0$). By (5') ($j=0$), $z^1(x)$ is also a solution of (5'') ($j=0$). (26) and Lemma 5 imply that $U_{01}(x, 0)=dw^0(y)/dy|_{y=0}=F_{01}(f, g)$. But by (19), $z^1(0)=F_{01}(f, g)$. Hence $U_{01}(x, 0)=z^1(x)$. If we use in natural order the remaining equations of the set (5''), a similar argument will establish that $U_{0j}(x, 0)=z^j(x)$, $j=1, \cdots, m-1$, the only difference in the discussion being that the identities $U_{0k}(x, 0)=z^k(x)$, $k=1, \cdots, j-1$, must be used together with (25). (5') ($j=m-1$) shows that $f(x)$ is a solution of (5'') ($j=m-1$), and (5; x, 0) ($j=m-1$) together with $U_{0j}(x, 0)=z^j(x)$, $j=0, \cdots, m-1$, show that $U_{0m}(x, 0)$ is also a solution. However, from (26), $U_{0m}(0, 0)=d^m w^0(y)/dy^m|_{y=0}$. But by prescription, that is, by (18'), $f(0)=F_{0m}(f, g)$, and hence, by Lemma 5, $f(0)=d^m w^0(y)/dy^m|_{y=0}$. Therefore $U_{0m}(x, 0)=f(x)$. It can be shown in a similar way that $U_{k0}(0, y)=g(y)$.

The slight change necessary in the case $mk=0$ is as follows. Say, for example, that $k=0$. An entity arising in much the same way and playing the same role as an entity (n) in the previous work will be denoted by ($\tilde{n}$).

Form the system

$$L_{0j}[u] = d_{0j} \qquad (j = 0, \cdots, m-1) \tag{$\tilde{2}$}$$

and the matrix of the u_{kl} occurring in ($\tilde{2}$)

$$\begin{pmatrix} u_{00} & u_{01} \cdots & u_{0m} \\ u_{10} & u_{11} \cdots & u_{1m} \end{pmatrix}. \tag{$\tilde{3}$}$$

($\tilde{2}$)* will be formed by transposing to the right-hand side in the system ($\tilde{2}$) those terms containing u_{1m} and u_{0m}. ($\tilde{5}$) will then be the same as (5). The $z^i(x)$, $i=0, \cdots, m-1$, will be obtainable as before, ($\tilde{5}$) playing the roles of both (2)* and (5). Lemmas 2, 3, and 5 will not be used. Solving (1) by the classical theory for the unique solution with initial conditions

$$u(x, 0) = z^0(x), \tag{$\widetilde{25}$}$$

$$u(0, y) = g(y), \tag{$\widetilde{26}$}$$

the rest of the proof will go through as before with Lemma 1 leading to uniqueness and $g(y)$ taking the place of $w^0(y)$.

CITY COLLEGE OF NEW YORK

NOTE ON A THEOREM DUE TO BORSUK

J. H. C. WHITEHEAD

1. **Introduction.** Let A, $B\subset A$ and B' be compacta, which are[1] ANR's (absolute neighbourhood retracts). Let $B'\subset A'$ where A' is a compactum, and let $f:(A, B)\rightarrow(A', B')$ be a map such that $f|(A-B)$ is a homeomorphism onto $A'-B'$. Thus A' is homeomorphic to the space defined in terms of A, B, B' and the map $g=f|B$ by identifying each point $b\in B$ with $gb\in B'$. K. Borsuk [3] has shown that A' is locally contractible. It is therefore an ANR if $\dim A'<\infty$. The main purpose of this note is to prove, without this restriction on $\dim A'$:

THEOREM 1. *A' is an ANR.*

We also derive some simple consequences of this theorem. For example, it follows that the homotopy extension theorem, in the form in which the image space is arbitrary, may be extended[2] from maps of polyhedra to maps of compact ANR's, P and $Q\subset P$. That is to say, if $f_0:P\rightarrow X$ is a given map, the space X being arbitrary, and if $g_t:Q\rightarrow X$ is a deformation of $g_0=f_0|Q$, then there is a homotopy $f_t:P\rightarrow X$, such that $f_t|Q=g_t$. For let $R=(P\times 0)\cup(Q\times I)\subset P\times I$ and let $h:R\rightarrow X$ be given by $h(p, 0)=f_0p$, $h(q, t)=g_tq$ $(p\in P,\ q\in Q)$. Since $Q\times I$ is (obviously) a compact ANR it follows from Theorem 1, with $A=Q\times I$, $B=Q\times 0$, $B'=P\times 0$, $A'=R$ that R is an ANR. Therefore R is a retract of some open set $U\subset P\times I$. If $\theta:U\rightarrow R$ is a retraction, then $h\theta:U\rightarrow X$ is an extension of $h:R\rightarrow X$ throughout U. This is all we need for the homotopy extension theorem (see [5, pp. 86, 87]). Thus we have the corollary:

COROLLARY. *A given homotopy, $g_t:Q\rightarrow X$, of $g_0=f_0|Q$, can be extended to a homotopy, $f_t:P\rightarrow X$, where P and $Q\subset P$ are compact ANR's and $f_0:P\rightarrow X$ is a given map of P in an arbitrary space X.*

We also use Theorem 1 to prove another theorem. We shall describe a map $\xi:X\rightarrow Y$ as a *homotopy equivalence* if, and only if, there is a map, $\eta:Y\rightarrow X$, such that $\eta\xi\simeq 1$, $\xi\eta\simeq 1$, where X and Y are any two spaces. Thus the statement that $\xi:X\rightarrow Y$ is a homotopy equivalence implies that X and Y are of the same homotopy type. Let A, B, A', B' and $f:(A, B)\rightarrow(A, B')$ be as in Theorem 1 and let $g=f|B$.

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[1] For an account of these spaces, on which this note is based, see [2]. Numbers in brackets refer to the references cited at the end of the paper.

[2] Cf. [4].

Then we shall prove:

THEOREM 2. *If* $g:B\to B'$ *is a homotopy equivalence so is* $f:A\to A'$.

For example B' may consist of a single point, in which case we describe the identification of B with B' as the operation of *shrinking* B *into a point*. Then it follows from Theorem 2 that any (compact) absolute retract, $B\subset A$, may be shrunk into a point, without altering the homotopy type of A. As another example let A and B' be cell complexes[3] and B a sub-complex of A. Then A' is also a cell complex, subject to suitable conditions on the map[4] $g=f|B$, and Theorem 2 shows that certain combinatorial operations do not alter the homotopy type of A. For example, if B is the n-section of A and if B' is any complex, of at most n dimensions, which is of the same homotopy type as B, then there is a complex, A', of the same homotopy type as B, whose n-section is B'.

2. **Another theorem.** We prove Theorem 1 by means of another theorem. Let X and $Y\subset X$ be compacta and let Y be an ANR. Given $\rho>0$ let $V_\rho\subset X$ be the subset consisting of points whose distances from Y are less than ρ. We assume that

(a) *given* $\epsilon>0$ *there is a* $\rho(\epsilon)>0$ *and an* ϵ*-homotopy*, $\theta_t:X\to X$, *such that* $\theta_0=1$, $\theta_t|Y=1$, $\theta_1 V_{\rho(\epsilon)}=Y$,

(b) *given* ϵ, $\rho>0$ *there is a* $u(\epsilon,\rho)>0$ *such that any partial realization*, $g:L\to X-V_\rho$, *whose mesh does not exceed* $u(\epsilon,\rho)$, *of a finite simplicial complex*, K, *can be extended to a full realization*, $f:K\to X$, *whose mesh does not exceed* ϵ, $u(\epsilon,\rho)$ *being independent of* K *and* L.

Then we prove:

THEOREM 3. *Subject to these conditions* X *is an ANR.*

For this we shall need a sharpened form of the homotopy extension theorem. Let P and $Q\subset P$ be compacta and let $f_0:P\to M$ be a given map of P in a metric space M. Let $g_t:Q\to M$ be an ϵ-deformation of $g_0=f_0|Q$. Assume that either

(1) M *is a (separable) ANR or that*

(2) P *is a finite polyhedron and* Q *a sub-polyhedron.*

Then we have:

LEMMA 1. *Given* $\epsilon'>0$ *there is an* $(\epsilon+\epsilon')$*-deformation*, $f_t:P\to M$, *such that* $f_t|Q=g_t$.

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[3] That is, a complex of the sort defined in [6], and in a forthcoming book by S. Eilenberg and N. E. Steenrod.

[4] For example, $gB^n\subset B'^n$ for each $n=0, 1, \cdots$, where K^n denotes the n-section of a complex, K, or $A^n\subset B$, $gB\subset B'^n$ for a particular value of n.

By way of proof it is sufficient to add a few comments to a standard proof of the homotopy extension. (See [5, pp. 86, 87].) Let $R=(P\times 0)\cup(Q\times I)\subset P\times I$ and let $h:R\to M$ be given by $h(p, 0)=f_0 p$, $h(q, t)=g_t q$ ($p\in P$, $q\in Q$). If P is a polyhedron and Q a sub-polyhedron, then R is a polyhedron and hence a neighbourhood retract of $P\times I$ (in fact R is a deformation retract of $P\times I$). Therefore h can be extended throughout some neighbourhood, $U\subset P\times I$, of R, as it can be if M is an ANR and P, Q arbitrary compacta. There is a neighbourhood, $V\subset P$, of Q such that $V\times I\subset U$. Since Q is compact we may take V to be the neighbourhood given by $\delta(p, Q)<\rho$, for some $\rho>0$, where $\delta(p, p')$ is a distance function in P. On following the argument given by Hurewicz and Wallman [5, pp. 86, 87] it is easily seen that the extension $f_t:P\to M$ is an $(\epsilon+\epsilon')$-deformation provided ρ is sufficiently small.

We now proceed to prove Theorem 3 by showing that X is LC^*, as defined by Lefschetz.[8] Given $\epsilon>0$ let $\eta'=\eta(\epsilon/2)/4$, $\rho'=\rho(\eta')/2$, where η is an extension function[8] for Y and $\rho(\eta')$ means the same as in the condition (a). Let

$$\xi_1(\epsilon) = \min\ (2\eta', \rho').$$

We shall prove that

$$\xi(\epsilon) = \kappa\{\xi_1(\epsilon), \rho'\}$$

is an extension function for X. Let K be a finite simplicial complex and $L\subset K$ a sub-complex, which contains all the vertices of K. Let $g:L\to X$ be a partial relization of K, whose mesh does not exceed $\xi(\epsilon)$. We first assume that $s\subset L$ if $g(s\cap L)\subset X-V_{\rho'}$, where s is the closure of any simplex in K. Let $K_1\subset K$ be the sub-complex consisting of all the (closed) simplexes, $s\in K$, such that $g(s\cap L)$ meets $V_{\rho'}$. Then $K=K_1\cup L$. Let $L_1=K_1\cap L$, $g_1=g|L_1$. Then it is sufficient to prove that g_1 can be extended to a full realization, $f_1:K_1\to X$, whose mesh does not exceed ϵ. For since $K_1\cap L=L_1$, $f_1|L_1=g|L_1$, the desired realization, $f:K\to X$, will be given by $f|L=g$, $f|K_1=f_1$. Clearly $\xi(\epsilon)\leqq\xi_1(\epsilon)$ and we shall prove this special case on the weaker assumption that the mesh of $g:L\to X$ does not exceed $\xi_1(\epsilon)$.

Since $\xi_1(\epsilon)\leqq\rho'$ we have $g_1 L_1\subset V_{2\rho'}=V_\rho$ where $\rho=\rho(\eta')$. Let $\theta_t:X\to X$ be the η'-deformation associated with V_ρ as in condition (a). Since $K_1^0\subset K^0\subset L$, $K_1^0\subset K_1$, we have $K_1^0\subset L_1$. Also $\theta_1 V_\rho\subset Y$. Therefore $\theta_1 g_1:L_1\to Y$ is a partial realization of K_1 in Y, whose mesh does not exceed

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[8] [2, pp. 82, 83, 84] (N.B. $K^0\subset L$).

$$\xi_1(\epsilon) + 2\eta' \leqq 2^{-1}\eta(\epsilon/2) + 2^{-1}\eta(\epsilon/2) = \eta(\epsilon/2).$$

Therefore $\theta_1 g_1: L_1 \rightarrow Y$ can be extended to a full realization, $f_0: K_1 \rightarrow Y$, whose mesh does not exceed $\epsilon/2$. By Lemma 1 there is an $(\eta'+\epsilon/8)$-homotopy, $f_t: K_1 \rightarrow X$, such that $f_t | L_1 = \theta_{1-t} g_1$. Clearly $\eta(\epsilon/2) \leqq \epsilon/2$, whence $\eta' + \epsilon/8 \leqq \epsilon/8 + \epsilon/8 = \epsilon/4$. Therefore the mesh of $f_1: K_1 \rightarrow X$ does not exceed $\epsilon/2 + 2(\eta' + \epsilon/8) \leqq \epsilon$ and $f_1 | L_1 = \theta_0 g_1 = g_1$. Therefore this special case is established.

In general let $K_0 \subset K$ be the sub-complex consisting of all the closed simplexes, $s \in K$, such that $g(s \cap L) \subset X - V_{\rho'}$. Let $L_0 = K_0 \cap L$. Then $g | L_0$ is a partial realization of K_0, whose mesh does not exceed $\xi(\epsilon) = u\{\xi_1(\epsilon), \rho'\}$. By condition (b) it can be extended to a full realization, $f_0: K_0 \rightarrow X$, of mesh at most $\xi_1(\epsilon)$. Since $K_0 \cap L = L_0$, $f_0 | L_0 = g | L_0$, a map, $g': K_0 \cup L \rightarrow X$, is defined by $g' | K_0 = f_0$, $g' | L = g$ and its mesh does not exceed $\xi_1(\epsilon)$. Therefore L may be replaced by $K_0 \cup L$ and the theorem follows from what we have already proved.

3. **Proof of Theorem** 1. We shall prove Theorem 1 by showing that the conditions (a) and (b) in §2 are satisfied when $X = A'$, $Y = B'$. Let $\delta(a_1, a_2)$ be a distance function in A and let $\epsilon > 0$ be given. Since A is compact there is a $\lambda(\epsilon) > 0$ such that $\delta'(fa_1, fa_2) < \epsilon$ provided $\delta(a_1, a_2) < \lambda(\epsilon)$, where $\delta'(a_1', a_2')$ is a distance function in A'. Let $U_\gamma \subset A$ be the neighbourhood of B which consists of all points, $a \in A$, such that $\delta(a, B) < \gamma$. As shown by Borsuk [3], there is a homotopy, $\phi_t: \overline{U}_\gamma \rightarrow A$, such that $\phi_0 = 1$, $\phi_t | B = 1$, $\phi_1 \overline{U}_\gamma = B$ for some $\gamma > 0$. By uniform continuity there is a $\mu > 0$ $(\mu \leqq \gamma)$ such that $\delta(\phi_t a, b) = \delta(\phi_t a, \phi_t b) < \lambda(\epsilon)/4$ if $\delta(a, b) \leqq \mu$. Hence $\phi_t | \overline{U}_\mu$ is a $\lambda(\epsilon)/2$-deformation. By Lemma 1, $\phi_t | \overline{U}_\mu$ can be extended to a $\lambda(\epsilon)$-deformation $\psi_t: A \rightarrow A$ $(\psi_0 = 1)$. Let $\theta_t: A' \rightarrow A'$ be given by $\theta_t | B' = 1$, $\theta_t | fA = f\psi_t f^{-1} | fA$. Since $f^{-1} | (A' - B')$ is single-valued and since $f^{-1}B' = B$ and $\psi_t | B = 1$ it follows that θ_t is single-valued. It is therefore continuous.[8] Since $\theta_t | B' = 1$ and the diameter of the trajectory, $\psi_t a$, of any point $a \in A$ is less than $\lambda(\epsilon)$ it follows that θ_t is an ϵ-deformation. Also $\theta_1(fU_\mu) = f\psi_1 U_\mu = fB \subset B'$. Therefore $\theta_1(B' \cup fU_\mu) = B'$. Since $f | (A - B)$ is a homeomorphism onto $A' - B'$ and $fB \subset B'$ it follows that $B' \cup fU_\mu$ is an open subset of A'. For

$$\begin{aligned} f(A - U_\mu) &= f\{(A - B) - (U_\mu - B)\} \\ &= A' - B' - f(U_\mu - B) \\ &= A' - (B' \cup fU_\mu). \end{aligned}$$

But $A - U_\mu$ is compact, whence $f(A - U_\mu)$ is closed and $B' \cup fU_\mu$ open.

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[8] See §5 below.

Therefore there is a $\rho(\epsilon)>0$ such that $V_{\rho(\epsilon)}\subset B'\cup fU_{\kappa}$, whence $\theta_1 V_{\rho(\epsilon)} = B'$. This establishes (a).

Let $\alpha(\epsilon')$ be an extension function for A. Since $f^{-1}|(A'-B')$ is a homeomorphism and $A'-V_\rho$ is a compact subset of $A'-B'$, for a given $\rho>0$, there is a $u(\epsilon, \rho)>0$ such that, if $\delta'(a', a'')<u(\epsilon, \rho)$ $(a', a''\subset A'-V_\rho)$, then $\delta(f^{-1}a', f^{-1}a'')<\alpha\{\lambda(\epsilon)\}$. If $\psi:L\to A'-V_\rho$ is a partial realization, of mesh at most $u(\epsilon, \rho)$, of a complex K, it follows that $f^{-1}\psi:L\to A$ is of mesh at most $\alpha\{\lambda(\epsilon)\}$. The latter can therefore be extended to a full realization, $\phi:K\to A$, of mesh at most $\lambda(\epsilon)$. Then $\phi'=f\phi:K\to A'$ is a realization of K, whose mesh does not exceed ϵ. Moreover $f\phi|L=ff^{-1}\psi=\psi$. Therefore (b) is satisfied and Theorem 1 is established.

4. **Proof of Theorem 2.** We first prove a lemma. Let X, Y be topological spaces[7]: let $X_0\subset X$, $Y_0\subset Y$ be closed subsets and let $\phi:(X, X_0)\to(Y, Y_0)$ be a map such that $\phi|X-X_0$ is a homeomorphism onto $Y-Y_0$. Moreover let the topology of Y be such that a subset $F\subset Y$ is closed if, and only if, $F\cap Y_0$ and $\phi^{-1}F$ are both closed.

LEMMA 2. *If X_0 is a deformation retract*[8] *of X, then Y_0 is a deformation retract of Y.*

After replacing X by a homeomorph, if necessary, we assume that it has no point in common with Y_0 and we unite X, Y_0 in the space, $Q=X\cup Y_0$, of which X and Y_0, each with its own topology, are closed subspaces. Then Y has the identification topology[6] determined by the map $\psi:Q\to Y$, where $\psi|X=\phi$, $\psi|Y_0=1$. Let $\xi_t:X\to X$ be a homotopy such that $\xi_0=1$, $\xi_t|X_0=1$, $\xi_1X=X_0$ and let ξ_t be extended throughout Q by taking $\xi_t|Y_0=1$. Let $\eta_t=\psi\xi_t\psi^{-1}:Y\to Y$. Clearly $\psi^{-1}|Y-Y_0$ is single-valued. Also $\psi^{-1}Y_0=X_0\cup Y_0$. Since $\xi_t|X_0\cup Y_0=1$ it follows that η_t is single-valued and hence continuous.[5] Obviously $\eta_0=1$, $\eta_t|Y_0=1$, $\eta_1Y=Y_0$, which establishes the lemma.

Notice that the topology of Y certainly satisfies the above condition if X is compact (that is, bi-compact) and if Y is a Hausdorff space. For let this be so and let $F\subset Y$ be such that $\phi^{-1}F$ and $F\cap Y_0$ are both closed. Then $\phi^{-1}F$ is compact, whence $\phi\phi^{-1}F$ is also compact and hence closed. But $\phi\phi^{-1}F=F\cap\phi X$ and

$$F = (F\cap\phi X)\cup(F\cap Y_0),$$

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[7] We do not need to assume that X and Y satisfy any separation axioms.

[8] Following Lefschetz [1, p. 40] we do not admit that X_0 is a deformation retract of X unless there is a retracting deformation throughout which each point of X_0 is held fixed (see [7]).

whence F is closed. The converse follows from the continuity of ϕ and the fact that Y_0 is closed.

We now turn to Theorem 2. We recall that $f:(A, B)\to(A', B')$ is such that $f|(A-B)$ is a homeomorphism onto $A'-B'$ and $g=f|B$ is a homotopy equivalence. Replacing A, A' by homeomorphs, if necessary, we assume that no two of the spaces A, A', $A\times I$ have a point in common. We form the mapping cylinder, Γ, of the map f by identifying $(a, 0)\in A\times I$ with a and $(a, 1)$ with $fa\in A'$ for each[9] $a\in A$. The theorem will follow when we have proved that A is a deformation retract[10] of Γ.

Let $C=(A\times 0)\cup(B\times I)$. Then C is an ANR, as shown in §1. Let $\delta_s:A\times I\to A\times I$ be the retracting deformation of $A\times I$ onto $A\times 0$, which is given by $\delta_s(a, t)=(a, t-st)$ $(0\leqq s\leqq 1)$. Then $\delta_s C\subset C$ and it follows that C is a deformation retract[10] of $A\times I$. Let $\phi:A\times I\to\Gamma$ be the map which is given by $\phi(a, 0)=a$, $\phi(a, 1)=fa$, $\phi(a, t)=(a, t)$ if $0<t<1$. Since $fB\subset B'$ and $f|(A-B)$ is a homeomorphism onto $A'-B'$ it follows that $\phi|(A\times I)-(B\times 1)$ is a homeomorphism onto $\Gamma-B'$. Therefore $\phi|(A\times I)-C$ is a homeomorphism onto $\Gamma-(B'\cup\phi C)$. It follows from Lemma 2 that $B'\cup\phi C$ is a deformation retract of Γ. Since $g=f|B$ is a homotopy equivalence, B is a deformation retract[10] of $\Gamma_g=B'\cup\phi(B\times I)$, which is the mapping cylinder of $g:B\to B'$. A homotopy, $\eta_s:\Gamma_g\to\Gamma_g$, such that $\eta_0=1$, $\eta_s|B=1$, $\eta_1\Gamma_g=B$, can be extended throughout $B'\cup\phi C=B'\cup\phi(B\times I)\cup A$ by writing $\eta_s|A=1$. The result is a retracting deformation of $B'\cup\phi C$ onto A. Therefore A is a deformation retract of $B'\cup\phi C$ and hence of Γ, which proves the theorem.

5. **Note on identification spaces.**[11] Let $\phi:P\to X$ be a map of a space P onto a space X, which has the *identification topology* determined by ϕ. That is to say a subset $X_0\subset X$ is closed (open) if, and only if, $\phi^{-1}X_0$ is closed (open). A subset $P_0\subset P$ is said to be *saturated* (with respect to ϕ) if, and only if, $P_0=\phi^{-1}\phi P_0$. Therefore $X_0\subset X$ is closed if, and only if, it is the image under ϕ of a saturated closed set $P_0=\phi^{-1}X_0$. If P is compact and if X is a Hausdorff space then X certainly has the identification topology determined by ϕ. For in this case, if $P_0\subset P$ is closed, and hence compact, ϕP_0 is compact, and hence closed, whether P_0 is saturated or not.

Let $f:P\to Z$ be a map of P in any space Z.

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[9] The points in A and A' shall retain their individualites in Γ, so that A, $A'\subset\Gamma$.

[10] See [7, Theorems 1.4 and 3.7] and [8].

[11] Cf. [9, pp. 61 et seq.] and [10, pp. 52 et seq.]. Concerning the theorem on p. 56 of [10] and Lemma 4 below see the correction at the beginning of [11].

LEMMA 3. *If X has the identification topology determined by ϕ, then the transformation $f\phi^{-1}:X\rightarrow Z$ is continuous if it is single-valued.*

If $p\in P$, then $p\in\phi^{-1}\phi p$, whence $fp\in f\phi^{-1}\phi p$. If $f\phi^{-1}$ is single-valued it follows that $fp=f\phi^{-1}\phi p$, or that $(f\phi^{-1})\phi=f$. Therefore the lemma follows from Theorem 1 on p. 53 of [10].

Let X have the identification topology determined by $\phi:P\rightarrow X$ and let $h:P\times I\rightarrow X\times I$ be given by $h(p, t)=(\phi p, t)$ $(p\in P, 0\leqq t\leqq 1)$. Then it follows from Lemma 4 below that $X\times I$ has the identification topology determined by h. Therefore we have the following corollary to Lemma 3, with P, X, ϕ and f replaced by $P\times I$, $X\times I$, h and $f:P\times I\rightarrow Z$, where $f(p, t)=f_tp$.

COROLLARY. *If $f_t:P\rightarrow Z$ is a given homotopy in any space, Z, then $f_t\phi^{-1}:X\rightarrow Z$ is continuous if it is single-valued.*

Let $\psi:Q\rightarrow Y$ be a map of a space, Q, onto a space, Y, which has the identification topology determined by ψ and which satisfies the following condition. Each point in any saturated open set, $V\subset Q$, is contained in a saturated open set, whose closure is a compact subset of V. This condition is satisfied if, for example, Q and Y are compacta. For in this case, if $q\in V$, there is a neighbourhood, $W\subset Y$, of ψq, such that $\overline{W}\subset\psi V$. Then $\psi^{-1}W$ is a saturated open set, whose (compact) closure is contained in V. In particular the condition is satisfied if $Q=Y=I$ and $\psi=1$.

Let X, Y have the identification topologies determined by maps $\phi:P\rightarrow X$, $\psi:Q\rightarrow Y$, which are onto X and Y, and let Y satisfy the above condition. Let $h:P\times Q\rightarrow X\times Y$ be given by $h(p, q)=(\phi p, \psi q)$ $(p\in P, q\in Q)$. Then we have:

LEMMA 4. *The space $X\times Y$ has the identification topology determined by h.*

Let $W\subset P\times Q$ be an open subset, which is saturated with respect to h, and let (x_0, y_0) be an arbitrary point in hW. Then we have to prove that there are open sets $U\subset P$, $V\subset Q$, which are saturated with respect to ϕ, ψ and are such that

$$(x_0, y_0)\in\phi U\times\psi V\subset hW.$$

Let $p_0\in\phi^{-1}x_0$, $q_0\in\psi^{-1}y_0$ and let

$$(p_0\times Q)\cap W=p_0\times Q_0.$$

Then it is easily verified that Q_0 is an open subset of Q, which is saturated with respect to ψ. Therefore q_0 is contained in a saturated open set, $V\subset Q$, such that $\bar{V}$ is a compact subset of Q_0. Let U be the totality of all points, $p\in P$, such that $p\times\bar{V}\subset W$. Then $p_0\in U$ and $U\times\bar{V}\subset W$, whence

$$(x_0, y_0) \in \phi U \times \psi V = h(U \times V) \subset hW$$

and the lemma will follow when we have proved that U is a saturated, open subset of P.

If X_0, Y_0 are any subsets of X, Y we have $h^{-1}(X_0\times Y_0)=\phi^{-1}X_0\times\psi^{-1}Y_0$, whence

$$\phi^{-1}\phi U \times \bar{V} \subset \phi^{-1}\phi U \times \psi^{-1}\psi\bar{V} = h^{-1}h(U \times \bar{V}) \subset h^{-1}hW = W.$$

Therefore $\phi^{-1}\phi U\subset U$, whence $\phi^{-1}\phi U=U$, that is, U is saturated. Let p be any point in U. Then $p\times\bar{V}\subset W$ and since W is open and $\bar{V}$ is compact it is easily proved that there is an open set, $N\subset P$, such that $p\in N$ and $N\times\bar{V}\subset W$. Therefore $N\subset U$. Therefore U is open and the lemma is established.

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ON THE HOMOTOPY TYPE OF ANR'S

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1. Introduction. If X and Y are any spaces and if $f: X \to Y$ and $g: Y \to X$ are maps such that $gf \simeq 1$, then g is called a *left homotopy inverse* of f and f a *right homotopy inverse*[1] of g. In this case we shall say that Y *dominates*[2] X. If Y dominates X and Z dominates Y then it is easily verified that Z dominates X. If g is both a right and left homotopy inverse of f it is called a *homotopy inverse* of f and f will be called a *homotopy equivalence*. Thus the assertion that $f: X \to Y$ is a homotopy equivalence claims that X and Y are of the same homotopy type and, moreover, that f has a homotopy inverse.

Two maps, $f_0, f_1: X \to Y$ are said (cf. [**1**, pp. 49, 50] and [**2**, p. 344]) to be n-homotopic if, and only if, $f_0\phi \simeq f_1\phi$ for every map, $\phi: P \to X$, of every (finite) polyhedron, P, of at most n dimensions. By an *n-homotopy inverse* of a map, $f: X \to Y$, or an *n-homotopy equivalence* we mean the same as a homotopy inverse or a homotopy equivalence with homotopy replaced by n-homotopy throughout the definition.

By a CR-space we shall mean a connected compactum, which is an ANR (absolute neighborhood retract). Any CR-space, X, is dominated by a finite simplicial complex [5, Theorems 12.2, 16.2, pp. 93, 99], even if its dimensionality is infinite. We shall use ΔX to denote the minimum dimensionality of all (finite, simplicial) complexes which dominate X. Then $\Delta X \leqq \dim X$ and we may think of ΔX as a kind of "quasi-dimensionality," noticing, however, that ΔX may be less than $\dim X$, even if X is itself a finite polyhedron.

Let X, Y be CR-spaces, and let $N = \max(\Delta X, \Delta Y)$. Let $f: X \to Y$ be a given map and let $f_n: \pi_n(X) \to \pi_n(Y)$ be the homomorphism induced by f. If f is a homotopy equivalence then f_n is an isomorphism onto for each $n \geqq 1$. In §3 below we prove a sharper theorem than the converse, namely:

THEOREM 1. *If $f_n: \pi_n(X) \to \pi_n(Y)$ is an isomorphism onto for each $n = 1, \cdots, N$, then $f: X \to Y$ is a homotopy equivalence.*[3]

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[1] Cf. [1]. Numbers in brackets refer to the references cited at the end of the paper.

[2] In this case the homomorphisms $H_n(Y) \to H_n(X)$ induced by $g: Y \to X$ are all onto, likewise the induced homomorphisms $\pi_n(Y) \to \pi_n(X)$, assuming X, Y to be arcwise connected. In fact $H_n(Y)$, or $\pi_n(Y)$ ($n \geqq 2$), may be represented as the direct sum of $H_n(X)$, or $\pi_n(X)$, and the kernel of this homomorphism.

[3] If X and Y are of the same homotopy type, then each dominates the other and $\Delta X = \Delta Y$. Theorem 1 is formulated with a view to applications in which it is possible to calculate separate upper bounds for ΔX, ΔY (for example, $\dim X$, $\dim Y$).

We shall also prove:

THEOREM 2. *The map $f:X\rightarrow Y$ is an $(N-1)$-homotopy equivalence if, and only if, $f_n:\pi_n(X)\rightarrow\pi_n(Y)$ is an isomorphism onto for each $n=1, \cdots, N-1$.*

If $\Delta X=\Delta Y=0$ then it is obvious that X and Y, being connected, are both absolute retracts. Therefore any map, $X\rightarrow Y$, is a homotopy equivalence and Theorem 1, likewise Theorem 3 below, is trivial. Similarly Theorems 2 and 4 are trivial if $N\leqq 1$. Therefore we shall assume that $N\geqq 1$ in Theorems 1 and 3 and $N\geqq 2$ in Theorems 2 and 4.

Theorem 2 is significant in the theory of polyhedra or cell complexes. For the $(n-1)$-homotopy type of the n-section is a homotopy invariant of a given complex K (that is, is the same for any complex of the same homotopy type). It is equivalent to what I have previously called the n-group (see [6] and [7]) of K, but now propose to call the n-type. These statements will be proved in a later paper in which the n-type of a complex will be further discussed.

A map $f:X\rightarrow Y$ is not necessarily an m-homotopy equivalence if $f_1, \cdots, f_m$ are isomorphisms onto, where $m<N-1$. For example, let Y be a complex projective plane, let $X\subset Y$ be a 2-sphere, which is a complex line in Y, and let $f:X\rightarrow Y$ be the identity. If $g: Y\rightarrow X$ were a 2-homotopy inverse of f, then $g|X=gf:X\rightarrow X$ would be of degree $+1$ and would therefore induce the identical automorphism of $\pi_3(X)$. But $f_3\pi_3(X)=0$. Therefore it would be absurd to suppose that $g|X$ can be extended to a map $g: Y\rightarrow X$.

Theorem 1, restricted to polyhedra and weakened by replacing N by max (dim $X+1$, dim Y), is essentially a restatement of parts of Theorems 15 and 17 in [6, pp. 273 and 277]. The generalization to CR-spaces was suggested by a theorem proved by Sze-Tsen Hu in [9]. However we do not actually use Hu's theorem. Instead we follow Lefschetz's approach to the subject and eventually deduce Hu's theorem in a modified form. Of course Theorem 1 does not mean that X and Y are necessarily of the same homotopy type if $\pi_n(X)\approx\pi_n(Y)$ for all values[4] of n. The crux of the matter is not merely that $\pi_n(X)\approx\pi_n(Y)$ but that a certain set of isomorphisms, $f_n:\pi_n(X)\rightarrow\pi_n(Y)$, can be "realized geometrically" by means of a map $f:X\rightarrow Y$.

Let $\tilde{X}$ be the universal covering space of a given CR-space, X, with base point $x_0\in X$. Then a point, $\tilde{x}\in\tilde{X}$, is a homotopy class of paths, with fixed end points, joining x_0 to the point $p\tilde{x} \in X$, where p is the projection, $p:\tilde{X} \to X$, which is thus defined. Thus $\tilde{X}$ contains a base point, $\tilde{x}_0 \in \tilde{X}$, which corresponds to the constant path on x_0. If $\pi_1(X) = 1$ we identify $\tilde{X}$ with X, taking $\tilde{x} = p\tilde{x}$. Let $H_n(\tilde{X})$ be the nth homology group[5] of $\tilde{X}$. Let Y be another CR-space, let $\tilde{Y}$, $H_n(\tilde{Y})$ be similarly defined and let p also denote the projection $p:\tilde{Y} \to Y$. Then a given map $f:X \to Y$ can be "lifted" into a unique map $\tilde{f}:\tilde{X} \to \tilde{Y}$, such that $fp = p\tilde{f}$, $\tilde{f}\tilde{x}_0 = \tilde{y}_0$, where $y_0 = fx_0$ and $\tilde{y}_0$ are the base points in Y and $\tilde{Y}$. The map $\tilde{f}$ induces homomorphisms $H_n(\tilde{X}) \to H_n(\tilde{Y})$, which we shall also describe as *induced* by $f:X \to Y$. In particular Y may be a finite, ΔX-dimensional polyhedron, which dominates X, and $f:X \to Y$ a map with a left inverse $g: Y \to X$. Let $\tilde{g}: \tilde{Y} \to \tilde{X}$ be the map obtained by lifting g. Then a homotopy $gf \to 1$ may be lifted into a homotopy $\tilde{g}\tilde{f} \to u$, where $u:\tilde{X} \to \tilde{X}$ is a transformation in the covering group (that is, $pu = p$). Therefore $u^{-1}\tilde{g}$ is a left homotopy inverse of $\tilde{f}$ (likewise $\tilde{f}u^{-1}$ is a right homotopy inverse of $\tilde{g}$) and $\tilde{Y}$ dominates $\tilde{X}$. It follows that $H_n(\tilde{X}) = 0$ if $n > \dim \tilde{Y} = \dim Y = \Delta X$. Therefore, if X, Y are any two CR-spaces, $H_n(\tilde{X}) = 0$, $H_n(\tilde{Y}) = 0$ if $n > \max (\Delta X, \Delta Y)$. We shall prove:

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[4] For example $\pi_n(P^4\times S^3)\approx\pi_n(S^5\times S^2)$ for every $n\geqq 1$, where P^4 is a complex projective plane and S^r is an r-sphere. This example is due to Hsien-Chung Wang. For other examples see [10].

THEOREM 3. *If X, Y are any CR-spaces, then a map $f:X \to Y$ is a homotopy equivalence if each of the induced homomorphisms $f_1:\pi_1(X) \to \pi_1(Y)$, $H_n(\tilde{X}) \to H_n(\tilde{Y})$ $(n = 2, 3, \cdots)$ is an isomorphism onto.*

As a corollary to this we have:

COROLLARY 1. *If X, Y are simply connected CR-spaces, then a map $f:X \to Y$ is a homotopy equivalence if each of the induced homomorphisms $H_n(X) \to H_n(Y)$ $(n = 2, 3, \cdots)$ is an isomorphism onto.*

Let X be a finite cell complex.[6] Then the groups $H_n(\tilde{X})$ may be defined in terms of chain groups,[7] $C_n(\tilde{X})$, which are free $\mathfrak{R}(X)$-modules, where $\mathfrak{R}(X)$ is the group ring of $\pi_1(X)$, with integral coefficients. According to Eilenberg and Steenrod a map, $f:X \to Y$, of X into another complex Y, is said to be *cellular* if, and only if, $fX^n \subset Y^n$ for each $n = 0, 1, \cdots$, where X^n, Y^n are the n-sections of

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[5] It is to be understood that all our homology groups are defined, as in [12], in terms of singular chains with integral coefficients.

[6] That is, a complex of the sort defined on p. 1235 of [7] or in a forthcoming book by S. Eilenberg and N. E. Steenrod.

[7] Cf. [11, chap IV, §17]. The generalization from ordinary polyhedral complexes to the more general cell complexes will be described in the book by Eilenberg and Steenrod. $C_n(\tilde{X})$ is the relative homology group $H_n(\tilde{X}^n, \tilde{X}^{n-1})$, where $\tilde{X}^r$ is the r-section of $\tilde{X}$. For an account of chain mappings and chain equivalences see [4] and [12].

X, Y. A cellular map, $f: X \to Y$, determines a chain mapping, $\gamma: C_n(\tilde{X}) \to C_n(\tilde{Y})$, which is an operator homomorphism, for each $n = 0, 1, \cdots$, in the sense that $\gamma(\rho c) = (\alpha\rho)\gamma c$, where $\rho \in \mathfrak{R}(X)$, $c \in C_n(\tilde{X})$ and $\alpha: \mathfrak{R}(X) \to \mathfrak{R}(Y)$ is the homomorphism induced by $f_1: \pi_1(X) \to \pi_1(Y)$.

A chain mapping $\gamma: C(\tilde{X}) \to C(\tilde{Y})$ of the family $C(\tilde{X}) = \{C_n(\tilde{X})\}$ into the family $C(\tilde{Y}) = \{C_n(\tilde{Y})\}$ is defined in purely algebraical terms as a homomorphism, $\alpha: \pi_1(X) \to \pi_1(Y)$, together with a family of operator homomorphisms, $\gamma: C_n(\tilde{X}) \to C_n(\tilde{Y})$, such that $\partial\gamma = \gamma\partial$, where ∂ is the boundary operator. If γ is the chain mapping, which is induced by some (cellular) map $f: X \to Y$, then f will be described as a *geometrical realization* of γ. From Theorem 2 we have the corollary:

COROLLARY 2. *If a given chain equivalence*[8] $\gamma: C(\tilde{X}) \to C(\tilde{Y})$ *has a geometrical realisation*, $f: X \to Y$, *then f is a homotopy equivalence.*

This corollary shows that the problem of determining conditions for a given chain mapping to have a geometrical realization is fundamental in the homotopy theory of complexes. In a later paper we shall prove that, if X is at most 3-dimensional, then any chain mapping, $\gamma: C(\tilde{X}) \to C(\tilde{Y})$, has a geometrical realization, subject to certain conditions on $\gamma: C_0(\tilde{X}) \to C_0(\tilde{Y})$.

Let X, Y, $f: X \to Y$ be as in Theorem 2 and let $N = \max(\Delta X, \Delta Y)$. Then we prove, as a companion to Theorem 2:

THEOREM 4. *The map $f: X \to Y$ is an $(N-1)$-homotopy equivalence if*

(a) *each of the induced homomorphisms* $f_1: \pi_1(X) \to \pi_1(Y)$, $H_n(\tilde{X}) \to H_n(\tilde{Y})$ $(n = 0, \cdots, N-2)$ *is an isomorphism onto,*

(b) *the induced homomorphism* $H_{N-1}(\tilde{X}) \to H_{N-1}(\tilde{Y})$ *is onto,*

(c) $f_{N-1}: \pi_{N-1}(X) \to \pi_{N-1}(Y)$ *is an isomorphism into.*

Conversely, if $f: X \to Y$ is an $(N-1)$-homotopy equivalence, so is the lifted map, $\tilde{f}: \tilde{X} \to \tilde{Y}$, and $H_n(\tilde{X}) \to H_n(\tilde{Y})$ is an isomorphism onto for $n = 0, 1, \cdots, N-1$.

2. **A lemma on mapping cylinders.** Let A, B be any two spaces and $A_0 \subset A$, $B_0 \subset B$ any subsets of A, B. We shall say that *the pair* (B, B_0) *dominates* (A, A_0) if, and only if, there are maps, $f: (A, A_0) \to (B, B_0)$ and $g: (B, B_0) \to (A, A_0)$, such that gf is deformable into the identity by a homotopy of the form $\xi_t: (A, A_0) \to (A, A_0)$.

Let X, Y be any compacta and $f: X \to Y$ a given map. We form the topological product $X \times I$ and, replacing X by a homeomorph, if necessary, assume that no two of X, Y, $X \times I$ have a point in common.

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[8] It is to be understood that the homomorphism $f_1: \pi_1(X) \to \pi_1(Y)$ associated with a chain equivalence is an automorphism onto.

Let Z be the mapping cylinder, which is formed by identifying[9] $(x, 0) \in X \times I$ with x and $(x, 1)$ with $fx \in Y$ for each $x \in X$. Let P, Q be compacta, which dominate X, Y, respectively, and let $\lambda : X \to P$, $\lambda' : P \to X$, $\mu : Y \to Q$, $\mu' : Q \to Y$ be maps such that $\lambda'\lambda \simeq 1$, $\mu'\mu \simeq 1$. Let R be the mapping cylinder of the map $\mu f \lambda' : P \to Q$. Then our lemma is:

LEMMA 1. *The pair* (R, P) *dominates* (Z, X).

Let $\xi_t : X \to X$ and $\eta_t : Y \to Y$ be homotopies such that $\xi_0 = \lambda'\lambda$, $\xi_1 = 1$, $\eta_0 = \mu'\mu$, $\eta_1 = 1$ and let $\nu : (Z, X) \to (R, P)$ be given by

$$\begin{aligned} \nu(x, t) &= (\lambda x, 2t) && (\text{if } 0 \leqq 2t \leqq 1) \\ &= \mu f \xi_{2t-1} x && (\text{if } 1 \leqq 2t \leqq 2), \\ \nu y &= \mu y && (x \in X,\ y \in Y). \end{aligned}$$

This is single-valued, hence continuous (see [8, §5]), since $(\lambda x, 1) = (\mu f \lambda')\lambda x = \mu f \xi_0 x$, $\mu f \xi_1 x = \mu f x$. Let $\nu' : (R, P) \to (Z, X)$ be given by

$$\begin{aligned} \nu'(p, t) &= (\lambda' p, 2t) && (\text{if } 0 \leqq 2t \leqq 1) \\ &= \eta_{2-2t} f \lambda' p && (\text{if } 1 \leqq 2t \leqq 2), \\ \nu' q &= \mu' q && (p \in P,\ q \in Q). \end{aligned}$$

This is single-valued since $(\lambda' p, 1) = f\lambda' p = \eta_1 f \lambda' p$ and $\eta_0 f \lambda' p = \mu' \cdot (\mu f \lambda') p$. The map $\nu'\nu : (Z, X) \to (Z, X)$ is given by

$$\begin{aligned} \nu'\nu(x, t) &= \nu'(\lambda x, 2t) && (\text{if } 0 \leqq 2t \leqq 1) \\ &= \nu' \mu f \xi_{2t-1} x && (\text{if } 1 \leqq 2t \leqq 2), \\ \nu'\nu y &= \nu' \mu y, \end{aligned}$$

or by

$$\begin{aligned} \nu'\nu(x, t) &= (\lambda'\lambda x, 4t) && (\text{if } 0 \leqq 4t \leqq 1) \\ &= \eta_{2-4t} f \lambda'\lambda x && (\text{if } 1 \leqq 4t \leqq 2) \\ &= \mu'\mu f \xi_{2t-1} x && (\text{if } 1 \leqq 2t \leqq 1), \\ \nu'\nu y &= \mu'\mu y. \end{aligned}$$

The desired homotopy, $\zeta_s : (Z, X) \to (Z, X)$, is given by

$$\begin{aligned} \zeta_s(x, t) &= (\xi_s x, (4 - 3s)t) && (\text{if } 0 \leqq t \leqq 1/(4 - 3s)) \\ &= \eta_{2-(4-3s)t} f \xi_s x && (\text{if } 1/(4 - 3s) \leqq t \leqq (2 - s)/(4 - 3s)) \\ &= \eta_s f \xi_{p(s,t)} x && (\text{if } (2 - s)/(4 - 3s) \leqq t \leqq 1), \\ \zeta_s y &= \eta_s y, \end{aligned}$$

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[9] The points in X, Y shall retain their individualities in Z, so that $X, Y \subset Z$.

where $\rho(s, t) = \{(4-3s)t+3s-2\}/2$. It is easy to verify that ζ_s is single-valued and that $\zeta_0 = \nu'\nu$, $\zeta_1 = 1$. Moreover $\zeta_s x = \xi_s x \in X$ if $x = (x, 0) \in X$. Therefore (R, P) dominates (Z, X).

3. **Proof of Theorem** 1. Let X, Y and $f: X \to Y$ satisfy the conditions of Theorem 1 and let Z be the mapping cylinder of f. The theorem will follow from [1, Theorem 3.7, p. 45] (see also [3]), when we have proved that X is a deformation retract of Z.

Let $g_s: Z \to Z$ be the deformation which is given by $g_s | Y = 1$, $g_s(x, t) = (x, s+t-st)$ $(0 \leq s \leq 1)$. Then $g_0 = 1$, $g_1 Z = Y$ and $g_1 x = fx$ for each $x = (x, 0) \in X$. Let $k: Z \to Y$ be the map which is given[10] by $kz = g_1 z$ for each $z \in Z$. Then $g_1 = jk$, $f = ki$, where i, j are the identical maps $i: X \to Z$, $j: Y \to Z$. Let $\pi_n(X)$, $\pi_n(Z)$, $\pi_n(Z, X)$ be referred to a base point $x_0 \in X$, and $\pi_n(Y)$ to $fx_0 \in Y$ as base point. Let $i_n: \pi_n(X) \to \pi_n(Z)$, $k_n: \pi_n(Z) \to \pi_n(Y)$ be the homomorphisms induced by i, k and let $j_n: \pi_n(Y) \to \pi_n(Z)$ be the homomorphism induced by j and the segment (cf. [13] and [6, pp. 279 et seq.]) (x_0, I), which joins x_0 to fx_0. Since $kj = 1: Y \to Y$, $jk = g_1 \simeq 1: Z \to Z$ and since $g_s x_0$ travels along the segment (x_0, I) in the homotopy g_s, it follows that j_n is an isomorphism onto and that k_n is its inverse. Since $f = ki$ we have $f_n = k_n i_n$. Therefore $i_n = j_n f_n$ and i_n, like f_n, is an isomorphism onto for $n = 1, \cdots, N$.

Let $2 \leq n \leq N$ and consider the homotopy sequence

$$(3.1) \qquad \pi_n(X) \xrightarrow{1} \pi_n(Z) \xrightarrow{2} \pi_n(Z, X) \xrightarrow{3} \pi_{n-1}(X) \xrightarrow{4} \pi_{n-1}(Z),$$

in which (1) is i_n and (4) is i_{n-1}. Since (1) is onto it follows from the exactness of the sequence that (2) maps $\pi_n(Z)$ into zero. Since (4) is an isomorphism it follows that (3) is into zero and (2) is onto. Therefore $\pi_n(Z, X) = 0$ for $n = 1, \cdots, N$, where $\pi_1(Z, X) = 0$ means that[11] $i_1: \pi_1(X) \to \pi_1(Z)$ is onto. Notice that if, in addition, i_{N+1} is onto, then it follows from (3.1) that $\pi_{N+1}(Z, X) = 0$.

Let P be a finite, ΔX-dimensional simplicial complex, which dominates X, and Q a finite, ΔY-dimensional simplicial complex which dominates Y. Let R, λ, μ, η_t, etc. mean the same as in §2. Since Z is (obviously) arcwise connected and since $\pi_n(Z, X) = 0$ for $n = 1, \cdots, N \geq \dim Q$ it follows from a standard argument[12] that there is a homotopy, $\delta_t: Q \to Z$, such that $\delta_0 = \nu' | Q = j\mu'$, $\delta_1 Q \subset X$. Therefore $\delta_t \mu: Y \to Z$ is a homotopy such that $\delta_0 \mu = j\mu'\mu = j\eta_0$, $\delta_1 \mu Y \subset \delta_1 Q \subset X$. Moreover we assume, as we obviously may, that $\delta_1 \mu f x_0 = x_0$, where x_0 is the base point of $\pi_n(X)$ and $\pi_n(Z)$. Let $h_t: Y \to Z$ be the homotopy which is given by

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[10] We distinguish between maps $u: A \to B$, $v: A \to C$, where $B \subset C$, $B \neq C$, even if $ua = va$ for each $a \in A$.

[11] Since X is arcwise connected this is equivalent to the condition that any arc in Z, with its end points in X, is deformable, with its end points held fixed, into an arc in X.

[12] [14, p. 526]. This argument is recapitulated, in a slightly more general form, in §8 below (Lemma 5).

$$\begin{aligned} h_t &= j\eta_{1-2t} && (\text{if } 0 \leqq 2t \leqq 1) \\ &= \delta_{2t-1}\mu && (\text{if } 1 \leqq 2t \leqq 2). \end{aligned}$$

Then $h_0 = j$, $h_1 Y \subset X$, $h_1 x_0 = x_0$. Therefore the resultant of the homotopy $g_t: Z \to Z$, followed by $h_t k: Z \to Z$, is a deformation, $\Delta_t: Z \to Z$, such that $\Delta_0 = 1$, $\Delta_1 Z \subset X$, $\Delta_1 x_0 = x_0$.

Let $c \in \pi_1(Z)$ be the element which is represented by the track of x_0 in the homotopy Δ_t. Since i_1 is onto we have $c = i_1 a$ for some $a \in \pi_1(X)$. Let $\theta_0: (S^n, p_0) \to (Z, x_0)$ be a map representing a given element $\gamma \in \pi_n(Z) (n \geqq 2)$, where p_0 is the base point in the standard n-sphere S^n. Then $\theta_t = \Delta_t \theta_0$ is a deformation of θ_0 into the map $\theta_1 = i\theta$, where $\theta: (S^n, p_0) \to (X, x_0)$ is given by $\theta p = \Delta_1 \theta_0 p$ $(p \in S^n)$. Therefore

$$\gamma = c i_n \alpha = (i_1 a)(i_n \alpha) = i_n(a\alpha),$$

where $\alpha \in \pi_n(X)$ is the element represented by θ and $ci_n\alpha$, $a\alpha$ are the images of $i_n\alpha$, α in the automorphisms ([13] and [6, pp. 279 et seq.]) determined by c, a. Hence i_n is onto for every $n \geqq 1$. Taking $n = N+1$, it follows from (3.1) that $\pi_{N+1}(Z, X) = 0$.

It follows from an extension of a theorem due to Borsuk ([15] and [8]) that Z is an ANR. Therefore the homotopy, $\delta_t: Q \to Z$, defined above, can be extended, first throughout $P \cup Q$ by defining $\delta_t p = \nu' p \in X$ if $p \in P$, and then to a homotopy $\delta_t': R \to Z$, such that $\delta_0' = \nu'$. Since $\pi_n(Z, X) = 0$ if $1 \leqq n \leqq N+1$ and since $N+1 \geqq \dim R$, it follows from repeated applications of Lemma 5, in §8 below (cf. the proof of Lemma 6), that there is a homotopy, $\rho_t: R \to Z$, rel. P (that is, $\rho_t | P = \rho_0 | P$), such that $\rho_0 = \delta_1'$, $\rho_1 R \subset X$. The resultant of δ_t', followed by ρ_t, is a homotopy, $\phi_t: R \to Z$, rel. P, such that $\phi_0 = \nu'$, $\phi_1 R \subset X$. Then $\phi_t \nu: Z \to Z$ is a homotopy such that $\phi_0 \nu = \nu'\nu = \zeta_0$, $\phi_1 \nu Z \subset \phi_1 R \subset X$. Therefore the resultant of $\zeta_{1-t}: (Z, X) \to (Z, X)$, followed by $\phi_t \nu$, is a homotopy, $\psi_t: (Z, X) \to (Z, X)$, such that $\psi_0 = 1$, $\psi_1 Z \subset X$. It follows from [1, Theorem 1.4, p. 42], and [3] that X is a deformation retract of Z and the proof is complete.

Notice that Theorem 1 follows more directly on the alternative hypothesis that $f_n: \pi_n(X) \to \pi_n(Y)$ is an isomorphism onto for $n = 1, \cdots, m-1$ and f_m is onto, where $m = \max(\Delta X + 1, \Delta Y) = \dim R$. For in this case it follows from (3.1) that $\pi_n(Z, X) = 0$ for $n = 1, \cdots, m$ and the paragraph showing that $\pi_{N+1}(Z, X) = 0$ is unnecessary.

Let Z and $X\subset Z$ be CR-spaces and let $\pi_n(Z, X)=0$ for $n=1, \cdots, m=\max(\Delta X+1, \Delta Z)$. Then it follows from the homotopy sequence (3.1) that $i_n:\pi_n(X)\to\pi_n(Z)$ is an isomorphism onto for $n=1, \cdots, m-1$ and that i_m is onto. Therefore the identity map $i:X\to Z$ is a homotopy equivalence, whence X is a deformation retract of Z. Hence we have the modified form of Hu's generalization of Hurewicz's theorem ([**14**, Theorem IV, p. 522] and [**9**]):

Corollary.[13] *If* $\pi_n(Z, X)=0$ *for* $n=1, \cdots, \max(\Delta X+1, \Delta Z)$, *then* X *is a deformation retract of* Z.

4. **Proof of Theorem 2.** Let $f_n:\pi_n(X)\to\pi_n(Y)$ be an isomorphism onto for $n=1, \cdots, N-1$. Using the same notation as in §3, we shall prove that $i:X\to Z$ is an $(N-1)$-homotopy equivalence. Since $k:Z\to Y$ is a homotopy equivalence and $f=ki$ it will then follow that f is an $(N-1)$-homotopy equivalence. It follows from (3.1) that $\pi_n(Z, X)=0$ for $n=1, \cdots, N-1$. Therefore there is a homotopy, $\delta_t:Q\to Z$, such that $\delta_0=j\mu'$, $\delta_1 Q^{N-1}\subset X$, where Q^n is the n-section of Q. Since i_{N-1} is an isomorphism it follows from an argument which is similar to one used in proving Lemma 6, in §8 below, that $\delta_1|Q^{N-1}$ can be extended to a map $i\theta$, where θ is of the form $\theta:Q\to X$. Then $i\theta|Q^{N-1}\simeq j\mu'|Q^{N-1}$. Since θ, μ, k are of the form $k:Z\to Y$, $\mu:Y\to Q$, $\theta:Q\to X$ they have a resultant $\theta\mu k:Z\to X$. I say that $\theta\mu k$ is an $(N-1)$-homotopy inverse of $i:X\to Z$. For let K be a finite polyhedron of at most $N-1$ dimensions and let $\phi:K\to X$ be a given map. Then $\mu ki\phi$ maps K into Q and is homotopic to a map, $\phi':K\to Q$, such that $\phi' K\subset Q^{N-1}$. Since $i\theta|Q^{N-1}\simeq j\mu'|Q^{N-1}$ it follows that $i\theta\phi'\simeq j\mu'\phi'$. Therefore

$$i\theta\mu ki\phi\simeq i\theta\phi'\simeq j\mu'\phi'\simeq j\mu'\mu ki\phi.$$

Since $\mu'\mu\simeq 1$, $jk\simeq 1$ it follows that

$$i\theta\mu ki\phi\simeq jki\phi\simeq i\phi$$

and hence that $\theta\mu ki\phi\simeq\phi$, according to Lemma 6 below. Therefore $\theta\mu ki\simeq_{N-1}1$. A similar but rather simpler argument shows that $i\theta\mu k\simeq_{N-1}1$. Therefore i, and hence $f:X\to Y$, is an $(N-1)$-homotopy equivalence.

Conversely[14] let $f:X\to Y$ have an $(N-1)$-homotopy inverse,

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[13] If $\dim(Z-X)<\Delta X+1$ this is weaker than Hu's version.

[14] This is nontrivial because of questions concerning the base point. For example, let $u, u':(S^n, p_0)\to(X, x_0)$ be maps representing two given elements of $\pi_n(X)=\pi_n(X, x_0)$. Assume that $gfx_0=x_0$. Then the images of p_0 will, in general, describe circuits, which represent different elements of $\pi_1(X, x_0)$, in the homotopies $gfu\simeq u$, $gfu'\simeq u'$.

$g: Y \to X$. Let vertices $p_0 \in P^0$, $q_0 \in Q^0$ be chosen as base points for all the groups $\pi_n(P)$, $\pi_n(Q)$ and let $x_0 = \lambda' p_0$, $y_0 = \mu' q_0$ be taken as base points for $\pi_n(X)$, $\pi_n(Y)$. After suitable homotopies of λ, μ, if necessary, we assume that $\lambda x_0 = p_0$, $\mu y_0 = q_0$. We then deform f so that $f x_0 = y_0$ and finally g so that $g y_0 = x_0$. Let $g_n: \pi_n(Y) \to \pi_n(X)$ be the homomorphism induced by g and let λ_n, μ_n, λ_n', μ_n' be the homomorphisms of $\pi_n(X)$, $\pi_n(Y)$, $\pi_n(P)$, $\pi_n(Q)$, which are induced by λ, μ, λ', μ'. Since $\lambda'\lambda \simeq 1$, $\mu'\mu \simeq 1$ we have $\lambda_n'\lambda_n = \alpha_n$, $\mu_n'\mu_n = \beta_n$, where α_n, β_n are automorphisms[15] of $\pi_n(X)$, $\pi_n(Y)$. Also $gf\lambda' | P^{N-1} \simeq \lambda' | P^{N-1}$, whence $g_n f_n \lambda_n' = \alpha_n' \lambda_n'$ if $1 \leqq n \leqq N-1$, where $\alpha_n': \pi_n(X) \to \pi_n(X)$ is an automorphism. Therefore

$$\begin{aligned} g_n f_n \alpha_n &= g_n f_n \lambda_n' \lambda_n = \alpha_n' \lambda_n' \lambda_n \\ &= \alpha_n' \alpha_n, \end{aligned}$$

whence $g_n f_n = \alpha_n'$. Similarly $f_n g_n = \beta_n'$ where $\beta_n': \pi_n(Y) \to \pi_n(Y)$ is an automorphism. Hence[16] it easily follows that f_n is an isomorphism onto and the theorem is established.

5. Lemmas on homology. Let A and $B \subset A$ be any arcwise connected spaces. Then we have:

LEMMA 2. *If the injection homomorphism* $i_1: \pi_1(B) \to \pi_1(A)$ *is an isomorphism (into) then* $\pi_2(A, B)$ *is Abelian.*

Consider the homotopy sequence

$$\pi_2(A) \xrightarrow{1} \pi_2(A, B) \xrightarrow{2} \pi_1(B) \xrightarrow{3} \pi_1(A).$$

If (3) is an isomorphism, then (2) is into zero and (1) is onto. Therefore the lemma follows from the fact that $\pi_2(A)$ is Abelian.

LEMMA 3. *If* $\pi_1(A) = 1$ *then the natural homomorphism* $\pi_2(A, B) \to H_2(A, B)$ *is onto and its kernel is the commutator subgroup of* $\pi_2(A, B)$. *If also* $\pi_1(B) = 1$ *and* $H_r(A, B) = 0$ *for* $r = 1, 2, \cdots, n-1$ $(n \geqq 2)$, *then the natural homomorphism* $\pi_n(A, B) \to H_n(A, B)$ *is an isomorphism onto.*

This is an extension of a theorem due to W. Hurewicz, to which it reduces in case B is a single point. S. Eilenberg [12, p. 443] has given a proof of Hurewicz's theorem which, with minor modifications, establishes Lemma 3.

On combining Lemmas 2 and 3 we have the following lemma.

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[15] These are inner automorphisms if $n = 1$ and are due to the operators in $\pi_1(X)$, $\pi_1(Y)$ if $n > 1$.

[16] Cf. (7.4) below.

LEMMA 4. *If $\pi_1(A)=1$, $\pi_1(B)=1$, $H_n(A, B)=0$ for every value of n, then $\pi_n(A, B)=0$ for each n.*

6. **Proof of Theorem 3.** Let $f: X\to Y$ be a map which induces isomorphisms of $\pi_1(X)$ and $H_n(\tilde{X})$ $(n=0, 1, \cdots)$ onto $\pi_1(Y)$ and $H_n(\tilde{Y})$ and let the notation be the same as in §3. Let $\tilde{Z}$ be the universal covering space of Z, with $x_0\in X$ as base point. Since $f_1:\pi_1(X)\to\pi_1(Y)$ is an isomorphism onto it follows from an argument used at the beginning of §3 that $i_1:\pi_1(X)\to\pi_1(Z)$ is an isomorphism onto. Therefore we may identify $\tilde{X}$, and similarly $\tilde{Y}$, with the sub-sets of $\tilde{Z}$ which cover X and Y respectively. Then it follows from arguments similar to those at the beginning of §3, including (3.1), with X, Y, Z replaced by $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and homotopy groups replaced by homology groups, that all the relative homology groups $H_n(\tilde{Z}, \tilde{X})$ $(n=1, 2, \cdots)$ are zero. Since $\pi_1(\tilde{X})=1$, $\pi_1(\tilde{Z})=1$ it follows from Lemma 4 that $\pi_n(\tilde{Z}, \tilde{X})=0$ for each $n\geqq 1$. Therefore $\pi_n(Z, X)=0$ $(n\geqq 2)$ and $i_1:\pi_1(X)\to\pi_1(Z)$ is onto. Therefore the theorem follows from the proof of the simpler version of Theorem 1, which was mentioned towards the end of §3.

7. **Proof of Theorem 4.** It follows from the conditions (a), (b) of Theorem 4 and from (3.1), with X, Y, Z replaced by $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$ and homotopy groups replaced by homology groups, that $H_n(\tilde{Z}, \tilde{X})=0$ for $n=2, \cdots, N-1$. Therefore $\pi_n(\tilde{Z}, \tilde{X})=0$ and hence $\pi_n(Z, X)=0$ for $n=2, \cdots, N-1$. Also f_1 is onto, whence $\pi_1(Z, X)=0$, and f_{N-1} is an isomorphism. Therefore the first half of Theorem 4 follows from the proof of Theorem 2.

The second half of Theorem 4 is trivial if $N=2$, since $\tilde{X}$ and $\tilde{Y}$ are simply connected. Therefore we assume that $N\geqq 3$. Using the same notation as before, let $\tilde{P}$ be the universal covering space of P, with a base point $p_0\in P^0$. As at the end of §4, let the base points $x_0\in X$, $y_0\in Y$, $q_0\in Q^0$ be such that

$$x_0=\lambda' p_0=gy_0, \qquad y_0=\mu' q_0=fx_0, \tag{7.1}$$

and let $\tilde{x}_0$, $\tilde{p}_0$, etc., be the base points in $\tilde{X}$, $\tilde{P}$, etc. Let $\tilde{\lambda}':\tilde{P}\to\tilde{X}$ be the map which *covers* $\lambda':P\to X$, meaning that $\tilde{\lambda}'\tilde{p}_0=\tilde{x}_0$ and $\lambda' p=p\tilde{\lambda}'$, where p denotes both projections $p:\tilde{P}\to P$, $p:\tilde{X}\to X$. According to a remark in §1 the map $\tilde{\lambda}'$ has a right homotopy inverse, $\tilde{\lambda}:\tilde{X}\to\tilde{P}$.

Let $h_0, h_1:\tilde{X}\to A$ be maps of $\tilde{X}$ in any space, A, and let

$$h_0\tilde{\lambda}' \mid \tilde{P}^{N-1}\simeq h_1\tilde{\lambda}' \mid \tilde{P}^{N-1}, \tag{7.2}$$

where $\tilde{P}^n$ is the n-section of $\tilde{P}$. Let $\phi:K\to\tilde{X}$ be a given map, where

K is a polyhedron and dim $K \leqq N-1$. Then $\tilde\lambda\phi: K \rightarrow \tilde P$ is homotopic to a map, ϕ', such that $\phi' K \subset \tilde P^{N-1}$. Since $\lambda'\tilde\lambda \simeq 1$ we have $h_i\phi \simeq h_i\lambda'\tilde\lambda\phi \simeq h_i\lambda'\phi'$ $(i=0, 1)$. Since $\phi' K \subset \tilde P^{N-1}$ it follows from (7.2) that $h_0\lambda'\phi' \simeq h_1\tilde\lambda'\phi'$ and hence that $h_0\phi \simeq h_1\phi$. Therefore

$$h_0 \simeq_{N-1} h_1, \tag{7.3}$$

in consequence of (7.2).

Let $f: X \rightarrow Y$ and $g: Y \rightarrow X$ be such that $gf \simeq_{N-1} 1$, $fg \simeq_{N-1} 1$ and let $\tilde f: \tilde X \rightarrow \tilde Y$, $\tilde g: \tilde Y \rightarrow \tilde X$ be the maps which cover f, g. Since $gf \simeq_{N-1} 1$ there is a homotopy, $\theta_t: P^{N-1} \rightarrow X$, such that $\theta_0 = gf\lambda' \mid P^{N-1}$, $\theta_1 = \lambda' \mid P^{N-1}$. It follows from (7.1) that $\theta_0 p_0 = \theta_1 p_0 = x_0$. Let θ_t be lifted into the homotopy, $\tilde\theta_t: \tilde P^{N-1} \rightarrow \tilde X$, such that $\tilde\theta_0 \tilde p_0 = \tilde x_0$, $\theta_t p = p\tilde\theta_t$. It follows from (7.1) that $\tilde\theta_0 = \tilde g\tilde f\lambda' \mid \tilde P^{N-1}$ and that $\tilde\theta_1 = u\lambda' \mid \tilde P^{N-1}$, where $u: \tilde X \rightarrow \tilde X$ is a transformation in the covering group. Hence it follows from (7.2) and (7.3), with $A = \tilde X$, $h_0 = \tilde g\tilde f$, $h_1 = u$, that $\tilde g\tilde f \simeq_{N-1} u$. Similarly $\tilde f\tilde g \simeq_{N-1} v$, where $v: \tilde Y \rightarrow \tilde Y$ is in the group of covering transformations. Let $\tilde g = u^{-1}\tilde g$. Then $\tilde g\tilde f \simeq_{N-1} 1$ and

$$\begin{aligned} \tilde f\tilde g &= (v^{-1}v)\tilde f\tilde g \simeq_{N-1} (v^{-1}\tilde f\tilde g)\tilde f\tilde g \\ &= v^{-1}\tilde f(\tilde g\tilde f)\tilde g \simeq_{N-1} v^{-1}\tilde f u\tilde g \\ &= v^{-1}\tilde f\tilde g \simeq_{N-1} 1. \end{aligned} \tag{7.4}$$

Therefore $\tilde f$ is an $(N-1)$-homotopy equivalence.

Since $\lambda'\tilde\lambda \simeq 1$ it follows that any (singular) cycle in $\tilde X$ is homologous to a continuous[17] cycle. Similarly any cycle in $\tilde Y$ is homologous to a continuous cycle. Therefore the homomorphism $H_n(\tilde X) \rightarrow H_n(\tilde Y)$, which is induced by $\tilde f$, is obviously an isomorphism onto if $n \leqq N-1$. This completes the proof.

8. **Two lemmas.** Let e^n $(n \geqq 1)$ be an n-cell, which is an open subset of a Hausdorff space, A, and let $A_0 = A - e^n$. Let $\bar e^n$, the closure of e^n, be the image of an n-simplex, σ^n, in a map, $\phi: (\sigma^n, \dot\sigma^n) \rightarrow (\bar e^n, \bar e^n \cap A_0)$, such that $\phi \mid (\sigma^n - \dot\sigma^n)$ is a homeomorphism onto e^n. Let B and $B_0 \subset B$ be arcwise connected spaces such that $\pi_n(B, B_0) = 0$ and let $f_0: (A, A_0) \rightarrow (B, B_0)$ be a given map.

LEMMA 5. *There is a homotopy,* $f_t: (A, A_0) \rightarrow (B, B_0)$, rel. A_0, *such that* $f_1 A \subset A_0$.

Since $\pi_n(B, B_0) = 0$ there is a homotopy, $\psi_t: (\sigma^n, \dot\sigma^n) \rightarrow (B, B_0)$, rel.

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[17] For example, to the image under $\tilde\lambda'$ of a continuous cycle in $\tilde P$. By a continuous, n-dimensional cycle is meant the image in a map, $K^n \rightarrow \tilde X$, of a cycle carried by an n-dimensional complex K^n.

$\dot{\sigma}^n$, such that $\psi_0 p = f_0\phi p$ $(p \in \sigma^n)$, $\psi_1\sigma^n \subset B_0$. Let $g_t:\bar{e}^n \to B$ be given by $g_t = \psi_t\phi^{-1}$. Clearly $\phi^{-1}e^n$, and hence $g_t | e^n$, is single-valued. If $a \in \bar{e}^n - e^n = \bar{e}^n \cap A_0$, then $\phi^{-1}a \subset \dot{\sigma}^n$ and $g_t a = \psi_0\phi^{-1}a = f_0\phi\phi^{-1}a = f_0 a$. Therefore g_t is single-valued and hence continuous (see [8, §5]). Moreover $g_t | \bar{e}^n \cap A_0 = f_0 | \bar{e}^n \cap A_0$. Therefore the requirements of the lemma are satisfied by f_t, which is given by $f_t | A_0 = f_0$, $f_t | \bar{e}^n = g_t$.

Let A be a closed subset of a separable metric space, A', let B be a separable metric ANR and let $f_0:A \to B$ have an extension $f_0':A' \to B$. Then the homotopy, f_t, of Lemma 5 can be extended to a homotopy $f_t':A' \to B$.

Now let B be a separable, metric ANR and let the homomorphism, $i_n:\pi_n(B_0) \to \pi_n(B)$, which is induced by the identical map $i:B_0 \to B$, be an isomorphism onto for $n = 1, \cdots, m$. Then it follows from (3.1), with X, Z replaced by B_0, B, that $\pi_n(Z, X) = 0$ for $n = 1, \cdots, m$. Let $f_0, f_1:K^m \to B_0$ be maps of an m-dimensional, simplicial complex, K^m, into B_0. Then we have:

LEMMA 6. *If* $if_0 \simeq if_1$ *(in* B*) then* $f_0 \simeq f_1$ *(in* B_0*).*

Let $g_t:K^m \to B$ be a deformation of $g_0 = if_0$ into $g_1 = if_1$ and let $g:K^m \times I \to B$ be given by $g(x, t) = g_t x$ $(x \in K^m)$. Let $C_m = (K^m \times 0) \cup (K^{m-1} \times I) \cup (K^m \times 1)$. Since $\pi_n(B, B_0) = 0$ $(n = 1, \cdots, m)$ it follows from repeated applications of Lemma 5, with $\bar{e}^n = \sigma_i^{n-1} \times I$, where $\sigma_1^{n-1}, \sigma_2^{n-1}, \cdots$ are the $(n-1)$-simplexes in K^m, that g is homotopic, rel. $(K^m \times 0) \cup (K^m \times 1)$, to a map $g':K^m \times I \to B$, such that $g'C_m \subset B_0$. Let $E_i^{m+1} = \sigma_i^m \times I$ $(i = 1, 2, \cdots)$. Then $g'\dot{E}_i^{m+1} \subset B_0$ and $g' | \dot{E}_i^{m+1}$ is contractible in B. Since $i_m:\pi_m(B_0) \to \pi_m(B)$ is an isomorphism it follows that $g' | \dot{E}_i^{m+1}$ is contractible in B_0. Therefore there is a map $h_i:E^{m+1} \to B_0$ such that $h_i p = g'p$ if $p \in \dot{E}_i^{m+1}$. Let $f:K^m \times I \to B_0$ be given by $fp = g'p$ if $p \in C_m$, $fp = h_i p$ if $p \in E_i^{m+1}$. Then $f_t:K^m \to B_0$, given by $f_t x = f(x, t)$, is a homotopy of f_0 into f_1, which completes the proof.

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TOPOLOGICAL CHARACTERIZATION OF FIELDS WITH VALUATIONS

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1. **Introduction.** A topological field is a (commutative) field which is also a topological space satisfying the separation axiom T_0, and in which addition, subtraction and multiplication are continuous, two-variable functions. For our purposes it is unnecessary to assume that division is continuous.

If F is any field, topological or not, we define a (nonarchimedean) valuation of F to be a function v on F to a linearly ordered group Γ with the symbol 0 adjoined, such that

$$v(xy) = v(x)v(y), \tag{1}$$

$$v(x+y) \leq \max\,[v(x), v(y)], \tag{2}$$

$$v(x) = 0 \text{ if and only if } x = 0, \tag{3}$$

for all x, y of F. It is understood that for every γ of Γ, $0<\gamma$ and $0\gamma=\gamma 0 = 0$. Such a valuation of a field defines a topology, with respect to which F is a topological field, when we specify that the neighborhoods of 0 in F are the sets $U(\gamma) = [x \in F \mid v(x) < \gamma]$, one for each γ in Γ. If F was a topological field to begin with and the topology defined by the valuation is the same as the original topology of F, we say that the valuation preserves the topology of F.

The question we intend to answer is, "Which topological fields have valuations preserving their topologies?"

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This problem was solved by Shafarevitch [3][1] and by Kaplansky [2] in the case where the assumption that Γ is an ordered group is strengthened to "Γ is the set of positive real numbers," and where the assumption (2) is weakened to $v(x+y) \leqq v(x)+v(y)$. Such a function v we shall call an *absolute value*. For a topological field F to have an absolute value preserving its topology, it is necessary and sufficient [2, Theorem 3] that F satisfy these two conditions:

(i_0) The set of nilpotent[2] elements is open.

(ii) If $A \subset F$ is bounded away from zero, then A^{-1} is bounded.[3]

For F to have a valuation preserving its topology, condition (ii) is still necessary. However, condition (i_0) is not; in fact, a field with a valuation may have no nilpotent elements besides 0 and yet the topology may be not discrete. Condition (i_0) essentially singled out the nilpotent elements for Shafarevitch and Kaplansky as the elements whose values are to be smaller than the identity element of Γ. But with our more general Γ, the set of elements of value smaller than the identity is not uniquely determined by the topological field F. What is the same thing, inequivalent valuations may give the same topology. Hence we are forced to replace (i_0) by a condition on subsets of F rather than on elements. We choose this existence condition:

(i) Some neighborhood of zero generates an additive group which is bounded.

It is easily seen that (i) is equivalent to the condition:

There is a bounded, open, additive subgroup of F.

With the aid of Theorem 1, it is equally easy to see that the following is also an equivalent condition:

There is a set of group-neighborhoods of zero in F and some open subset of F is bounded.[4]

We shall show that a topological field F has a valuation preserving its topology if and only if F satisfies (i) and (ii). We shall see in the course of the proof that fields satisfying (i) alone are plentiful and that the topology in such a field is closely linked to the arithmetic properties of the field (see Theorem 1 and its corollary).

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] An element x is called nilpotent if $\lim_{n\to\infty} x^n = 0$. If $A \subset F$ then "A is bounded away from zero" means "A is disjunct from some neighborhood of zero." By A^{-1} we denote the set of inverses of elements of A. And A is said to be bounded in case, for every open set V containing 0, there is an open set U about 0 such that $AU \subset V$. Here and elsewhere the product of two subsets of a field is the set of all products of two elements, one from each of the given subsets.

[4] The referee has pointed out that the existence of an open, bounded subset, that is, local boundedness, follows from (ii); and that our main theorem could also be proved by first exploiting (ii) and using (i) only in the final stages.

Some preliminary remarks on boundedness of subsets of F may be in order. In a field with a valuation or an absolute value, a set is bounded exactly when it is bounded in the usual sense: the values of its elements are bounded above. It follows, then, in well known fashion that an absolute value is a valuation (that is, is nonarchimedean) if and only if the additive group generated by 1 is bounded. In general, every subset of a bounded set is bounded. And, in our definition of boundedness, the notion is unchanged if we change the range of the variables U or V or both from the class of open sets about zero to an equivalent set of neighborhoods of zero. Every discrete field is itself bounded (in fact, these are the only bounded fields) and so conditions (i) and (ii) are satisfied. But such a field has a trivial valuation preserving its trivial topology: $v(x)=1$ if $x\neq 0$, $v(0)=0$, so that all our theorems will be true for discrete fields.

2. **Condition (i) alone.** Let F be a topological field satisfying (i).

There is an open, bounded, additive subgroup of F which contains 1. For there is an open, bounded, additive subgroup G of F which is not $\{0\}$; if b is a nonzero element of G, then $b^{-1}G$ is the required group.

There is an open, bounded subring R of F containing 1. For if G is an open, bounded, additive group containing 1, let R be $[a\in F \mid Ga\subset G]$. Clearly R is a ring and contains 1. Since G is both open and bounded, there is some neighborhood U of zero such that $GU\subset G$. That is, $U\subset R$ and R is open. Since $1\in G$, $R\subset G$ and R is bounded.

If F is not discrete, any open, bounded subring of F which contains 1 *is an order.*[5] For if $0\neq a\in F$, then Ra is an open set about zero. Since R is bounded there is a neighborhood U of zero such that $RU\subset Ra$. Choose an element $b\in R\cap U$ with $b\neq 0$ and have $b\in Rb\subset Ra$, $b=ca$, $a=c^{-1}b$, with b and c in R.

It is now clear that (i) is equivalent to the existence of a bounded, open order in F. Let us temporarily turn our attention to fields and their orders.

If F is a field and R is an order in F, we may topologize F by designating the principal R-ideals as neighborhoods of zero. Whether or not we include fractional ideals and whether or not we limit ourselves to principal ideals makes no real difference. The one restriction we do make is that $\{0\}$ is not to be counted as an R-ideal unless $R=F$. It is easy to check that this designation makes F a topological field.

On the other hand, R can be used to define a "semivaluation" in the usual fashion: If N is the multiplicative group of units of R and if F^* is the multiplicative group of nonzero elements of F, then we may order F^*/N by writing $fN \leqq gN$ when $fg^{-1} \in R$. This makes F^*/N a partially ordered group or o-group in the sense of Everett and Ulam [1],[6] and the natural homomorphism w of F^* onto F^*/N has the following properties:

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[5] An order in F is a subring of F which contains 1 and whose quotient field is F.

$$w(xy) = w(x)w(y), \tag{4}$$

$$w(x - y) \leqq \max [w(x), w(y)]. \tag{5}$$

Of course, (5) is to be interpreted to mean: If $\delta \in F^*/N$ and $\delta \geqq w(x)$, $w(y)$, then $\delta \geqq w(x-y)$. We usually adjoin the symbol 0 to F^*/N with the provision that $0\delta = \delta 0 = 0$, $0 < \delta$ for all $\delta \in F^*/N$. Then (4) and (5) are still true when we extend w by defining $w(0) = 0$, and

$$w(x) = 0 \text{ if and only if } x = 0. \tag{6}$$

Any function w that maps F onto an o-group Δ with 0 adjoined and satisfies (4), (5) and (6) will be called a semivaluation. (Note that the only difference between a valuation and a semivaluation is that the group of values of the latter need not be linearly ordered.) Not only does each order in F define a semivaluation, but each semivaluation w defines an order, namely, $[x \in F \mid w(x) \leqq \epsilon]$, where ϵ is the identity element in Δ. This is actually a one-to-one correspondence between orders in F and semivaluations of F, but we shall omit the proofs.

Every semivaluation of F defines a topology in F exactly as a valuation would. Specifically, for each $\delta \in \Delta$, define a neighborhood of zero in F by $[x \in F \mid w(x) < \delta]$. It is a straightforward job to prove that this definition converts F into a topological field whose topology is the same as that defined by the order corresponding to the semivaluation w.

Thus in a field the orders are in one-to-one correspondence with the semivaluations and a topology may be defined either by an order or by the corresponding semivaluation. "Close to zero" in such a topology means "divisible by many elements of the order." It may also be of interest to point out that the resultant topologies are not in one-to-one correspondence with the orders in the field, but rather with the classes of equivalent orders.[7]

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[6] An o-group is a group which is also a partially ordered set with respect to a relation, $\leqq$, with the further properties: If $a \leqq b$, then $ac \leqq bc$; and for any a, b, there is an element c with $a \leqq c$, $b \leqq c$.

[7] Two orders R and S are equivalent in case $R \subset aS$ and $S \subset bR$ for some nonzero elements a and b in F.

THEOREM 1. *The following conditions on an order R in a topological field F are equivalent.*

(7) *The topology defined by R is the same as that of F.*

(8) *The semivaluation corresponding to R preserves the topology of F.*

(9) *The order R is bounded and open.*

PROOF. We have already remarked that R and its semivaluation define the same topology in F. Hence (7) is equivalent to (8). Since R is bounded and open in the topology it defines, (7) implies (9). Lastly, if R is open, every R-ideal Ra is open. But since R is also bounded, every open set V about zero contains RU for a suitable open set U containing 0; hence V contains the R-ideal Ra where $a\in R\cap U$ and where $a\neq 0$ when $R\neq F$. This proves Theorem 1.

COROLLARY. *Each of the following conditions on a topological field F is equivalent to condition* (i).

(i′) *There is bounded, open order in F.*

(i″) *There is an order in F which defines the same topology as that of F.*

(i‴) *There is a semivaluation of F preserving the topology of F.*

3. **Conditions (i) and (ii).** We now prove our principal theorem.

THEOREM 2. *A topological field F has a valuation preserving its topology if and only if F satisfies* (i) *and* (ii).

PROOF. Necessity is obvious. To prove sufficiency, let R be any open, bounded order in F, and define X to be $[x\in F \mid x\notin R$ and $x^{-1}\notin R]$.

The set X is bounded. For $X^{-1}=X$ is disjunct from R which is a neighborhood of zero.

If we assume condition (i), a set A is bounded exactly when $A\subset Rb$ for some $b\neq 0$ in F. For there is some R-ideal Rc such that $ARc\subset R$, and we may choose $b=c^{-1}$, except when $R=F$ in which case any nonzero b will do. We shall call such an element b a bound for A. If b is a bound for A, so is bc^{-1} for any nonzero c in R. Hence we may choose a bound p for X with the property that $p^{-1}\in R$, and define Y to be the set of all p^n $(n=1, 2, \cdots)$.

If Y is not bounded then $\lim_{n\to\infty} p^{-n}=0$. For, because of (ii), Y^{-1} cannot be bounded away from zero, every neighborhood Ra of zero contains an element p^{-k}. But if $k'\geq k$ then $p^{-k'}\in Rp^{-k}\subset Ra$. Thus $\lim p^{-n}=0$.

If Y is not bounded, the set of nilpotent elements in F is open. For it contains Rp^{-1} which is a neighborhood of zero, and contains the open set Rq about q whenever it contains a nonzero element q.

If Y is not bounded, Theorem 2 is true. For Kaplansky's theorem [2, Theorem 3] asserts that F has an absolute value preserving its topology. This absolute value is in fact a valuation since the additive group generated by 1, being a subset of R, is bounded.

If Y is bounded we shall produce a valuation. Let S be the ring generated by R and X, that is, the set of all polynomials in elements of X with coefficients in R.

If Y is bounded, S is an open, bounded order. Since S contains R, S is clearly an open order. But if $a\prod_{i=1}^{n} x_i$ is any monomial in S with $a\in R$ and $x_i\in X$. then $a\prod x_i\in R(Rp)^n=Rp^n\subset Rq$, where q is a bound for Y. Since Rq is closed under addition, $S\subset Rq$.

If Y is bounded, Theorem 2 is true. The required valuation is the semivaluation v corresponding to the order S. By Theorem 1, this semivaluation preserves the topology of F. It only remains to show that v is a valuation, that is, that $v(F^*)$ is linearly ordered. This follows from an obvious property of S: if $x\in F$ then $x\in S$ or $x^{-1}\in S$. For if x and y are nonzero elements of F, then either $x^{-1}y\in S$ or $y^{-1}x\in S$. That is, either $v(y)\leqq v(x)$ or $v(x)\leqq v(y)$. This completes the proof.

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THE INSTITUTE FOR ADVANCED STUDY

THE ASYMPTOTIC DISTRIBUTION OF THE SUM OF A RANDOM NUMBER OF RANDOM VARIABLES

HERBERT ROBBINS

1. Introduction. If a random variable (r. v.) Y is the sum of a large but constant number N of independent components

$$Y = X_1 + \cdots + X_N, \tag{1}$$

then under appropriate conditions on the X_j it follows from the central limit theorem that the distribution of Y will be nearly normal. In many cases of practical importance, however, the number N is itself a r. v., and when this is so the situation is more complex.

We shall consider the case in which the X_j $(j=1, 2, \cdots)$ are independent r. v.'s with the same distribution function (d. f.) $F(x) = P[X_j \leqq x]$, and in which the non-negative integer-valued r. v. N is independent of the X_j. The d. f. of N we shall assume to depend on a parameter λ, so that the d. f. of Y is a function of λ which may have an asymptotic expression as $\lambda \to \infty$. In the degenerate case in which for any integer λ, N is certain to have the value λ, the problem reduces to the ordinary central limit problem for equi-distributed components.

In the general case the d. f. of N for any λ is determined by the values $\omega_k = P[N=k]$ $(k=0, 1, \cdots)$, where the ω_k are functions of λ such that for all λ,

$$\omega_k \geqq 0, \qquad \sum_0^\infty \omega_k = 1.$$

We shall use Greek letters to denote functions of the parameter λ; in particular we define

$$\begin{aligned} \alpha &= E(N) = \sum_0^\infty \omega_k \cdot k, \\ \beta^2 &= E(N^2) = \sum_0^\infty \omega_k \cdot k^2 \qquad \text{(assumed finite for all } \lambda\text{)}, \\ \gamma^2 &= \operatorname{Var}(N) = \sum_0^\infty \omega_k \cdot (k-\alpha)^2 = \beta^2 - \alpha^2, \\ \theta(t) &= E(e^{i(N-\alpha)t/\gamma}) = \sum_0^\infty \omega_k \cdot e^{i(k-\alpha)t/\gamma}, \end{aligned} \tag{2}$$

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the last being the characteristic function (c. f.) of the normalized r. v.

$$M = (N - \alpha)/\gamma. \tag{3}$$

We shall use Latin letters to denote quantities independent of λ; in particular we define

$$\begin{aligned} a &= E(X_j) = \int x dF(x), \\ b^2 &= E(X_j^2) = \int x^2 dF(x), \\ c^2 &= \mathrm{Var}\,(X_j) = \int (x-a)^2 dF(x) = b^2 - a^2 \quad (0 < c^2 < \infty), \\ f(t) &= E(e^{iX_j t}) = \int e^{ixt} dF(x). \end{aligned} \tag{4}$$

We then have for the r. v. (1),

$$\begin{aligned} E(Y) &= \sum_0^\infty \omega_k \cdot E(X_1 + \cdots + X_k) = \sum_0^\infty \omega_k \cdot ka = \alpha a, \\ E(Y^2) &= \sum_0^\infty \omega_k \cdot E(X_1 + \cdots + X_k)^2 \\ &= \sum_0^\infty \omega_k \{ kb^2 + k(k-1)a^2 \} = \alpha c^2 + \beta^2 a^2, \\ \sigma^2 &= \mathrm{Var}\,(Y) = \alpha c^2 + \gamma^2 a^2. \end{aligned} \tag{5}$$

We shall be concerned with the normalized r. v.

$$Z = \frac{Y - E(Y)}{(\mathrm{Var}\,(Y))^{1/2}} = \frac{(X_1 + \cdots + X_N) - \alpha a}{\sigma}, \tag{6}$$

whose c. f. is

$$\begin{aligned} \phi(t) = E(e^{iZt}) &= \sum_0^\infty \omega_k \cdot E(e^{i[(X_1 + \cdots + X_k) - \alpha a/\sigma]t}), \\ &= \sum_0^\infty \omega_k \cdot e^{-i\alpha a t/\sigma} \cdot f^k\left(\frac{t}{\sigma}\right). \end{aligned} \tag{7}$$

By definition, Z has the limiting d. f. $H(x)$ if whenever x is a continuity point of the d. f. $H(x)$, $\lim_{\lambda \to \infty} P[Z \leq x] = H(x)$, or, equivalently, setting

$$h(t) = \int e^{itx} dH(x),$$

if for every t,

$$\lim_{\lambda \to \infty} \phi(t) = h(t). \tag{8}$$

In particular, if (8) holds for $h(t) = e^{-t^2/2}$, then for every x,

$$\lim_{\lambda \to \infty} P[Z \leqq x] = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-u^2/2} du = H_0(x),$$

and Y is said to be asymptotically normal $(\alpha a, \sigma)$.

2. Some general results.

Theorem 1. *Let*

$$\delta = \frac{\gamma a}{\sigma} = \left(\frac{\gamma^2 a^2}{\alpha c^2 + \gamma^2 a^2} \right)^{1/2} \qquad (0 \leqq \delta \leqq 1). \tag{9}$$

If, as $\lambda \to \infty$,

$$\sigma^2 \to \infty, \qquad \gamma = o(\sigma^2), \tag{10}$$

then

$$\phi(t) = \theta(\delta t) \cdot e^{-t^2(1-\delta^2)/2} + o(1). \tag{11}$$

Proof. Since from (10),

$$E\left(\frac{N-\alpha}{\sigma^2}\right) = 0, \qquad E\left(\frac{N-\alpha}{\sigma^2}\right)^2 = \frac{\gamma^2}{\sigma^4} \to 0,$$

it follows that $(N-\alpha)/\sigma^2 \to 0$ in probability as $\lambda \to \infty$. Hence for any $d > 0$,

$$2 \cdot P\left[\left| \frac{N-\alpha}{\sigma^2} \right| > d \right] = o(1) \qquad \text{as } \lambda \to \infty. \tag{12}$$

We now write (7) in the form

$$\phi(t) = \sum_{0}^{\infty} \omega_k \cdot e^{i(k-\alpha)at/\sigma} \cdot \left\{ e^{-iat/\sigma} f\left(\frac{t}{\sigma}\right) \right\}^k, \tag{13}$$

and define

$$\phi_1(t) = \left\{ \sum_{0}^{\infty} \omega_k \cdot e^{i(k-\alpha)at/\sigma} \right\} \cdot \left\{ e^{-iat/\sigma} f\left(\frac{t}{\sigma}\right) \right\}^{\alpha};\tag{14}$$

then

$$\phi(t) - \phi_1(t) = \sum_0^\infty \omega_k \cdot e^{i(k-\alpha)at/\sigma}\left[\left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^k - \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^\alpha\right],$$

whence

$$\begin{aligned}(15)\qquad \left|\phi(t) - \phi_1(t)\right| &\leqq \sum_0^\infty \omega_k\left|\left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^k - \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^\alpha\right| \\ &= \sum_{|(k-\alpha)/\sigma^2|>d} + \sum_{|(k-\alpha)/\sigma^2|\leqq d} \\ &\leqq 2\sum_{|(k-\alpha)/\sigma^2|>d} \omega_k + \left|e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right|^\alpha \\ &\quad \cdot \sum_{|(k-\alpha)/\sigma^2|\leqq d} \omega_k\left|\left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{k-\alpha} - 1\right| \\ &\leqq 2P\left[\left|\frac{N-\alpha}{\sigma^2}\right| > d\right] \\ &\quad + \operatorname*{Max}_{|r|\leqq d}\left|\left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\sigma^2 r} - 1\right|.\end{aligned}$$

From (4) we have as $t \to 0$,

$$f(t) = 1 + iat - \frac{b^2t^2}{2} + o(t^2);$$

hence as $\sigma^2 \to \infty$

$$\begin{aligned} e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right) &= \left\{1 - \frac{iat}{\sigma} - \frac{a^2t^2}{2\sigma^2} + o\left(\frac{1}{\sigma^2}\right)\right\} \\ &\quad \cdot \left\{1 + \frac{iat}{\sigma} - \frac{b^2t^2}{2\sigma^2} + o\left(\frac{1}{\sigma^2}\right)\right\} \\ &= 1 - \frac{c^2t^2}{2\sigma^2} + o\left(\frac{1}{\sigma^2}\right), \end{aligned}$$

$$\begin{aligned}\left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\sigma^2} &= \left\{1 - \frac{c^2t^2}{2\sigma^2} + o\left(\frac{1}{\sigma^2}\right)\right\}^{\sigma^2} = e^{-c^2t^2/2} + o(1) \\ &= \left\{e^{-t^2/2} + o(1)\right\}^{c^2}.\end{aligned}$$

Thus

$$\left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{a} = \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\sigma^2\alpha/\sigma^2} \tag{16}$$

$$= \left\{e^{-t^2/2} + o(1)\right\}^{\sigma^2\alpha/\sigma^2},$$

and

$$\left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{\sigma^2 r} = \left\{e^{-t^2/2} + o(1)\right\}^{\sigma^2 r}. \tag{17}$$

Now fix t and $\epsilon > 0$. Choose $d > 0$, until now arbitrary, so that

$$\left| z - e^{-t^2/2} \right| < d, \qquad |r| < d \tag{18}$$

imply that

$$\left| z^{\sigma^2 r} - 1 \right| < \frac{\epsilon}{2}. \tag{19}$$

Then choose λ_1 so that $\lambda > \lambda_1$ implies that

$$2P\left[\left|\frac{N-\alpha}{\sigma^2}\right| > d\right] < \frac{\epsilon}{2} \tag{20}$$

and that the $o(1)$ in (17) satisfies the inequality

$$|o(1)| < d. \tag{21}$$

Then it follows from (15) that for $\lambda > \lambda_1$,

$$|\phi(t) - \phi_1(t)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon;$$

since ϵ was arbitrary we conclude that

$$\phi(t) = \phi_1(t) + o(1). \tag{22}$$

We can write $\phi_1(t)$ in the form

$$\phi_1(t) = \left\{\sum_0^\infty \omega_k \cdot e^{i(k-a)at/\sigma}\right\} \cdot \left\{e^{-iat/\sigma}f\left(\frac{t}{\sigma}\right)\right\}^{a}$$

$$= \theta\left\{\frac{\gamma a t}{\sigma}\right\} \cdot \left\{e^{-t^2/2} + o(1)\right\}^{\alpha\sigma^2/\sigma^2} = \theta(\delta t) \cdot \left\{e^{-t^2/2} + o(1)\right\}^{1-\delta^2} \tag{23}$$

$$= \theta(\delta t) \cdot \left\{e^{-t^2/2}(1 + o(1))\right\}^{1-\delta^2}$$

$$= \theta(\delta t) \cdot e^{-t^2(1-\delta^2)/2} + o(1),$$

which, with (22), completes the proof of the theorem.

COROLLARY 1. *If* (10) *holds, and if as* $\lambda \to \infty$,

$$a^2\gamma^2 = o(\alpha), \tag{24}$$

then

$$\lim_{\lambda\to\infty} \phi(t) = e^{-t^2/2}, \tag{25}$$

so that Z *has the limiting* d. f. $H_0(x)$ *and* Y *is asymptotically normal* $(\alpha\, a, \sigma)$.

PROOF. From (24) it follows that as $\lambda \to \infty$, $\delta \to 0$. Moreover, considering the r. v.

$$M_1 = \frac{(N-\alpha)\delta}{\gamma}$$

we have $E(M_1) = 0$, $E(M_1^2) = \delta^2 \to 0$, so that $M_1 \to 0$ in probability. It follows that

$$E(e^{iM_1t}) = \sum_0^\infty \omega_k \cdot e^{i(k-\alpha)\delta t/\gamma} = \theta(\delta t) \to 1, \tag{26}$$

while

$$e^{-t^2(1-\delta^2)/2} \to e^{-t^2/2}, \tag{27}$$

so that (25) follows from (11).

Until now we have not assumed that the normalized r. v. M defined by (3) has a limiting d. f. $G(x)$ as $\lambda \to \infty$.

COROLLARY 2. *If* (10) *holds, and if* N *is asymptotically normal* (α, γ), *then* Z *has the limiting* d. f. $H_0(x)$ *and* Y *is asymptotically normal* $(\alpha a, \sigma)$.

PROOF. In this case we have

$$\lim_{\lambda\to\infty} \theta(\tau) = e^{-\tau^2/2},$$

and the convergence is uniform in the interval $0 \leqq \tau \leqq t$. Since $0 \leqq \delta \leqq 1$ it follows that as $\lambda \to \infty$,

$$\theta(\delta t) = e^{-(\delta t)^2/2} + o(1), \tag{28}$$

and therefore from (11),

$$\phi(t) = \left\{e^{-\delta^2t^2/2} + o(1)\right\} \cdot \left\{e^{-(1-\delta^2)t^2/2}\right\} + o(1) = e^{-t^2/2} + o(1).$$

The assumption (10) is actually superfluous in this case as we shall see later (Corollary 4).

Let us now consider the case in which M has a non-normal limiting d. f.

Corollary 3. *If* (10) *holds, and if M has a non-normal limiting* d. f. *$G(x)$, so that*

$$\lim_{\lambda\to\infty} \theta(t) = g(t) = \int e^{ixt}dG(x) \neq e^{-t^2/2}, \tag{29}$$

and if

$$\lim_{\lambda\to\infty} \frac{c^2\alpha}{a^2\gamma^2} = s \tag{30}$$

exists, $0 \leq s < \infty$, *then*

$$\lim_{\lambda\to\infty} \phi(t) = g\left(\frac{t}{(1+s)^{1/2}}\right)\cdot e^{-(t(s/(1+s))^{1/2})^2/2} \neq e^{-t^2/2}, \tag{31}$$

so that Z has the non-normal limiting d. f.

$$H(x) = G(t(1+s)^{1/2})_* H_0\left(x\left(\frac{1+s}{s}\right)^{1/2}\right), \tag{32}$$

*where * denotes the operation of convolution.*

Proof. In this case as $\lambda\to\infty$, $\delta\to(1+s)^{-1/2}$, whence (31) follows as before.

If $s=0$ (that is, if $\alpha=o(a^2\gamma^2)$) then $\lim_{\lambda\to\infty} \phi(t)=\lim_{\lambda\to\infty} \theta(t)=g(t)$, so that Y has the same asymptotic distribution as N. If $0<s<\infty$ then the limiting d. f. of Z is the convolution of a normal with a non-normal d. f. If $s=\infty$ we refer to Corollary 1.

Lemma 1. *If M has a limiting d. f. $G(x)$ such that $G(x)>0$ for every finite x, then* (10) *holds.*

Proof. First we shall show that $\gamma=o(\alpha)$ as $\lambda\to\infty$. Suppose not. Then there exists a constant $B>0$ such that for any λ_1 there exists a $\lambda>\lambda_1$ such that

$$\alpha/\gamma < B. \tag{33}$$

We may assume that $-B$ is a continuity point of $G(x)$. Now choose λ_1 so that for all $\lambda>\lambda_1$,

$$P\left[\frac{N-\alpha}{\gamma} \leq -B\right] > G(-B) - \frac{G(-B)}{2} = \frac{G(-B)}{2} > 0; \tag{34}$$

then for some $\lambda>\lambda_1$ we have both (33) and (34), whence

$$0 = P[N < 0] = P\left[\frac{N - \alpha}{\gamma} < \frac{-\alpha}{\gamma}\right] \geqq P\left[\frac{N - \alpha}{\gamma} \leqq -B\right] > 0,$$

a contradiction. It follows that $\gamma = o(\alpha)$ and hence $\gamma = o(\sigma^2)$.

We shall now show that $\alpha \to \infty$. If not, then since $\gamma = o(\alpha)$, it follows that $\gamma \to 0$, which we shall show to be impossible.

From Tchebychef's inequality, $\gamma \to 0$ implies that

$$P[|N - \alpha| < 1/2] \to 1.$$

But there is at most one integer k satisfying $|k - \alpha| < 1/2$; denoting this integer by k_λ we have

$$P[N = k_\lambda] \to 1.$$

Define

$$L = \liminf_{\lambda \to \infty} \left\{\frac{k_\lambda - \alpha}{\gamma}\right\}.$$

Either $L > -\infty$ or $L = -\infty$. If the former, let $x < L$ be a continuity point of $G(x)$. Then

$$G(x) = \lim_{\lambda \to \infty} P\left[\frac{N - \alpha}{\gamma} \leqq x\right].$$

But for sufficiently large λ,

$$\frac{k_\lambda - \alpha}{\gamma} > x,$$

whence

$$P\left[\frac{N - \alpha}{\gamma} \leqq x\right] < 1 - P[N = k_\lambda].$$

It follows that $G(x) = 0$, a contradiction. On the other hand suppose $L = -\infty$. Then for any x and sufficiently large λ,

$$\frac{k_\lambda - \alpha}{\gamma} < x,$$

whence

$$P\left[\frac{N - \alpha}{\gamma} \leqq x\right] \geqq P[N = k_\lambda].$$

It follows that $G(x) = 1$. Since x was arbitrary, $G(x)$ is not a d. f.

Thus $\alpha \to \infty$ and hence $\sigma^2 \to \infty$. This completes the proof.

It follows that in Corollary 3 we may drop the assumption (10) provided $G(x) > 0$. Moreover, Corollary 2 may now be given its final form.

COROLLARY 4. *If N is asymptotically normal* (α, γ) *then Y is asymptotically normal* $(\alpha a, \sigma)$.

We shall conclude this section with a theorem concerning the "singular" case in which α and γ are of the same order as $\lambda \to \infty$, and $a = 0$, so that (10) does not hold.

THEOREM 2. *Let $a = 0$. If as $\lambda \to \infty$*

$$\alpha \to \infty, \qquad \gamma/\alpha \to r \qquad (0 < r < \infty), \tag{35}$$

and if M has a limiting d. f. $G(x)$ (necessarily such that $G(x) = 0$ for some x), then

$$\lim_{\lambda \to \infty} \phi(t) = \int_0^\infty e^{-t^2 y/2} dG_1(y) = g_1\left(\frac{it^2}{2}\right), \tag{36}$$

where

$$G_1(x) = G\left(\frac{x-1}{r}\right), \qquad g_1(t) = \int_0^\infty e^{itx} dG_1(x). \tag{37}$$

Thus the limiting d. f. of Z is

$$H(x) = \int_0^\infty H_0\left(\frac{x}{y^{1/2}}\right) dG_1(y). \tag{38}$$

PROOF. We have for $a = 0$,

$$\phi(t) = \sum_0^\infty \omega_k \cdot f^k\left(\frac{t}{(\alpha c^2)^{1/2}}\right) = \sum_0^\infty \omega_k \left\{ f^\gamma\left(\frac{t}{(\alpha c^2)^{1/2}}\right)\right\}^{k/\gamma}.$$

Now as $\lambda \to \infty$,

$$f^\gamma\left(\frac{t}{(\alpha c^2)^{1/2}}\right) = \left\{ f^\alpha\left(\frac{t}{(\alpha c^2)^{1/2}}\right)\right\}^{\gamma/\alpha} \to e^{-rt^2/2}.$$

Set

$$\zeta = \frac{-i \log f^\gamma(t/(\alpha c^2)^{1/2})}{r};$$

then as $\lambda \to \infty$,

$$\zeta \to it^2/2, \tag{39}$$

and

$$\phi(t) = \sum_0^\infty \omega_k \cdot e^{ir\zeta k/\gamma}. \tag{40}$$

Now let

$$M_1 = rN/\gamma;$$

then for any x such that $(x-1)/r$ is a continuity point of $G(x)$,

$$P[M_1 \leq x] = P\left[\frac{rN}{\gamma} \leq x\right] = P\left[\frac{N-\alpha}{\gamma} \leq \frac{x}{r} - \frac{\alpha}{\gamma}\right]$$
$$\to G\left(\frac{x-1}{r}\right) = G_1(x),$$

where $G_1(x)$ is defined by (37). It follows that

$$E(e^{isM_1}) = \sum_0^\infty \omega_k \cdot e^{isrk/\gamma} \to g_1(s) = \int_0^\infty e^{isy} dG_1(y)$$

uniformly for every s in some neighborhood of $s = it^2/2$. Hence from (39) and (40),

$$\lim_{\lambda \to \infty} \phi(t) = g_1\left(\frac{it^2}{2}\right) = \int_0^\infty e^{-t^2 y/2} dG_1(y).$$

Since

$$\int_0^\infty e^{-t^2 y/2} dG_1(y) = \int_0^\infty \int_{-\infty}^\infty e^{itx} d_x H_0\left(\frac{x}{y^{1/2}}\right) dG_1(y)$$
$$= \int_{-\infty}^\infty e^{itx} d_x \left\{ \int_0^\infty H_0\left(\frac{x}{y^{1/2}}\right) dG_1(y) \right\},$$

it follows that the limiting d. f. of Z is given by (38). This completes the proof of Theorem 2.

From the relation $M_1 = rM + (r\alpha)/\gamma$ it follows that $g_1(t) = e^{it} \cdot g(rt)$, where $g(t)$ is defined by (29). Hence (36) may be written in the equivalent form

$$\lim_{\lambda \to \infty} \phi(t) = e^{-t^2/2} \cdot g\left(\frac{irt^2}{2}\right).$$

3. **Some examples.** (i) Let N have a Poisson distribution with

parameter λ, so that

$$\omega_k = e^{-\lambda}\cdot(\lambda^k/k!) \qquad (k = 0, 1, \cdots);$$

then

$$\alpha = \gamma^2 = \lambda, \qquad \sigma^2 = \lambda b^2.$$

From Corollary 4 it follows that Y is asymptotically normal $(\lambda a, b\,\lambda^{1/2})$. Note that (10) holds but (24) does not.

(ii) Let N have a binomial distribution with parameters λ, p, where λ is an arbitrary positive integer and p and $q=1-p$ are constants, so that

$$\omega_k = \frac{\lambda!}{k!(\lambda-k)!}\,p^k q^{\lambda-k} \qquad (k = 0, 1, \cdots, \lambda);$$

then

$$\alpha = \lambda p, \qquad \gamma^2 = \lambda p q, \qquad \sigma^2 = \lambda p(c^2 + q a^2).$$

Again it follows from Corollary 4 that Y is asymptotically normal $(\lambda p a, (\lambda p(c^2+qa^2))^{1/2})$.

(iii) For any integer λ suppose that N can assume the two values λ, 2λ, with probability 1/2 in each case. Then

$$\alpha = \frac{3\lambda}{2}, \qquad \gamma^2 = \frac{\lambda^2}{4}, \qquad \sigma^2 = \frac{3\lambda}{2}c^2 + \frac{\lambda^2}{4}a^2.$$

First suppose $a\neq 0$. Then as $\lambda\to\infty$ (10) holds, and the quantity s of Corollary 3 is 0. Moreover, $\theta(t) = \cos t$, so that M has the non-normal limiting distribution for which $P[M=-1]=P[M=1]=1/2$. It follows from Corollary 3 that Z has the same limiting distribution.

The case is quite different when $a=0$, for then $\gamma\neq o(\sigma^2)$, and Theorem 2 applies. We have

$$r = \lim_{\lambda\to\infty}\frac{\gamma}{\alpha} = \frac{1}{3}, \qquad g_1(t) = \frac{1}{2}\left\{e^{2it/3} + e^{4it/3}\right\},$$

so that

$$\lim_{\lambda\to\infty}\phi(t) = \frac{1}{2}\left\{e^{-t^2/3} + e^{-2t^2/3}\right\}.$$

Thus the limiting d. f. of Z is a mixture of two normal d. f.'s with means 0 and variances 2/3 and 4/3.

UNIVERSITY OF NORTH CAROLINA

ASYMPTOTIC DISTRIBUTION OF THE MAXIMUM CUMULATIVE SUM OF INDEPENDENT RANDOM VARIABLES

KAI LAI CHUNG

The limiting distribution of the maximum cumulative sum[1] of a sequence of independent random variables has been discussed recently by Erdös-Kac[2] and Wald.[3] Erdös and Kac treated the case where each random variables has zero mean, while Wald considered more general cases.

We shall show that the problem can be treated by a uniform method starting with a classical combinatorial formula due to De Moivre.[4] A careful application of Stirling's formula does the trick in all cases, but it is interesting that quite different transformations are needed in different cases. It is also to be noted that this is the usual method of arriving at the normal approximation to the binomial distribution given in elementary textbooks (frequently without rigor).

By this method we obtain easily a remainder term to the approximation. In the Bernoullian case this is the best possible order of magnitude. In this general case we have to use a recent theorem of H. Bergström[5] concerning the remainder term in the k-dimensional central limit theorem. Although it was A. C. Berry and Esseen (independently of each other) who first obtained this result in the one-dimensional case and their methods can be extended to higher dimensions, Bergström's result is more precise in the determination of the dependence on k.

For the sake of simplicity we assume that each variable has unit variance. It is easy to remove this restriction but it is cumbersome to do so.

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[1] We use the term "maximum cumulative sum" for max S_k; we have used the term "maximum partial sum" for max $|S_k|$ (*On the maximum partial sums of sequences of independent random variables*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 205–253).

[2] P. Erdös and M. Kac, *On certain limit theorems of the theory of probability*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 292–302.

[3] A. Wald, *Limit distribution of the maximum and minimum of successive cumulative sums of random variables*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 142–153.

[4] See, for example, Uspensky, *Introduction to mathematical probability*, New York, McGraw-Hill, 1937, p. 153.

[5] H. Bergström, *On the central limit theorem in the space R_k, $k>1$*, Skandinavisk Aktuarietidskrift vol. (1945) pp. 106–127. The result there was stated for random variables having the same distribution, but no change is needed to carry over into the present case. I owe to Dr. Bergström the present form of his result.

We state our results as follows. Let $X_1, \cdots, X_n$ be a sequence of independent random variables with

$$E(X_k) = \mu_k, \qquad E(X_k^2) = 1 + \mu_k^2,$$
$$E(|X_k - \mu_k|^3) = \gamma_k = O(1).$$

Let

$$S_n = \sum_{k=1}^{n} X_k, \qquad \bar{S}_n = \max_{1 \leq k \leq n} S_k.$$

Then the asymptotic distributions of $\bar{S}_n$ as n tends to infinity are given below:

Case (i): $\mu_k = 0$, $k = 1, \cdots, n$.

$$\Pr(\bar{S}_n < an^{1/2}) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^a e^{-x^2/2} dx + O(n^{-1/26} \log n).$$

Case (ii): $\mu_k = dn^{-1/2}$ $(d \neq 0)$, $k = 1, \cdots, n$.

$$\Pr(\bar{S}_n < an^{1/2}) = \int_{d^2/2}^{\infty} \frac{ade^{ad}}{2\pi^{1/2}x^{3/2}} e^{-a^2d^2/4x - x} dx + O(n^{-1/26} \log n).$$

Case (iii): $\mu_k = \mu > 0$, $k = 1, \cdots, n$.

$$\Pr(\bar{S}_n < n\mu + an^{1/2}) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{a} e^{-x^2/2} dx + O(n^{-1/26} \log n).$$

In the Bernoullian case the remainder term in each case is to be replaced by $O(n^{-1/2})$.

1. The Bernoullian case. Let each X_ν, $\nu = 1, \cdots, n$, be distributed as follows:

$$X_\nu = \begin{cases} +g & \text{with probability} \quad p = 2^{-1}(1 + \mu/g), \\ -g & \text{with probability} \quad q = 2^{-1}(1 - \mu/g), \end{cases} \tag{1}$$

where

$$g^2 = 1 + \mu^2. \tag{2}$$

Then

$$E(X_\nu) = \mu, \qquad E(X_\nu - \mu)^2 = 1.$$

For definitiveness we assume that n and b are both even. From a classical formula[6] we have

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[6] See Uspensky, op. cit. p. 151, formula (12), p and q being interchanged.

$$\Pr(\overline{S}_n \geqq bg) = p^b \sum_{r=0}^{(n-b)/2} \frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r. \tag{3}$$

Since

$$\lim_{n\to\infty} \Pr(\overline{S}_n \geqq bg) = 1$$

we have

$$\Pr(\overline{S}_n < bg) = p^b \sum_{r>(n-b)/2} \frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r. \tag{4}$$

Using Stirling's formula we have

$$\begin{aligned} \log\left[\frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r p^b\right] \\ &= \log b + \left(b + 2r - \frac{1}{2}\right)\log(b+2r-1) \\ &\quad - \left(r+\frac{1}{2}\right)\log r - \left(b+r+\frac{1}{2}\right)\log(b+r) \\ &\quad + 1 - \frac{1}{2}\log 2\pi + r\log pq + b\log p + O\left(\frac{1}{b+r}\right). \end{aligned} \tag{5}$$

Assuming $b = o(r)$, $\mu = o(1)$, we can write (5) as

$$\begin{aligned} &\log b + \left(b+2r-\frac{1}{2}\right)\left[\log 2r + \frac{b-1}{2r} - \frac{(b-1)^2}{8r^2} + O\left(\frac{b^3}{r^3}\right)\right] \\ &\quad - \left(r+\frac{1}{2}\right)\log r \\ &\quad - \left(b+r+\frac{1}{2}\right)\left[\log r + \frac{b}{r} - \frac{b^2}{2r^2} + O\left(\frac{b^3}{r^3}\right)\right] \\ &\quad + 1 - \frac{1}{2}\log 2\pi - r(\log 4 + \mu^2 + O(\mu^4)) \\ &\quad - b\log 2 + b\left[\frac{\mu}{g} + O(\mu^2)\right] + O\left(\frac{1}{b+r}\right) \\ &= \log\frac{b}{2\pi^{1/2}r^{3/2}} - \frac{b^2}{4r} + \frac{b\mu}{g} - r\mu^2 + O\left(\frac{b^3}{r^2} + \frac{b}{r} + r\mu^4 + b\mu^2\right). \end{aligned} \tag{6}$$

Now let

$$\mu = dn^{-1/2}, \qquad b = an^{1/2},$$

where a and d are constants, $a>0$. Then if $2^{-1}n \leqq r \leqq n^{3/2}$, (6) gives

$$
\begin{aligned}
&\frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r p^b \\
&\qquad = \frac{an^{1/2}}{2\pi^{1/2}r^{3/2}g}\exp\left(-\frac{a^2n}{4g^2r}-\frac{d^2r}{n}+\frac{ad}{g^2}\right)\cdot(1+O(n^{-1/2})) \qquad (7)\\
&\qquad = \frac{an^{1/2}}{2\pi^{1/2}r^{3/2}}\exp\left(-\frac{a^2n}{4r}-\frac{d^2r}{n}+ad\right)\cdot(1+O(n^{-1/2})).
\end{aligned}
$$

If $r>n^{3/2}$, (6) gives

$$\frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r p^b = O(\exp(-Ar^{1/2})), \tag{8}$$

where A, as later, is a positive constant which need not be the same each time it appears. Therefore we obtain from (7) and (8)

$$
\begin{aligned}
&\sum_{r>(n-b)/2}\frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r p^b \\
&\qquad = \sum_{n/2\leqq r\leqq n^{3/2}}\frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r p^b + O\left(\frac{bn^{1/2}}{n^{3/2}}e^A\right) \\
&\qquad\quad + \sum_{r>n^{3/2}}\exp(-Ar^{1/2}) \qquad (9)\\
&\qquad = \sum_{r\geqq n/2}\frac{an^{1/2}}{2\pi^{1/2}r^{3/2}}\exp\left(-\frac{a^2n}{4r}-\frac{d^2r}{n}+ad\right)+O(n^{-1/2}) \\
&\qquad = \int_{n/2}^{\infty}\frac{an^{1/2}}{2\pi^{1/2}y^{3/2}}\exp\left(-\frac{a^2n}{4y}-\frac{d^2y}{n}+ad\right)dy+O(n^{-1/2}).
\end{aligned}
$$

If $d\neq 0$, letting

$$x = d^2y/n$$

we can write (9) as

$$
\begin{aligned}
&\int_{d^2/2}^{\infty}\frac{ade^{ad}}{2\pi^{1/2}x^{3/2}}\exp\left(-\frac{a^2d^2}{4x}-x\right)dx+O(n^{-1/2}) \qquad (10)\\
&\qquad\qquad = I(a)+O(n^{-1/2}).
\end{aligned}
$$

If $d=0$, letting

$$a^2n/2r = y^2$$

we can write (9) as

$$\left(\frac{2}{\pi}\right)^{1/2}\int_0^{\alpha} e^{-y^2/2}dy + O(n^{-1/2}). \tag{11}$$

2. The method of Erdös-Kac. Let X_ν, $\nu=1, \cdots, n$, be independent, and

$$E(X_\nu) = \mu = dn^{-1/2}, \quad E(X_\nu - \mu)^2 = 1, \quad E(|X_\nu - \mu|^3) = \gamma_\nu.$$

We also assume that

$$\gamma_\nu = O(1).$$

Let

$$n_j = \left[\frac{j}{k}n\right], \qquad j = 1, \cdots, k.$$

We shall take

$$k = k(n) \sim n^{1/13},$$

$$\epsilon = \epsilon(n) = \left(\frac{\log n}{k}\right)^{1/2} \sim \frac{(\log n)^{1/2}}{n^{1/26}}.$$

According to the method of Erdös-Kac, we have

$$\begin{aligned}\Pr(\overline{S}_n < an^{1/2}) \geqq \Pr\left(\max_{1\leqq j<k} S_{n_j} < (a-\epsilon)n^{1/2}\right)\\ + \max_{1\leqq i<k}\ \max_{n_i<r\leqq n_{i+1}} \Pr(|S_{n_{i+1}} - S_r| \geqq \epsilon n^{1/2}).\end{aligned} \tag{12}$$

Let $n_i<r\leqq n_{i+1}$. Let $B=B_n$ be a function of n to be determined later.

If $n_{i+1}-r\leqq B$, then

$$\begin{aligned}\Pr(|S_{n_{i+1}} - S_r| \geqq \epsilon n^{1/2}) &\leqq \frac{n_{i+1} - r + (n_{i+1}-r)^2\mu^2}{\epsilon^2 n}\\ &\leqq \frac{B + B^2\mu^2}{\epsilon^2 n}.\end{aligned} \tag{13}$$

If $n_{i+1}-r>B$, then

$$\begin{aligned}\Pr(|S_{n_{i+1}} - S_r| \geqq \epsilon n^{1/2}) = \frac{1}{(2\pi)^{1/2}}\left\{\int_{v_1}^{\infty} + \int_{-\infty}^{-v_2} e^{-u^2/2}du\right\}\\ + O((n_{i+1}-r)^{-1/2})\end{aligned} \tag{14}$$

where

$$v_1 = \frac{\epsilon n^{1/2} - (n_{i+1} - r)\mu}{(n_{i+1} - r)^{1/2}}, \qquad v_2 = \frac{\epsilon n^{1/2} + (n_{i+1} - r)\mu}{(n_{i+1} - r)^{1/2}}.$$

If $n > n_0$, a fixed constant, we have

$$n_{i+1} - r \leqq n_{i+1} - n_i < 2n/k,$$
$$k\epsilon > 4d;$$

hence

$$\epsilon n^{1/2} - (n_{i+1} - r)\mu = \epsilon n^{1/2}\left[1 - \frac{(n_{i+1} - r)d}{\epsilon n}\right]$$
$$\geqq \epsilon n^{1/2}\left[1 - \frac{2d}{k\epsilon}\right] \geqq \frac{\epsilon}{2} n^{1/2},$$
$$v_2 > v_1 > \frac{\epsilon n^{1/2}}{2(2nk^{-1})^{1/2}} \geqq \frac{k^{1/2}\epsilon}{2^{3/2}} \geqq \left(\frac{\log n}{8}\right)^{1/2},$$
$$\left(\int_{v_1}^{\infty} + \int_{-\infty}^{-v_2}\right) e^{-u^2/2} du = O\left(\exp\left(-\frac{\log n}{16}\right)\right) = O(n^{-1/16}).$$

Now we choose B so that

$$\frac{B}{\epsilon^2 n} = \frac{1}{B^{1/2}}, \qquad B = \epsilon^{4/3} n^{2/3}.$$

Then

$$\frac{B + B^2\mu^2}{\epsilon^2 n} = O(n^{-1/4}), \qquad B^{-1/2} = O(n^{-1/4}).$$

Thus combining (13) and (14) we obtain

$$\Pr\left(\left| S_{n_{i+1}} - S_r \right| \geqq \epsilon n^{1/2}\right) = O(n^{-1/16}). \tag{15}$$

Hence from (12)

$$\Pr\left(\overline{S}_n < a n^{1/2}\right) \geqq \Pr\left(\max_{1 \leqq j \leqq k} S_{n_j} < (a - \epsilon) n^{1/2}\right) + O(n^{-1/16}). \tag{16}$$

3. The estimate of Bergström. Write

$$V_i^{(p)} = \sum_{j \equiv p \pmod{n_1},\, j \leqq n_i} X_j$$

and consider the k-dimensional vector

$$V^{(p)} = (V_1^{(p)}, \cdots, V_k^{(p)});$$

we have

$$\sum_{p=1}^{n_1} V^{(p)} = (S_{n_1}, \cdots, S_{n_k}). \tag{17}$$

Now

$$E(|V_i^{(p)} - E(V_i^{(p)})|^3) \leq Ak^2 \sum_{j \equiv p \pmod{n_1}, j \leq n_i} E(|X_j - \mu|^3).$$

$$E(|V_i^{(p)} - E(V_i^{(p)})|^2) = \sum_{j \equiv p \pmod{n_1}, j \leq n_i} 1.$$

Let $F(x_1, \cdots, x_k)$ be the distribution of (17) and let $\phi(x_1, \cdots, x_k)$ be that of the k-dimensional normal distribution having the same first and second order moments as F. According to the result of Bergström,[4] we have

$$|F - \phi| < \frac{C(k)}{n_1^{1/2}} \max_p \sum_{i=1}^{k} \frac{E(|V_i^{(p)} - E(V_i^{(p)})|^3)}{E(|V_i^{(p)} - E(V_i^{(p)})|^2)}$$
$$< \frac{C(k)}{n^{1/2}} A k^{5/2} \max_{1 \leq j \leq n} \gamma_j < A \frac{C(k) k^{5/2}}{n^{1/2}}.$$

The $C(k)$ in this result can be taken to be $Ak^{7/2} \log k$, hence

$$|F - \phi| \leq A \frac{k^6 \log k}{n^{1/2}} \leq A \frac{\log n}{n^{1/26}}. \tag{18}$$

Remembering that

$$F(x, \cdots, x) = \Pr\left(\max_{1 \leq j \leq k} S_{n_j} \leq x\right),$$

we obtain from (16) and (18) the following inequality

$$\Pr(\bar{S}_n < an^{1/2}) \geq \phi((a - \epsilon)n^{1/2}, \cdots, (a - \epsilon)n^{1/2}) + An^{-1/26} \log n. \tag{19}$$

On the other hand, we have

$$\Pr(\bar{S}_n < an^{1/2}) \leq \Pr\left(\max_{1 \leq j \leq k} S_{n_j} < an^{1/2}\right)$$
$$= \phi(an^{1/2}, \cdots, an^{1/2}) + An^{-1/26} \log n, \tag{20}$$

where the last equation follows from (18).

4. **The final result.** As before let $k \sim n^{1/13}$. Let $\phi_k(x)$ denote $\phi(x_1, \cdots, x_k)$ with $x_1 = \cdots = x_k = x$. Then we have obtained in (19) and (20)

$$\begin{aligned} \phi_k((a-\epsilon)n^{1/2}) - \frac{A \log n}{n^{1/26}} &\leqq \Pr(\bar{S}_n < an^{1/2}) \\ &\leqq \phi_k(an^{1/2}) + \frac{A \log n}{n^{1/26}}. \end{aligned} \tag{21}$$

Applying these inequalities to the Bernoullian case in §1, and considering the case $d \neq 0$ for the sake of definitiveness, we obtain

$$\begin{aligned} \phi_k((a-\epsilon)n^{1/2}) - \frac{A \log n}{n^{1/26}} &\leqq I(a) + O\left(\frac{1}{n^{1/2}}\right) \\ &\leqq \phi_k(an^{1/2}) + \frac{A \log n}{n^{1/26}}. \end{aligned}$$

Consequently we have

$$I(a-\epsilon) - \frac{A \log n}{n^{1/26}} \leqq \Pr(\bar{S}_n < an^{1/2}) \leqq I(a+\epsilon) + \frac{A \log n}{n^{1/26}}. \tag{22}$$

Since it is easy to see that

$$I(a) - I(a \pm \epsilon) = O(\epsilon) = O\left(\frac{(\log n)^{1/2}}{n^{1/26}}\right), \tag{23}$$

we obtain from (22) and (23)

$$\Pr(\bar{S}_n < an^{1/2}) = I(a) + O\left(\frac{\log n}{n^{1/26}}\right).$$

5. **The case** $\mu_k = \mu \neq 0$. We have left out this case in order not to interrupt the argument. We shall now briefly indicate the transformations needed here. Instead of making the variance one in this case we shall follow the more usual way and assume that

$$X_\nu = \begin{cases} +1 & \text{with probability } p, \\ -1 & \text{with probability } q \end{cases}$$

where

$$p - q = \mu.$$

Then

$$E(X_\nu) = \mu, \qquad E(X_\nu - \mu)^2 = 4pq.$$

In formula (3) we write

$$\begin{aligned} b &= n\mu + a(4pqn)^{1/2}, \\ r &= nq + s = nq + t(pqn)^{1/2} \end{aligned} \tag{24}$$

where c is a constant and t varies between $-(p^{-1}qn)^{1/2}$ and $-a$.

We find, upon using Stirling's formula, that if $t = O(n^{1/4})$

$$\frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r p^b = \left(\frac{p\mu^2}{2\pi qn}\right)^{1/2} \exp\left(-\frac{(\mu t - 2aa)^2}{2}\right) \cdot\left(1 + O\left(\frac{t}{n^{1/2}}\right)\right)$$

and if $n^{1/4} = O(t)$,

$$\frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r p^b = O(\exp(-An^{1/2})).$$

Thus we can write, t running through the values defined by (24) as r runs through integers,

$$\begin{aligned} &\sum_{r=0}^{(n-b)/2} \frac{b(b+2r-1)!}{r!(b+r)!}(pq)^r p^b \\ &= \sum_{r \leq n^{3/4}} + \sum_{r > n^{3/4}} = \sum_{-n^{1/4} < t < -a} \left(\frac{p\mu^2}{2\pi qn}\right)^{1/2} \exp\left(-\frac{(\mu t - 2qa)^2}{2}\right) \\ &\quad \cdot\left(1 + O\left(\frac{t}{n^{1/2}}\right)\right) + O(n \exp(-An^{1/2})) \\ &= \frac{1}{(2\pi)^{1/2}} \int_a^\infty e^{-x^2/2} dx + O\left(\frac{1}{n^{1/2}}\right). \end{aligned}$$

PRINCETON UNIVERSITY

COMPOSITION OF BINARY QUADRATIC FORMS

GORDON PALL

1. **Introduction.** The composition of quadratic forms, as originated by Gauss,[1] is based on bilinear transformations. Thus, if a quadratic form $f_1 = \sum a_{ij} x_i x_j$ is expressible as a product of two forms $f_2(y_1, \cdots, y_n)$ and $f_3(z_1, \cdots, z_n)$ by means of a bilinear substitution $x_\alpha = \sum a_{\alpha\beta\gamma} y_\beta z_\gamma$, and if the determinants of order n in the n-by-n^2 matrix $(a_{\alpha\beta\gamma})$ are relative prime, f_1 is called the compound, or product under composition, of f_2 and f_3. There are few examples of composition except for quadratic forms, and there it is confined to certain classes of forms in two, four, and eight variables.

Now there is evidence that quadratic forms not admitting composition have certain properties akin to those which are most easily established in the case of binaries by use of composition. This suggests that the use of bilinear transformations is too restrictive, and that other useful definitions of composition may be possible. Dirichlet[2] did in fact base a theory of composition of binary quadratic forms on the representation of numbers. However, bilinear transformations appear (loc. cit., p. 159, formula (5)) in his proof of the uniqueness of the product class. Again, Brandt[3] gave a theory of composition for binaries, based on integral linear transformations of a Grundform into multiples of the binary quadratic forms of a given discriminant. The extension of this to n variables appears to be difficult.

In this article we define a compound of binary quadratic forms in a manner basically related to that of Dirichlet; and prove the uniqueness of the product class without using bilinear transformations. We also show that the basic lemma (due to Gauss) can be extended to quadratic forms in n variables. All the usual consequences of composition of binary quadratic forms can be derived from our present approach, some of them more simply. But we shall not enter into these details here.

2. **Gauss's lemma and its generalization.** The basic lemma of

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[1] C. F. Gauss, *Disquisitiones Arithmeticae*, 1801, Articles 235–249 et. seq.

[2] G. L. Dirichlet, *De formarum binarum secundi gradus compositione*, Berlin, 1851. Reprinted in Journal fur Mathematik vol. 47 (1854) pp. 155–160; Werke, II, 1897, pp. 105–114. French translation, Journal de Mathematik (2) vol. 4 (1859) pp. 389–398. Also, Dirichlet-Dedekind, *Zahlentheorie*, Supplement X, §§145–9, 1871, 1879, 1894.

[3] H. Brandt, Journal für Mathematik vol. 150 (1919) pp. 1–46.

Gauss gives a criterion for equivalence of binary quadratic forms. Let $[a, b, c]$ denote the real form $ax^2+bxy+cy^2$. If $[a, b, c]$ is carried into $[a', b', c']$ by the unimodular transformation

$$(1) \qquad x = \alpha x' + \beta y', \qquad y = \gamma x' + \delta y', \qquad \alpha\delta - \beta\gamma = 1,$$

then $c' = a\beta^2+b\beta\delta+c\delta^2$, and

$$(2) \qquad a' = a\alpha^2 + b\alpha\gamma + c\gamma^2, \qquad b' = 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta.$$

From (2) it is easily seen that

$$(3) \qquad \begin{array}{l} \{a\alpha + 2^{-1}(b + b')\gamma\}/a' \quad \textit{and} \\ \{2^{-1}(b - b')\alpha + c\gamma\}/a' \quad \textit{are integral;} \end{array}$$

indeed these expressions are equal, respectively, to δ and $-\beta$. Gauss's lemma is as follows:

LEMMA 1. *The real forms* $[a, b, c]$ *and* $[a', b', c']$ *with* $a' \neq 0$ *are equivalent if and only if their discriminants are equal and there exist two integers* α *and* γ *satisfying* (2_1) *and* (3).

Indeed, if the expressions in (3) are denoted by δ and $-\beta$, then $\alpha\delta+\gamma(-\beta) = (a\alpha^2+b\alpha\gamma+c\gamma^2)/a' = 1$, and $0 = a'(\beta\delta+\delta(-\beta)) = a\alpha\beta +2^{-1}b(\alpha\delta+\beta\gamma)+c\gamma\delta-2^{-1}b'$, in agreement with (2). Hence the transformation (1) replaces $[a, b, c]$ by $[a', b', c'']$, where $b'^2-4a'c'' = b'^2-4a'c'$, $c''=c'$.

To extend this criterion to n-ary quadratic forms, consider a symmetric, nonsingular matrix A of order n. Apply to A the unimodular transformation of matrix $T = (T_1\, T_2)$, where T_1 has $n-1$ columns, and T_2 one column, and obtain

$$(4) \qquad B = T'AT = \begin{bmatrix} B_1 & K' \\ K & B_2 \end{bmatrix}, \qquad \text{where} \quad B_1 = T_1'AT_1,\ K = T_2'AT_1,\ B_2 = T_2'AT_2.$$

Thus, if A and B are equivalent matrices, then the leading minor matrix B_1 of order $n-1$ of B is represented primitively by A, the representation being T_1. Also, if $S' = T^{-1}$, $S = (S_1\, S_2)$ can be partitioned similarly to T, with S_2 a single column. It should be noted that S_2 is uniquely determined by T_1 alone, since S_2 is the column of cofactors in T of its last column. Also,

$$T_1'S_1 = I_1, \quad T_1'S_2 = 0, \quad T_2'S_1 = 0, \quad T_2'S_2 = 1,$$

where I_1 is the identity matrix of order $n-1$. Finally, notice that

$$T'AT_1 = \begin{bmatrix} B_1 \\ K \end{bmatrix}, \quad \text{hence} \quad AT_1 = S_1B_1 + S_2K.$$

Thus $(AT_1 - S_2K)B_1^{-1}$ is an integral matrix. This is the analogue of condition (3) above. We are now ready to state and prove the generalization of Lemma 1:

THEOREM 1. *Let A and B denote symmetric nonsingular matrices of order n, of equal determinants. Partition B as follows, with B_1 of order $n-1$ and B_2 a number:*

$$B = \begin{bmatrix} B_1 & K' \\ K & B_2 \end{bmatrix}. \tag{5}$$

Let T_1 (with n rows, $n-1$ columns) be an integral matrix such that $B_1 = T_1'AT_1$. Denote by S_2 the column vector of cofactors consisting of the minor determinants of order $n-1$ of T_1 taken with appropriate signs. Assume that $(AT_1 - S_2K)B_1^{-1}$ is an integral matrix. Then A and B are equivalent, and it is possible to construct T_2 so that $(T_1\ T_2)$ is a unimodular transformation of A into B.

PROOF. Set $(AT_1 - S_2K)B_1^{-1} = S_1$. It is not clear whether $(S_1\ S_2)$ is then unimodular. However, the equation $AT_1 = S_1B_1 + S_2K$ yields $T_1'AT_1 = T_1'S_1B_1 + T_1'S_2K$, $B_1 = T_1'S_1B_1$, and since B_1 is assumed to be nonsingular, $T_1'S_1 = I_1$. This implies that T_1 is primitive, that is, the minor determinants of order $n-1$ of T_1 are relatively prime. Hence, the most general integral matrix R_1 (with n rows and $n-1$ columns) satisfying $T_1'R_1 = I_1$ is given by $R_1 = S_1 + S_2H$, where H is an arbitrary integral matrix of one row and $n-1$ columns. Indeed, if $R_1 - S_1 = X$, then each column x_i of X is a solution of $T_1'x_i = 0$; since T_1 is primitive, this solution is $x_i = S_2h_i$ where h_i is an integer. Since T_1 is primitive, there exists a column T_2 such that $(T_1\ T_2)$ is unimodular, and $(T_1\ T_2)^{-1} = (R_1\ S_2)'$, where R_1 is thus a solution of $T_1'R_1 = I_1$. Hence $R_1 = S_1 + S_2H$, for some integral H, and $(S_1\ S_2)' = (T_1\ T_1H' + T_2)$. Accordingly, we can rename $T_1H' + T_2$ as T_2, and have $(S_1\ S_2)' = (T_1\ T_2)^{-1}$. Then $T_2'AT_1 = T_2'(S_2K + S_1B_1) = K$. The value of B_2 is fixed by the equality of the determinants of A and B. The theorem follows.

This theorem opens the way to a possible extension of the methods of this article to n variables.

3. **Preliminary lemmas.** As a first application of Lemma 1, we have the following lemma.

LEMMA 2. *Let a, a_1, a_2, b, c be integers, $aa_2 \neq 0$. Then $[a_1, b, aa_2c] \sim [a_2, b, aa_1c]$ implies that $[aa_1, b, a_2c] \sim [aa_2, b, a_1c]$. If $(a, b, c) = 1$,* *then the equivalence of the latter two forms implies that of the former.*

PROOF. By Lemma 1, the equivalence of the first two forms is tantamount to the existence of integers α and ν satisfying

$$(6)\qquad a_2 = a_1\alpha^2 + b\alpha\nu + aa_2c\nu^2,\ a_1\alpha + b\nu \equiv 0,\ aa_2c \equiv 0 \pmod{a_2};$$

and the equivalence of the last two forms amounts to

$$(7)\quad aa_2 = aa_1\alpha^2 + b\alpha\gamma + a_2c\gamma^2,\ aa_1\alpha + b\gamma \equiv 0,\ a_2c\gamma \equiv 0 \pmod{aa_2},$$

with some integers α, γ. If (6) holds, defining $\gamma = a\nu$ yields (7). If (7) holds, then $a|b\gamma$ and $a|c\gamma$, and hence if $(a, b, c) = 1$, $a|\gamma$, and $\nu = \gamma/a$ is an integer.

The importance of this lemma may be seen from

LEMMA 3. *For any primitive binary quadratic forms $\phi_1, \cdots, \phi_q$ of the same discriminant d, there can be found integers $b, s, a_1, \cdots, a_q$ such that*

$$(8)\qquad \phi_i \sim [a_i, b, sa_1 \cdots a_q/a_i] \qquad (i = 1, \cdots, q).$$

Furthermore, these integers can be chosen so that $a_1, \cdots, a_q$, and $2d$ are coprime in pairs.

PROOF. A primitive form represents primitively integers prime to any assigned integer, and any integer primitively represented can be taken to be the first coefficient of an equivalent form. Choose for a_1 any integer primitively represented by ϕ_1 and prime to $2d$; for a_2 any integer primitively represented by ϕ_2 and prime to $2a_1d$; $\cdots$; and, finally, for a_q any integer primitively represented by ϕ_q and prime to $2a_1 \cdots a_{q-1}d$. Then the ϕ_i are equivalent to respective forms $[a_i, b_i, c_i]$ $(i=1, \cdots, q)$. By the Chinese Remainder Theorem, an integer b can be chosen to satisfy

$$(9)\qquad b \equiv b_i \pmod{2a_i} \qquad (i = 1, \cdots, q).$$

Then $\phi_i \sim [a_i, b, k_i]$, where $d = b^2 - 4a_ik_i$ $(i = 1, \cdots, q)$, and hence since $a_1, \cdots, a_q, 2$ are coprime in pairs, $d = b^2 - 4a_1 \cdots a_q r$, with r an integer.

4. Composition of binary quadratic forms. By the preceding lemma, there can be constructed within any two primitive classes C_1 and C_2, not necessarily distinct, of binary quadratic forms of the same discriminant, *united* forms of the type

$$(10)\qquad \phi_1 = [a_1, b_1, a_2c_1] \text{ in } C_1, \qquad \phi_2 = [a_2, b_1, a_1c_1] \text{ in } C_2.$$

This is easily seen to be true when the classes are not primitive, if merely their divisors are coprime. The divisors are integers, t_1 and t_2, such that ϕ_1/t_1 and ϕ_2/t_2 are primitive forms. The *product*, or *compound*, of the forms ϕ_1 and ϕ_2 will be defined to be the form $[a_1a_2, b_1, c_1]$, and will be denoted by $\phi_1\phi_2$. The significance of this definition lies in the fact that, when t_1 and t_2 are coprime, *it defines a unique product class.*

THEOREM 2. *Let the divisors of the classes* C_1 *and* C_2 *of discriminant* d *be assumed coprime. Then, for all choices of united forms* (10), *the form* $[a_1a_2, b_1, c_1]$ *belongs to a unique class.*

PROOF. Consider, besides the forms ϕ_1 and ϕ_2, a second pair, $\phi_3=[a_3, b_2, a_4c_2]$ in C_1, $\phi_4=[a_4, b_2, a_3c_2]$ in C_2. It is to be proved that $\phi_1\phi_2=[a_1a_2, b_1, c_1]\sim[a_3a_4, b_2, c_2]=\phi_3\phi_4$. The difficulty in applying Lemma 2 immediately lies in the circumstance that a_1a_2 and a_3a_4 may not be coprime, and hence that it may not be possible to obtain equal middle coefficients by merely adding multiples of $2a_1a_2$ to b_1 and $2a_3a_4$ to b_2. To circumvent this difficulty, we introduce intermediate forms, with coefficients prime to both. Thus, an integer a_5 can be chosen, which is primitively represented by C_1, and such that a_5/t_1 is prime to $2a_1a_2a_3a_4$; and an integer a_6, primitively represented by C_2, such that a_6/t_2 is prime to $2a_1a_2a_3a_4a_5$. Construct $\phi_5=[a_5, b_3, c_3]$ in C_1, $\phi_6=[a_6, b_4, c_4]$ in C_2. Since $2a_1a_2$, a_5/t_1, and a_6/t_2 are coprime in pairs, an integer b_5 can be found to satisfy

$$(11)\quad b_5\equiv b_1 \bmod 2a_1a_2,\ b_5/t_1\equiv b_3/t_1 \bmod 2a_5/t_1,\ b_5/t_2\equiv b_4/t_2 \bmod 2a_6/t_2.$$

Then $d-b_5^2$ is divisible by each of $4a_1a_2$, a_5/t_1, and a_6/t_2, and hence

$$d-b_5^2=4a_1a_2a_5a_6c_5/t,\qquad t=t_1t_2,\quad \text{with } c_5 \text{ integral.}$$

Hence $\phi_1\sim[a_1, b_5, a_2a_5a_6c_5/t]\sim[a_5, b_5, a_1a_2a_6c_5/t]$, $\phi_1\phi_2\sim[a_1a_2, b_5, a_5a_6c_5/t]$. By Lemma 2, $\phi_1\phi_2\sim[a_5a_2, b_5, a_1a_6c_5/t]$. Similarly, since $\phi_2\sim\phi_6$, the last displayed form is equivalent to $[a_5a_6, b_5, a_1a_2c_5/t]$. We now choose an integer b_6 such that

$$b_6/t\equiv b_2/t \bmod 2a_3a_4/t,\qquad b_6/t\equiv b_5/t \bmod 2a_5a_6/t.$$

Then ϕ_3, ϕ_4, ϕ_5, ϕ_6, $\phi_3\phi_4$, and $\phi_5\phi_6$ are equivalent to new forms with a common middle coefficient b_6, respective first coefficients a_3, a_4, a_5, a_6, a_3a_4, a_5a_6; while the last coefficients are determined from $d=b_6^2-4a_3a_4a_5a_6c_6/t$, with c_6 an integer. By Lemma 2, $\phi_3\sim\phi_5$, $\phi_3\phi_6\sim\phi_5\phi_6$; $\phi_4\sim\phi_6$, $\phi_3\phi_4\sim\phi_3\phi_6\sim\phi_5\phi_6\sim\phi_1\phi_2$. This completes the proof.

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SEMILATTICES AND A TERNARY OERATION IN MODULAR LATTICES

S. A. KISS

Before discussing the subject matter proper it is necessary to introduce the following:[1]

LEMMA 1. *The inequality*

$$(1)\quad \{(x \cap (y \cap (v \cup z))\} \cup (v \cap z) \subseteq v \cup (y \cap (x \cup z)) \cup (x \cap z)$$

is identically satisfied in any lattice.

PROOF. $x \cap y \cap (v \cup z) \subseteq x \cap y \subseteq (x \cup z) \cap y \subseteq v \cup (y \cap (x \cup z)),$

$$v \cap z \subseteq v \subseteq v \cup (y \cap (x \cup z))$$

and from these two inequalities follows

$$\begin{aligned}(x \cap y \cap (v \cup z)) \cup (v \cap z) &\subseteq v \cup (y \cap (x \cup z)) \\ &\subseteq v \cup (y \cap (x \cup z)) \cup (x \cap z).\end{aligned}$$

For purposes of facility of expression the concept of *semilattice* is here introduced following Klein-Barmen [1]:[2]

DEFINITION 1. A semilattice L_s is a partially ordered system in which a relation $x\sigma y$ is defined which satisfies

S1: For all x, $x\sigma x$,

S2: If $x\sigma y$ and $y\sigma x$, then $x=y$,

S3: If $x\sigma y$ and $y\sigma z$, then $x\sigma z$,

and in which any two elements x and y have a greatest lower bound or meet xmy.

It then follows that xmy or any binary operation xoy which is closed, idempotent, commutative and associative defines, by means of the convention that $x\sigma y$ means $xmy=x$ or $xoy=x$, a semilattice L_s in which xmy or xoy is the greatest lower bound of x and y.

LEMMA 2. *The ternary operation*

$$(2)\quad [x, t, y] = (x \cap (t \cup y)) \cup (t \cap y) = (x \cup (t \cap y)) \cap (t \cup y)$$

on the elements of a modular lattice L is closed and is an idempotent and

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[1] The author is indebted to Garrett Birkhoff for the proof of Lemma 1, and for helpful criticism.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

associative operation for a constant t. The expression $[x, t, y]$, for $t =$ const., is said to determine the "operational plane" t.

(3) The *idempotent law*

$$[x, t, x] = x \text{ for all } x \text{ and } t$$

holds because of the absorption law in L:

$$[x, t, x] = (x \cap (t \cup x)) \cup (t \cap x) = x \cup (t \cap x) = x.$$

The proof of the *associative law* is somewhat longer, proceeding as follows:

Expanding the expression of the associative law

$$[[x, t, y], t, z] = [x, t, [y, t, z]] \tag{4}$$

one obtains

$$\begin{aligned}\{\{x \cap (t \cup y)) \cup (t \cap y)\} \cap (t \cup z)\} \cup (t \cap z) \\ = x \cap \{t \cup \{(y \cap (t \cup z)) \cup (t \cap z)\}\} \\ \cup \{t \cap \{(y \cap (t \cup z)) \cup (t \cap z)\}\}.\end{aligned}$$

Some of the expressions on the right-hand side may be simplified by employing the absorption law and Dedekind's modular identity.

$$\begin{aligned}t \cup \{(y \cap (t \cup z)) \cup (t \cap z)\} &= t \cup (t \cap z) \cup (y \cap (t \cup z)) \\ &= t \cup (y \cap (t \cup z)) = (t \cup y) \cap (t \cup z), \\ t \cap \{(y \cap (t \cup z)) \cup (t \cap z)\} &= t \cap (t \cup z) \cap (y \cup (t \cap z)) \\ &= t \cap (y \cup (t \cap z)) = (t \cap y) \cup (t \cap z).\end{aligned}$$

Therefore,

$$\begin{aligned}\{\{(x \cap (t \cup y)) \cup (t \cap y)\} \cap (t \cup z)\} \cup (t \cap z) \\ = \{x \cap (t \cup y) \cap (t \cup z)\} \cup \{(t \cap y) \cup (t \cap z)\}.\end{aligned}$$

Putting $X = x \cap (t \cup y)$, $Y = t \cup z$, $U = t \cap y$, $V = t \cap z$ where $V \subseteq Y$ and $U \subseteq t \subseteq Y$, the above formula becomes

$$\{(X \cup Y) \cap Y\} \cup V = (X \cap Y) \cup (U \cap V)$$

and, in view of $(X \cup U) \cap Y = (U \cup X) \cap Y = U \cup (X \cap Y)$,

$$\{U \cup (X \cap Y)\} \cup V = (X \cap Y) \cup (U \cup V)$$

which is an identity, thus concluding the proof.

An immediate consequence, then, is:

THEOREM 1. *The commutative "products" $[x, t, y]$ for a constant t,* *that is, those which satisfy* $[x, t, y]=[y, t, x]$, *form a semilattice in which the element* $[x, t, y]$ *is the greatest lower bound of* x *and* y.

THEOREM 2. *Whether commutative or not, the "product"* $[x, t, y]$ *is always determined for a modular lattice L and in the ternary operational system thus obtained any two operational planes* u *and* v *satisfy the following identical equation, namely,*

$$[[x, u, [y, v, z]], v, [y, u, z]] = [[x, v, [y, u, z]], u, [y, v, z]]. \tag{5}$$

To prove this formula, it is expanded by means of (2) above and is subsequently shown to be an identity. Putting

$$\begin{aligned} Y &= u \cup (y \cap (v \cup z)) \cup (v \cap z), \\ Z &= u \cap ((y \cap (v \cup z)) \cup (v \cap z)), \\ U &= v \cup (y \cap (u \cup z)) \cup (u \cap z), \\ V &= v \cap ((y \cap (u \cup z)) \cup (u \cap z)) \end{aligned}$$

where $Z \subseteq Y$, $V \subseteq U$ and, in view of Lemma 1, $Z \subseteq U$ and $V \subseteq Y$, the equation (5) becomes

$$(((x \cap Y) \cup Z) \cap U) \cup V = (((x \cap U) \cup V) \cap Y) \cup Z$$

or, in view of Dedekind's modular identity,

$$(V \cup (x \cap Y) \cup Z) \cap U = (Z \cup (x \cap U) \cup V) \cap Y.$$

Since $Z \subseteq U$ and $V \subseteq U$ give $Z \cup V \subseteq U$ and, similarly, $Z \subseteq Y$ and $V \subseteq Y$ give $Z \cup V \subseteq Y$, the last equation becomes, in view of Dedekind's modular identity,

$$Z \cup V \cup (x \cap Y \cap U) = Z \cup V \cup (x \cap U \cap Y).$$

(5) is thus proven to be an identity.

EXAMPLE. Designating the elements of the nondistributive modular lattice L_5 by 0 (least), a, b, c, e (greatest), the commutative "products" of the operational plane $[x, a, y]$ define a semilattice which is not a lattice, similar remarks applying to $[x, b, y]$ and $[x, c, y]$. The noncommutative products, namely, $[b, a, c]=b$, $[c, a, b]=c$, $[a, b, c]=a$, $[c, a, b]=c$, $[a, c, b]=a$, $[b, c, c]=b$ do not belong to the semilattices.

Being partially ordered systems, semilattices may be represented by diagrams. In L_5 links are preserved in all semilattices defined by (2) with constant t; it is the author's conjecture that this rule holds for the semilattices defined in any modular lattice.

When a lattice is distributive in addition to being modular, the expression (2) becomes

$$[x, t, y] = (x \cap t) \cup (t \cap y) \cup (y \cap x). \tag{6}$$

This is the ternary operation (x, t, y) which was independently introduced by Grau[3] [2] for Boolean algebras and by Birkhoff and Kiss [3] for distributive lattices in general.

It is obvious from the expression (2) of $[x, t, y]$ that $[x, t, y] = [x, y, t]$; on the other hand the above example shows that, in some cases at least, $[x, t, y] \neq [y, t, x]$ and also $[x, t, y] \neq [t, x, y]$.

Complementation in distributive lattices has been defined by Birkhoff and Kiss [3] and can now be extended to modular lattices by means of the following:

DEFINITION 2. The elements x, x' of a modular lattice L are called "strictly complementary" if and only if

$$[x, t, x'] = t \text{ for all } t. \tag{7}$$

THEOREM 3. *Strict complementation in a modular lattice is unique.*

PROOF. If x has two complements, x' and x'', then $x'' = [x, x'', x'] = [x, x', x''] = x'$.

THEOREM 4. *The* 0 *(least) and* e *(greatest) elements of a modular lattice are always strictly complementary; furthermore, the* $[x, 0, y]$ *and* $[x, e, y]$ *planes of the ternary lattice give the* $x \cap y$ *and* $x \cup y$ *operations, respectively.*

PROOF. $[0, t, e] = (0 \cap (t \cup e)) \cup (t \cap e) = 0 \cup t = t,$

$$[x, 0, y] = (x \cap (0 \cup y)) \cup (0 \cap y) = (x \cap y) \cup 0 = x \cap y,$$

$$[x, e, y] = (x \cap (e \cup y)) \cup (e \cap y) = (x \cap e) \cup y = x \cup y.$$

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[3] Grau uses the notation $x^t y$ for (x, t, y).

A NOTE ON HOMOMORPHIC MAPPINGS OF QUASIGROUPS INTO MULTIPLICATIVE SYSTEMS

GRACE E. BATES AND FRED KIOKEMEISTER

The study of normality theories for general quasigroups and loops became productive when that study was restricted to a study of homomorphisms of quasigroups on quasigroups.[1] The existence of a loop with homomorphic image which is not a quasigroup is then pertinent to this study. In this note we exhibit such a loop and show that certain properties of our example are necessary. In particular, if the homomorphic image of a quasigroup is a finite or an associative multiplicative system, this image is a quasigroup. A deeper statement is that of Theorem 4—finiteness of the kernel of a loop homomorphism into a multiplicative system is a sufficient condition for the image of this homomorphism to be a loop.

We make use of the following definitions: A *multiplicative system* M is a nonvacuous set of elements $a, b, c, \cdots$ such that to each ordered pair of elements a, b, there corresponds in M a uniquely defined element ab called the product. If the product is defined for a (possibly vacuous) subset of the set of ordered pairs, then M is called a *partial multiplicative system*. If M_1 and M_2 are partial multiplicative systems, M_1 is said to be *imbedded* in M_2 if $M_1 \subseteq M_2$ and products in M_2 coincide with those in M_1 whenever they are defined in M_1. A partial multiplicative system has an *identity element* e if the products ea and ae are defined for each element a and $ea = ae = a$. A mapping of a multiplicative system M on a multiplicative system $\overline{M}$ which preserves products is called a *homomorphism* of M. A multiplicative system G in which the equations $ax = b$ and $ya = b$ have unique solutions for each pair of elements a, b in G is called a *quasigroup*. A *loop* is a quasigroup with identity element.

THEOREM 1. *If J is a partial multiplicative system, then J can be imbedded in a multiplicative system M which has the additional properties*:

(1) *If a and b are elements of M, there is at least one x and at least one y in M such that $ax = b$ and $ya = b$.*

(2) *If $x \neq y$ but $ax = ay$ (or $xa = ya$) in M, then a, x, y, and $ax = ay$ (or $xa = ya$) are all in J.*

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[1] Cf. [1, p. 513]; [2, p. 450]; [3, p. 769]; [4, Theorem 10A]; and [5]. (Numbers in brackets refer to the bibliography.)

(Note that (1) asserts that all equations are solvable in M, while (2) states that the solutions are unique except possibly for those equations possessing in J more than one solution. Hence, in particular, if J has both cancellation laws, M is a quasigroup.)

We define first an elementary extension K of a partial multiplicative system J as follows: Let K consist of all elements of J together with new elements z_{ab}, x_{ab}, y_{ab} defined in the following manner:

Each ordered pair of elements a, b in J for which ab is not in J gives rise to an element z_{ab} in K, the element z_{ab} being uniquely defined by the relation $z_{ab}=ab$ and the requirement that $z_{ab}=z_{cd}$ if and only if $a=c$ and $b=d$. Similarly, to each ordered pair of elements in J for which there is no x in J satisfying $xa=b$, there corresponds an element x_{ab} in K for which $(x_{ab})a=b$ is the defining relation, and to each ordered pair of elements a, b in J for which there is no y in J satisfying $ay=b$, there corresponds an element y_{ab} in K defined by the relation $a(y_{ab})=b$. Again, $x_{ab}=x_{cd}$ or $y_{ab}=y_{cd}$ if and only if $a=c$ and $b=d$.

The set K is a partial multiplicative system having the following properties:

(i) If a, b is an ordered pair of elements in J, ab is a uniquely defined element of K.

(ii) If a and b are elements of J, there is at least one x and at least one y in K such that $ax=b$ and $ya=b$.

(iii) If $x\neq y$, but $ax=ay$ (or $xa=ya$) in M, then a, x, y, and $ax=ay$ (or $xa=ya$) are all in J.

Consider the chain of partial multiplicative systems

$$J = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_i \subseteq J_{i+1} \subseteq \cdots$$

where J_{i+1} is an elementary extension of J_i for $i=1, 2, \cdots$. Let

$$M = \bigcup J_i$$

be the set-theoretic sum[2] of the J_i.

If a and b are elements of M, there exists an integer k such that a and b are elements of J_k, and therefore there exists a unique element ab in J_{k+1}. Thus to each ordered pair of elements a, b in M there corresponds a product ab in M. Furthermore, if a and b are in J_k, then J_{k+1} contains elements x and y such that $ax=b$ and $ya=b$.

If $x\neq y$, but $ax=ay$ in M, then a, x, y, and $ax=ay$ all lie in some J_k and $x\neq y$ in J_k. It follows that a, x, y, and $ax=ay$ are elements of J_i, $i=k-1$, $k-2, \cdots, 0$; that is, these elements lie in J. This proves the theorem.

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[2] Clearly M is countable if J is countable.

COROLLARY 1. *If J is a partial multiplicative system with identity element e, then J may be imbedded in a multiplicative system M with identity e, having properties* (1) *and* (2) *of the theorem.*

For, in the construction of the elementary extension K of J, we require now only the additional condition imposed on the symbols z_{ab}, x_{ab}, and y_{ab} that they satisfy the relations $(z_{ab})e = e(z_{ab}) = z_{ab}$, and so forth.

Suppose, now, that J is commutative (that is, ab is in J if and only if ba is in J, and $ab = ba$). Then if ab is undefined in J, ba is also undefined, and we may let $z_{ab} = ab = ba$ in K. Similarly, if there is in J no solution x of the equation $xa = b$, there is no solution of $ay = b$, and we may define $s_{ab} = x_{ab} = y_{ab}$ in K by the relations $(s_{ab})a = a(s_{ab}) = b$. Hence we have the following corollary.

COROLLARY 2. *If J is a partial multiplicative system which is commutative, then J may be imbedded in a commutative multiplicative system M having properties* (1) *and* (2) *of the theorem.*

We shall employ Theorem 1 in constructing an example as follows:

Let J be the set consisting of the four elements β_1, β_2, β_3, β_4 with the following products defined:

$$\beta_1\beta_k = \beta_k\beta_1 = \beta_k, \qquad k = 1, 2, 3, 4,$$

$$\beta_2\beta_2 = \beta_2\beta_4 = \beta_4\beta_2 = \beta_4\beta_4 = \beta_3.$$

It is to be noted that J is a commutative partial multiplicative system with identity element β_1. The equation $\beta_2 x = \beta_3$ has two distinct solutions in J, and thus J cannot be imbedded in a quasigroup.

Let J be imbedded in a system M, as in Theorem 1, with elements β_k, $k = 1, 2, 3, \cdots$. By Corollary 1, β_1 may be taken to be the identity element of M. Then M is a multiplicative system with the following properties:

(1) There exist positive integers h and k such that for each pair β_i and β_m, $\beta_i\beta_h = \beta_m$ and $\beta_k\beta_i = \beta_m$.

(2) If $\beta_i\beta_h = \beta_i\beta_k$, or if $\beta_h\beta_i = \beta_k\beta_i$, where $h < k$, then $h = 2$, $k = 4$, and $i = 2$ or $i = 4$.

Let A be a countably infinite loop[3] with elements α_1, α_2, α_3, $\cdots$ where α_1 is the identity of A. We construct a system G whose elements are the ordered pairs of elements (β_i, α_j), $i, j = 1, 2, 3, \cdots$, with β_i in M and α_j in A.[4] The product of two elements in G is defined by:

---

[3] A may be a group.

[4] A similar construction has been employed by Bruck; see [4, p. 166].

$$\text{(P)} \qquad (\beta_i, \alpha_j)(\beta_h, \alpha_k) = (\beta_i\beta_h, \alpha_n)$$

where the subscript n of α_n is determined in the following way: Let α_q be the uniquely determined element $\alpha_j\alpha_k$ in A; then in (P)

(1) If $i=h=2$, or if $i=h=4$, let $n=2q-1$,
(2) If $i=2$, $h=4$, or if $i=4$, $h=2$, let $n=2q$,
(3) In all other cases, let $n=q$.

It is easily verified that G is a loop with identity (β_1, α_1). The set H of elements (β_1, α_i), $i=1, 2, 3, \cdots$, is a loop isomorphic with A under the correspondence

$$(\beta_1, \alpha_i) \leftrightarrow \alpha_i, \qquad i = 1, 2, 3, \cdots .$$

The definition of product in G implies that the correspondence

$$(\beta_i, \alpha_j) \rightarrow \beta_i, \qquad i, j = 1, 2, 3, \cdots ,$$

is a homomorphism of G on M. The kernel of this homomorphism is H.

By Corollary 2 of Theorem 1, M may be chosen to be commutative. If, furthermore, A is commutative, then G will have the same property. Commutativity, then, is not a sufficient condition that a quasigroup have only quasigroup images.

It is to be noted that in the example M is neither finite nor associative, and H is not finite. We shall show that these are necessary properties of the example.

In general, let the quasigroup G be homomorphic to the multiplicative system G'. If the elements of G are $a, b, c, \cdots$, let $a', b', c', \cdots$ be their corresponding images in G'. We have immediately the following lemmas:

Lemma 1. *If a' and b' are any two elements of G', then x' and y' exist in G' such that $a'x'=b'$ and $y'a'=b'$.*

Lemma 2. *The system G' is a quasigroup if and only if $a'b'=a'c'$ and $b'a'=c'a'$ each implies $b'=c'$.*

Theorem 2. *If the homomorphic image G' of a quasigroup G is finite, then G' is a quasigroup.*[8]

Let $b_1', b_2', \cdots, b_n'$ be the elements of G', n a positive integer. Then by Lemma 1, for given i, $b_i'b_1', b_i'b_2', \cdots, b_i'b_n'$ are n distinct elements of G'. Right-hand cancellation is similarly established.

Theorem 3. *If the homomorphic image G' of a quasigroup G is associative, then G' is a quasigroup.*

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[8] Cf. [5, Theorem 4.13].

A multiplicative system which satisfies Lemma 1 and is associative is known to be a group (see [7, p. 19]).

If S is a subset of G, let $O[S]$ be the cardinal number of elements in S. Obviously

$$O[S] = O[aS] = O[Sa] \tag{A}$$

if a is any element of G.

We define $R(a)$ to be the set of elements g in G such that $g' = a'$ in G'. Then

$$\begin{aligned} &R(a)b \subseteq R(ab), \quad \text{and} \\ &aR(b) \subseteq R(ab); \end{aligned} \tag{B}$$

for if $g = d_1 b$ where $d_1' = a'$, then $g' = d_1' b' = a'b' = (ab)'$, and g lies in $R(ab)$. The second statement follows in the same way.

Let a and b be any elements of G. There exists x in G such that $a = bx$. By (B), $R(b)x \subseteq R(bx) = R(a)$, and thus $O[R(b)x] \leqq O[R(a)]$. By (A), $O[R(b)] = O[R(b)x]$. It follows that $O[R(b)] \leqq O[R(a)]$. But a and b were any elements of G. We have proved the following lemma.

LEMMA 3. *If a and b are any elements of G, then $O[R(a)] = O[R(b)]$.*[6]

LEMMA 4. *If G' is the homomorphic image of the quasigroup G, then G' is a quasigroup if and only if $R(ab) = aR(b) = R(a)b$.*

Let $R(ab) = aR(b) = R(a)b$. If $a'b' = a'c'$, then ac is an element of $R(ab) = aR(b)$, and c lies in $R(b)$, that is, $c' = b'$. By Lemma 2, G' is a quasigroup.

Conversely, let G' be a quasigroup. Let $g = xb$ be an element of $R(ab)$. Then $g' = x'b' = a'b'$, and by Lemma 2, $x' = a'$. Thus x lies in $R(a)$, and g lies in $R(a)b$. It follows that $R(a)b \supseteq R(ab)$. By (B), however, $R(a)b \subseteq R(ab)$. Then $R(a)b = R(ab)$, and by a similar argument $aR(b) = R(ab)$.

LEMMA 5. *If $O[R(a)]$ is finite, G' is a quasigroup.*

By (B), $R(ab) \supseteq aR(b)$. By Lemma 3 and (A), $O[R(ab)] = O[R(b)] = O[aR(b)]$. Since this order is finite, $R(ab) = aR(b)$. Similarly, $R(ab) = R(a)b$, and by Lemma 4, G' is a quasigroup.

THEOREM 4. *If G is a loop homomorphic to the multiplicative system G', and if the kernel of the homomorphism is finite, then G' is a loop.*

If e is the identity of G, then $R(e)$ is the kernel of the homomorphism. The theorem follows by Lemma 5.

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[6] Cf. [3, p. 770].

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A CONJECTURE OF KRISHNASWAMI

D. H. LEHMER

Let $T(N)$ denote the number of right triangles whose perimeters do not exceed $2N$, and whose sides are relatively prime integers. A list of all such triangles whose perimeters do not exceed 10000 has been given by A. A. Krishnaswami.[1] On the basis of this table he conjectured that

$$T(N) \sim N/7. \tag{1}$$

The asymptotic formula

$$T(N) \sim \pi^{-2} N \log 4 \tag{2}$$

follows from the general theory of "totient points," as developed by D. N. Lehmer in 1900. A statement equivalent to (2) will be found in his paper[2] (p. 328).

The conjecture (1) is not far wrong since

$$\pi^2/\log 4 = 7.11941466.$$

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[1] A. A. Krishnaswami, *On isoperimetrical Pythagorean triangles*, Tôhoku Math. J. vol. 27 (1926) pp. 332–348. Two omissions in Table I may be noted: For $s=3450$, $a=50$, $b=19$; for $s=3465$, $a=55$, $b=8$. This table is the basis for the one at the end of the present paper.

[2] D. N. Lehmer, *Asymptotic evaluation of certain totient sums*, Amer. J. Math. vol. 22 (1900) pp. 293–335.

In this paper we give a short proof of the fact that

$$T(N) = \pi^{-2}N \log 4 + O(N^{1/2} \log N). \tag{3}$$

The actual values of the error term for $N=500(500)5000$ are given in a short table at the end of this paper. The proof of (3) is based on the following lemmas.

LEMMA 1. *Let $\phi(m)$ denote the number of positive integers not exceeding m and prime to m. Then*

$$\Phi(x) \equiv \sum_{m \leq x} \phi(m) = 3\pi^{-2}x^2 + O(x \log x). \tag{4}$$

A proof of this well known result will be found for example in Hardy and Wright[3] (p. 266).

LEMMA 2. *Let $\Phi_e(x)$ and $\Phi_0(x)$ be defined by*

$$\Phi_e(x) = \sum_{m \leq x, m \text{ even}} \phi(m), \qquad \Phi_0(x) = \sum_{m \leq x, m \text{ odd}} \phi(m).$$

Then

$$\Phi_e(x) = \pi^{-2}x^2 + O(x \log x), \tag{5}$$

$$\Phi_0(x) = 2\pi^{-2}x^2 + O(x \log x). \tag{6}$$

PROOF. Since (6) follows from (4) and (5) it suffices to prove (5). To this effect we note that if m is even

$$\phi(m) = \begin{cases} \phi(m/2), & m \equiv 2 \pmod 4, \\ 2\phi(m/2), & m \equiv 0 \pmod 4. \end{cases} \tag{7}$$

Hence

$$\Phi_e(x) = \Phi_0(x/2) + 2\Phi_e(x/2) = \Phi(x/2) + \Phi_e(x/2).$$

Therefore

$$\Phi_e(x) = \sum_{\lambda=1}^{p} \Phi(2^{-\lambda}x) \qquad (p = [\log x/\log 2]).$$

Applying Lemma 1 we have

$$\Phi_e(x) = 3\pi^{-2}x^2 \sum_{\lambda=1}^{p} 4^{-\lambda} + O(x \log x)$$

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[3] G. H. Hardy and E. M. Wright, *Introduction to the theory of numbers*, Oxford, 1938. Lemma 1 appears to be due to Mertens, Journal für Mathematik vol. 77 (1871) pp. 289–291.

$$= \pi^{-2}x^2 + O\left(x^2\int_p^\infty 4^{-t}dt\right) + O(x \log x).$$

Since $p > \log x/\log 4$, the integral is $O(1/x)$. Hence (5) follows.

LEMMA 3. *Let* $0<\theta<1$, *and define* $F(\theta, x)$, $F_e(\theta, x)$ *and* $F_0(\theta, x)$ *by*

$$F(\theta, x) = \sum_{\theta x<m\leqq x} m^{-2}\phi(m), \qquad F_e(\theta, x) = \sum_{\theta x<m\leqq x,\, m \text{ even}} m^{-2}\phi(m),$$

$$F_0(\theta, x) = \sum_{\theta x<m\leqq x,\, m \text{ odd}} m^{-2}\phi(m).$$

Then as $x\to\infty$, *with* θ *fixed,*

$$F(\theta, x) = -6\pi^{-2}\log\theta + O(x^{-1}\log x), \tag{8}$$

$$F_e(\theta, x) = -2\pi^{-2}\log\theta + O(x^{-1}\log x), \tag{9}$$

$$F_0(\theta, x) = -4\pi^{-2}\log\theta + O(x^{-1}\log x). \tag{10}$$

PROOF. Since (10) follows from (8) and (9) it suffices to prove (8) and (9). Now

$$\begin{aligned} F(\theta, x) &= \sum_{\theta x<m\leqq x} m^{-2}\phi(m) = \sum_{\theta x<m\leqq x} \{\Phi(m) - \Phi(m-1)\} m^{-2} \\ &= \sum_{\theta x<m\leqq x} \Phi(m)\{m^{-2} - (m+1)^{-2}\} \\ &\quad - \Phi(\theta x)[\theta x + 1]^{-2} + \Phi(x)[x+1]^{-2}. \end{aligned}$$

By Lemma 1 these last two terms cancel to some extent and together contribute only $O(x^{-1}\log x)$. As for the rest

$$\begin{aligned} &\sum_{\theta x<m\leqq x} \Phi(m)\{m^{-2} - (m+1)^{-2}\} \\ &\quad = 3\pi^{-2}\sum (1-(1+m^{-1})^{-2}) + O\left(\sum m^{-1}(1-(1+m^{-1})^{-2})\log m\right) \\ &\quad = 3\pi^{-2}\sum 2m^{-1}(1+O(m^{-1})) + O\left(\sum m^{-2}\log m\right) \\ &\quad = 6\pi^{-2}\int_{\theta x}^{x} t^{-1}dt + O(x^{-1}) + O\left(\int_{\theta x}^{x} t^{-2}\log t\,dt\right) \\ &\quad = -6\pi^{-2}\log\theta + O(x^{-1}\log x), \end{aligned}$$

which gives (8). To prove (9) we note from (7) that

$$F_e(\theta, x) = F_0(\theta, x/2)/4 + F_e(\theta, x/2)/2 = F(\theta, x/2)/4 + F_e(\theta, x/2)/4.$$

Hence

$$F_e(\theta, x) = \sum_{\lambda=1}^{p} F(\theta, x/2^\lambda)4^{-\lambda} \qquad (p = [\log x/\log 2]).$$

Using (8) we find

$$F_s(\theta, x) = -6\pi^{-2}\log\theta\sum_{\lambda=1}^{\infty} 4^{-\lambda} + O\left(\int_x^{\infty} 4^{-t}dt\right) + O(x^{-1}\log x).$$

Since the integral is $O(x^{-1})$, (9) follows at once. This completes the proof of Lemma 3.

LEMMA 4. *Let $\phi(x, m)$ denote the number of integers $\leqq x$ and prime to m. Then*

$$\left|\,\phi(x, m) - xm^{-1}\phi(m)\,\right| < d(m)$$

where $d(m)$ is the number of divisors of m.

This follows easily from a familiar theorem of Legendre to the effect that

$$\phi(x, m) = \sum_{\delta \mid m} [x\delta^{-1}]\mu(\delta) \tag{11}$$

where μ is the Möbius function and the sum extends over all the divisors of m. In fact if we write

$$[x\delta^{-1}] = x\delta^{-1} - \epsilon(x, \delta)$$

so that

$$0 \leqq \epsilon(x, \delta) < 1,$$

then (11) becomes

$$\phi(x, m) = x\sum \delta^{-1}\mu(\delta) - \sum \epsilon(x, \delta)\mu(\delta).$$

The first sum is $m^{-1}\phi(m)$ and the second is less than

$$\sum_{\delta \mid m} 1 = d(m)$$

in absolute value. This proves the lemma.

Finally we need one more lemma.

LEMMA. 5.

$$\sum_{m \leqq x} d(m) = O(x \log x).$$

This is a very weak corollary of a famous result of Dirichlet (see Hardy and Wright,[3] p. 262–263).

We are now in a position to prove the following theorem.

THEOREM. *Let $T(N)$ denote the number of integral right triangles whose perimeters do not exceed $2N$ and whose sides are relatively prime, then*

$$T(N) = \pi^{-2}N \log 4 + O(N^{1/2} \log N).$$

PROOF. It is well known that all integral right triangles (a, b, c) are given by the parametric equations

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2$$

where m, n are integers with

$$n \leqq m. \tag{12}$$

Since the perimeter is supposed not to exceed $2N$ we have

$$mn + m^2 \leqq N. \tag{13}$$

In order to avoid the cases in which a, b, c have a common factor it is necessary to suppose that we choose m, n so that

$$m, n \text{ are coprime and not both odd.} \tag{14}$$

$T(N)$ is then merely the number of pairs of positive integers (m, n) such that (12), (13) and (14) hold. In case $m \leqq (N/2)^{1/2}$, (13) is a consequence of (12). In case $(N/2)^{1/2} < m \leqq N^{1/2}$, (12) is a consequence of (13). Hence if we define

$$\psi(m) = \begin{cases} 1 & \text{if } m \leqq (N/2)^{1/2}, \\ m^{-2}N - 1 & \text{if } (N/2)^{1/2} < m \leqq N^{1/2} \end{cases} \tag{15}$$

then the number of integers n that go with a given m is the number of integers prime to m not exceeding $m\psi(m)$ or $m\psi(m)/2$ according as m is even or odd. Hence if we set $x = N^{1/2}$

$$T(N) = \sum_{m \leqq x,\, m \text{ even}} \phi(m\psi(m), m) + \sum_{m \leqq x,\, m \text{ odd}} \phi\left(\frac{m\psi(m)}{2}, m\right).$$

By Lemma 4,

$$\begin{aligned} T(N) &= \sum_{m \leqq x,\, m \text{ even}} \psi(m)\phi(m) + 2^{-1} \sum_{m \leqq x,\, m \text{ odd}} \psi(m)\phi(m) + R(N) \\ &= \sum\nolimits_1 + 2^{-1} \sum\nolimits_2 + R(N) \end{aligned} \tag{16}$$

where

$$|R(N)| \leqq \sum_{m \leqq x} d(m) = O(x \log x) = O(N^{1/2} \log N).$$

By (15) with $\theta = 2^{-1/2}$ we can write

$$\sum\nolimits_1 = \Phi_e(\theta x) + NF_e(\theta, x) - \Phi_e(x) + \Phi_e(\theta x),$$

$$\sum\nolimits_2 = \Phi_0(\theta x) + NF_0(\theta,\, x) - \Phi_e(x) + \Phi_0(\theta x),$$

so that

$$\sum\nolimits_1 + 2^{-1} \sum\nolimits_2 = \Phi(\theta x) + \Phi_e(\theta x) - 2^{-1}\{\Phi(x) + \Phi_e(x)\} + 2^{-1}N\{F(\theta,\, x) + F_e(\theta,\, x)\}.$$

By Lemmas 1, 2, and 3 therefore we obtain after simplification

$$T(N) = \pi^{-2}(2 \log 2)x^2 + O(x \log x) = \pi^{-2}(\log 4)N + O(N^{1/2} \log N).$$

The following small table illustrates the error in (3):

$$E(N) = T(N) - \pi^{-2}N \log 4.$$

The function $C(N)$ is defined by

$$C(N)N^{1/2} \log N = 10^3\, E(N)$$

and gives some idea of the possible constant implied by the O term of (3).

N	$T(N)$	ΔT	$\pi^{-2} N \log 4$	$E(N)$	$C(N)$
500	70	70	70.23049	−0.23049	−1.6596
1000	140	71	140.46099	−0.46099	−2.1103
1500	211	69	210.69148	+0.30852	+1.0893
2000	280	69	280.92197	−0.92197	−2.7123
2500	349	73	351.15246	−2.15246	−5.5022
3000	422	70	421.38296	−0.61704	−1.4071
3500	492	68	491.61345	+0.38655	+0.8007
4000	560	71	561.84394	−1.84394	−3.5152
4500	631	72	632.07444	−1.07444	−1.9041
5000	703		702.30493	+0.69507	+1.1541

UNIVERSITY OF CALIFORNIA

CORRECTION: DERIVATIVES OF INFINITE ORDER

R. P. BOAS, JR., AND K. CHANDRASEKHARAN

It has been pointed out to us by S. Mandelbrojt that our statement[1] that M^{c}_{n+1}/M^{c}_{n} is nondecreasing is incorrect except on the interval $(-\infty, \infty)$ (where M^{c} must be replaced by M^{e}) and that for a finite interval there are in fact quasianalytic classes $C\{M_n\}$ which do not contain the class $C\{1\}$. However, Mandelbrojt has shown that our Theorem 2 is nevertheless correct; with his permission, we give his proof here. Theorem 2 states that, if $f(x)$ belongs to a quasianalytic class $C\{M_n\}$ in $a<x<b$ and if $f^{(n)}(x_0)\to L$ for one x_0 in (a, b), then $f(x)$ is analytic in (a, b) and consequently $f^{(n)}(x)\to Le^{x-x_0}$ in $a<x<b$. There are two cases: either $\liminf M_n^{1/n}>0$ or $\liminf M_n^{1/n}=0$. In the first case $C\{1\}\subset C\{M_n\}$ trivially and our original proof applies. In the second case, let $\{n_j\}$ be a sequence such that $M_j^{1/n_j}\to 0$. Since $|f^{(n_j)}(x_0)|<k^{n_j}M_{n_j}\to 0$ and $f^{(n)}(x_0)\to L$, we must have $L=0$. Given $\epsilon>0$, there exist p and i such that $|f^{(n)}(x_0)|<\epsilon$ for $n>p$ and $k^{n_j}M_{n_j}<\epsilon$ for $j>i$. For $n>p$ let $j>i$ and $n_j>p$; then for x in (a, b) and $|x-x_0|<1$,

$$f^{(n)}(x) = f^{(n)}(x_0) + (x - x_0)f^{(n+1)}(x_0) + \cdots$$
$$+ f^{(n_j)}(x')(x - x_0)^{n_j-n}/(n_j - n)!,$$

where x' is between x_0 and x. Then $|f^{(n)}(x)|\leq\epsilon\sum_{k=0}^{\infty}|x-x_0|^k/k!+\epsilon=\epsilon(e^{|x-x_0|}+1)$, which shows that $f^{(n)}(x)\to 0$ uniformly between x_0 and x (and so, by a repetition of the argument, if necessary, in (a, b)), and also that $f(x)$ is analytic.

In line 9 of page 523, replace ae^n by ke^n.

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Received by the editors September 10, 1948.

[1] R. P. Boas, Jr. and K. Chandrasekharan, *Derivatives of infinite order*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 523–526; p. 524.

ERRATA, VOLUME 13

R. D. Carmichael, *On Euler's ϕ-function*, pp. 241–243.

The author has been informed by Professor E. A. Hedburg that the second theorem may be stated as follows:

The number of solutions of $\phi(x)=2^n$ is $n+2$ when $n\leqq 31$, and exactly 32 when $31<n<1024$.

This corrects the result as originally stated by having 32 in place of 33 (as formerly given) and extends the result by having 1024 as the upper bound for n instead of 256 as in the initial paper. In making the extension, Professor Hedburg uses the now known facts that the numbers $2^{2^8}+1$ and $2^{2^9}+1$ are composite.

ERRATA, VOLUME 53

S. M. Shah, *On real continuous solutions of algebraic difference equations.*

p. 550, line 17. For "$\{f(x_0)\}^{B_p}$" read "$\{f(x_0)\}^{B^p}$."

p. 558, last line. For "(30)" read "(29)."

Index of Volume 53.

p. 1198, Hall, M. For "(1111)" read "(1110)."

p. 1199, Stoll, R. R. For "(1113)" read "(1112)."

ERRATA, VOLUME 54

S. P. Avann, *Ternary distributive semi-lattices.* Abstract 54-1-86.

p. 79, line 5 of the abstract. For "$(\mathfrak{T})\mathfrak{J}$ of $\mathfrak{T}$" read "$\mathfrak{J}(\mathfrak{T})$ of $\mathfrak{T}$."

Bjarni Jónsson and Alfred Tarski, *Representation problems for relation algebras.* Abstract 54-1-89.

p. 80, line 3 of abstract. For "$a|\overline{a|b}$" read "$\breve{a}\,|\,\overline{a|b}$."

p. 80, line 6 of abstract. For "$a\in U$" read "$(a, a)\in U$."

p. 80, line 11 of abstract. For "a" read "a", twice.

Albert Newhouse, *On finite extending groups.*

p. 563, line 6. For "$x^{3\cdot 1^m}$" read "$x^{3\ 1^m}$."

P. M. Morse, *Mathematical problems in operations research.*

p. 603, last line. For "mean" read "men."

J. S. Frame, *Group decomposition by double coset matrices.*

p. 748, display (4.15). For "$(Q_j^{t\prime})$" read "$(Q_j^t)'$."

Notes. p. 870, line 11. For "Rundberg" read "Rudberg."

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